ON THE FOURIER COEFFICIENTS OF A FUNCTION OF 
\(\Lambda - BOUNDED \ VARIATION\)

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**Definition.** \(f\) is of \(\Lambda\)-bounded variation on the interval \(I = [a, b], (\Lambda - BV),\) if

\[
\sum_{i=1}^{\infty} |f(I_i)|/\lambda_i < \infty
\]

for any decomposition \(\{I_i\}\) of \(I\), where \(\Lambda = \{\lambda_i\}\) is an increasing sequence of positive numbers such that \(\sum \lambda_i^{-1} = \infty\) and

\[
f(I_i) = f(b_i) - f(a_i) \quad \text{for} \quad I_i = [a_i, b_i].
\]

The fundamental properties of function of this class are given in the following.

[I] \(\Lambda - BV \subset L^\infty\).

[II] The function of \(\Lambda - BV\) has only discontinuous points of the first kind, so, at most denumerable. \(\Lambda - BV \subset W\), (c.f. B. I. Golubov [2]).

[III] The Helly’s selection theorem holds for these functions.

[IV] The followings are equivalent.

(i) \(f \in \Lambda - BV\).

(ii) There exists a \(M > 0\) such that \(\sum |f(I_i)|/\lambda_i < M\) for every decomposition \(\{I_i\}\) of \(I\),

(iii) There exists a \(M > 0\) such that for every finite collection \(\{I_i\}\) \((i = 1, 2, \ldots, N) \subset I\),

\[
\sum_{I}^{N} |f(I_i)|/\lambda_i < M.
\]
[V] $\Lambda - BV$ is a Banach space with the norm

$$||f||_{\Lambda - BV} + |f(a)| \leq V_\lambda(b),$$

where $V_\lambda(b) = \sup\{\sum |f(I_i)|/|\lambda_i|; \{I_i\} \text{ such that } I = \bigcup I_i\}$.

[VI] If $\{\lambda_i\}$ is a strictly sequence, $BV \subset \Lambda - BV$.

[VII] $BV = \bigcup\{\Lambda - BV; \Lambda\}$.

[VIII] $\Lambda - BV \cap C$ is a closed subspace of $\Lambda - BV$.

2. Let $f$ be an $2\pi$-periodic integrable function on $[0, 2\pi]$ and $\{a_n\}$ and $\{b_n\}$ are Fourier coefficients of $f$. At first we show the order of the magnitude $\{a_n\}$ and $\{b_n\}$ of $f \in \Lambda - BV$.

**Lemma.** If $A \in \Lambda - BV$, then

$$a_n, b_n = O(\lambda_n/n).$$

**Corollary.** If $f \in \{n^\alpha\} - BV$, $0 \leq \alpha \leq 1$, then

$$a_n, b_n = O(1/n^{1-\alpha}).$$

**Proof of Lemma.** From (ii) of [IV], we have

$$\sum_{i=1}^{2N} f(I_i^x)/\lambda_i < M$$

for some $M > 0$, where $I_i^x = x + (i - 1)\pi/N, x + i\pi/N$ ($i = 1, 2, \ldots, 2N$), that is

$$\sum_{i=1}^{2N} |f(I_i^x)| = O(\lambda_{2N}).$$

From the properties of $\lambda_n$, we assume that $\lambda_{2n} = O(\lambda_n)$, so,

$$\sum_{i=1}^{2N} |f(I_i^x)| = O(\lambda_N).$$

(3)

It is well known (c.f. N. K. Bari [1] and M. and S. Izumi [3])

$$|a_N| \leq (1/2\pi) \int_{0}^{2\pi} |f(x + \pi/N) - f(x)|dx$$

$$\leq (1/2\pi) \int_{0}^{2\pi} |f(I_i^x)|dx, \ (i = 1, 2, \ldots, 2N).$$
Adding such inequalities for $i = 1, 2, \ldots, 2N$, we have (1) by (3). Similarly, we have $b_n = O(\lambda_n/n)$.

Now, we give the necessary condition for continuity of $\Lambda - BV$.

(5) \[ I_N = (N/\lambda_N) \sum_{k=1}^{\infty} \rho_k^2 \sin^2(n\pi/2N) = o(1). \]

(6) \[ J_N = (N/\lambda_N)^{-1} \sum_{k=1}^{N} n^2 \rho_k^2 = o(1). \]

(7) \[ T_N = N^{-1} \lambda_N^{-1/2} \sum_{k=1}^{N} n \rho_k = o(1). \]

(8) \[ S_N = (\log N)^{-1} \lambda_N^{-1/2} \sum_{k=1}^{N} \rho_k = o(1). \]

(9) \[ H_N = N \lambda_N^{-1} \sum_{k=1}^{N} \rho_k^2 = o(1). \]

**Theorem 1.** If $f \in \Lambda - BV$, then we have

(i) \[ f \in C \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8). \]

(ii) \[ (9) \Rightarrow (6), \]

where $p_n = \{a_n^2 + b_n^2\}^{1/2}$.

(5') \[ I_N^{(\alpha)} = N^{1-\alpha} \sum_{k=1}^{\infty} \rho_k^2 \sin^2(n\pi/2N) = o(1). \]

(6') \[ J_N^{(\alpha)} = N^{-(1+\alpha)} \sum_{k=1}^{N} n^2 \rho_k^2 = o(1). \]

(7') \[ T_N^{(\alpha)} = N^{-(1+\alpha)/2} \sum_{k=1}^{N} n \rho_k = o(1). \]

(8') \[ S_N^{(\alpha)} = \begin{cases} N^{-\