ON SEHГАL’S MAPS
WITH A CONTRACTIVE ITERATE AT A POINT

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Abstract. Let $(X, d)$ be a complete metric space and $T$ a mapping of $X$ into itself. Suppose that for each $x \in X$ there exists a positive integer $n = n(x)$ such that for all $y \in X$,

$$d(T^n x, T^n y) \leq \alpha \max\{d(x, y), d(x, T y), d(x, T^2 y), \ldots, d(x, T^n y), d(x, T^n x)\},$$

holds for some $\alpha < 1$. With these assumptions our main result states that $T$ has a unique fixed point. This generalizes an earlier result of V. M. Sehgal and a recent result of the author.

1. We shall prove the following theorem, which is a generalization of Sehgal’s Theorem [3].

Theorem 1. Let $(X, d)$ be a complete metric space and $T : X \to X$ a mapping. If for each $x \in X$ there exists a positive integer $n = n(x)$ such that

(1) $$(T^n x, T^n y) \leq \alpha \cdot \max\{d(x, y), d(x, T y), d(x, T^2 y),
\quad d(x, T^3 y), \ldots, d(x, T^n y) d(x, T^n x)\}$$

holds for some $\alpha < 1$ and all $y \in X$, then $T$ has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $\lim_{m \to \infty} T^m x = u$.

Proof. First we shall show that for every $x \in X$, the orbit $\{T^m x\}_{m=0}^\infty$ is bounded. To prove this assertion, we shall show that for any $x \in X$

(2) $r(x) = \sup\{m > 0 : d(x, T^m x) \leq \max\{d(x, T^r x) : 0 < r \leq n(x)\}/(1 - \alpha)\}$.

Let $m$ be any, but fixed, positive integer and $k (k = k(x, m))$ a positive integer such that

(3) $d(x, T^k x) = \max\{d(x, T^r x) : 0 < r \leq m\}$.


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We may suppose that \( m > n(x) \) and \( k > n(x) \). Then by triangle inequality and by (1) we have

\[
d(x,T^k) \leq d(x,T^n x) + d(T^n x, T^n T^{k-n} x) \leq d(x,T^n x) + \alpha \cdot \max\{d(x,T^{k-n} x), d(x,T^{k-n+1} x), \ldots, d(x,T^k x), d(x,T^n x)\} \leq d(x,T^n x) + \alpha \cdot \max\{d(x,T^r x) : 0 < r \leq m\}.
\]

Using (3) we obtain \( d(x,T^k x) \leq d(x,T^n x) + \alpha \cdot d(x,T^k x) \) and hence

\[
d(x,T^k x) \leq d(x,T^n x)/(1-\alpha), \quad \text{i.e.,} \quad \max\{d(x,T^r x) : 0 < r \leq m\} \leq d(x,T^n x)/(1-\alpha).
\]

Since \( m \) was arbitrary, this implies

\[
\sup_{m > n(x)} \{d(x,T^m x) \leq d(x,T^n x)/(1-\alpha)\}.
\]

The relation (2) now follows immediately. Consequently, the orbit \( \{T^m x\}_{m=0}^\infty \) is bounded.

Now, let \( x_0 = x \in X, \ n_0 = n(x_0), \ x_1 = T^{n_0} x_0 \) and inductively

\[
n_k = n(x_k), \quad x_{k+1} = T^{n_k} x_k \quad (k = 1, 2, \ldots).
\]

Evidently, \( \{x_k\} \) is a subsequence of the orbit \( \{T^m x_0\}_{m=0}^\infty \). Using this subsequence we shall show that \( \{T^m x_0\}_{m=0}^\infty \) is a Cauchy sequence.

Let \( x_k \) be any fixed member of \( \{x_k\}_{k=1}^\infty \) and let \( x_p = T^p x_0 \) and \( x_q = T^q x_0 \) be any two members of the orbit \( \{T^m x_0\}_{m=0}^\infty \) which follow after \( x_k \). Then \( x_p = T^r x_k \) and \( x_q = T^s x_k \) for some \( r \) and \( s \), respectively. Now, using (1) we get

\[
d(x_k,x_p) = d(x_k,T^r x_k) = d(T^{n_k-1} x_{k-1},T^{n_k-1} x_{k-1}) \leq \alpha d(x_{k-1},T^{r_1} x_{k-1})
\]

where

\[
d(x_{k-1},T^{r_1} x_{k-1}) = \max\{d(x_{k-1},T^{r_1} x_{k-1}),d(x_{k-1},T^{r_1+1} x_{k-1}),\ldots,d(x_{k-1},T^{r_1+n_k-1} x_{k-1}),d(x_{k-1},T^{n_k-1} x_{k-1})\}.
\]

Similarly, \( d(x_{k-1},T^{r_1} x_{k-1}) \leq \alpha d(x_{k-2},T^{r_2} x_{k-2}) \), where

\[
d(x_{k-2},T^{r_1} x_{k-2}) = \max\{d(x_{k-2},T^{r_1} x_{k-2}),d(x_{k-2},T^{r_1} x_{k-2}),\ldots,d(x_{k-2},T^{n_k-1} x_{k-1})\}.
\]

Repeating this argument \( k \)-times we get

\[
d(x_k,x_p) \leq \alpha d(x_{k-2},T^{r_2} x_{k-2}) \leq \alpha^2 d(x_{k-2},T^{r_2} x_{k-2}) \leq \alpha^k d(x_0,T^{r_k} x_0).
\]

Hence \( d(x_k,x_p) \leq \alpha^k r(x) \). Similarly, \( d(x_k,x_q) = d(x_k,T^s x_k) \leq \alpha^k r(x) \).

Therefore,

\[
(4) \quad d(x_p,x_q) \leq d(x_k,x_p) + d(x_k,x_q) \leq \alpha^k \cdot 2r(x).
\]
Since $\alpha < 1$, (4) implies that the orbit $\{T^m x_0\}_{m=0}^{\infty}$ is a Cauchy sequence.

By the completeness of $X$ there is $u \in X$ such that $u = \lim_{m \to \infty} T^m x_0$. We shall show that $T^{n(0)} u = u$. For $m \geq n = n(u)$, we now have
\[
d(T^u, T^n T^{m} x_0) \leq \alpha \cdot \max\{d(u, T^m x_0), d(u, T^{m+1} x_0), \ldots,
\]
\[
d(u, T^{m+n} x_0), d(u, T^n u)\}
\]
and on letting $m$ tend to infinity it follows that
\[
d(T^u u, u) \leq \alpha d(u, T^n u).
\]
Since $\alpha < 1$, we see that $u$ is a fixed point of $T^n u$.

To show that $u$ is a fixed point of $T$, let us assume that $T u \neq u$ and let
\[
d(u, T^k u) = \max\{d(u, T^n u) : 0 < r \leq n = n(u)\}.\]

Then
\[
d(u, T^k u) = d(T^u, T^n T^u) = d(T^n u, T^n T^k u) \leq
\]
\[
\alpha \cdot \max\{d(u, T^k u), d(u, T^{k+1} u), \ldots, d(u, T^{k+n} u), d(u, T^n u)\} \leq
\]
\[
\alpha d(u, T^k u).
\]
Since $\alpha < 1$, it follows that $d(u, T^k u) = 0$, which implies that $u$ is a fixed point of $T$. The uniqueness of a fixed point of $T$ follows immediately from (1). This completes the proof of the Theorem.

2. If we suppose that $T$ is continuous, then we may prove the following theorem.

**Theorem 2.** Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a continuous mapping which satisfies the following condition: for each $x \in X$ there is a positive integer $n = n(x)$ such that for all $y \in X$,
\[
d(T^n x, T^n y) \leq \alpha \cdot \max\{d(x, y), d(x, T y), d(x, T^2 y), \ldots, d(x, T^n y), d(x, T x), d(x, T^2 x), \ldots, d(x, T^n x)\},
\]
where $0 \leq \alpha < 1$. Then $T$ has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $\lim_{k \to \infty} T^k x = u$.

**Proof.** Let $x$ be an arbitrary point in $X$. Then, as in the proof of Theorem 1, the orbit $\{T^m x\}_{m=0}^{\infty}$ is bounded and is a Cauchy sequence in the complete metric space $X$ and so it has a limit $u$ in $X$. Since by the hypothesis $T$ is continuous, it follows that $T^n u$ is continuous, which implies that
\[
T^n u = T^n u \lim_{m \to \infty} T^m x = \lim_{m \to \infty} T^{m+n(u)} x = u.
\]
Therefore, $u$ is a fixed point of $T^n u$. By the same arguments as in the proof of Theorem 1, it follows that $u$ is a unique fixed point of $T$. This completes the proof of the Theorem.
Remark. The condition that $T$ be continuous in Theorem 2 may be relaxed by the following: $T^n(x)$ is continuous at a point $x \in X$.

We now note that the condition that $T^n(u)$ be continuous at $u$ is necessary in Theorem 2. This is easily seen by letting $X$ be the closed interval $[0,1]$ with the usual metric. $X$ is then complete. Define a discontinuous mapping $T$ on $X$ by $T(0) = 1$ and $Tx = x/2$, if $x \neq 0$. We then have

$$d(T^2x, T^2y) \leq \max\{d(x, Ty), d(x, Tx)\}/2$$

for all $x$ and $y$ in $X$ and so $T$ satisfies (5) with $\alpha = 1/2$. $T$ however has no fixed point, because $T^n$ is not continuous at 0 for any $n = 2, 3, \ldots$

REFERENCES


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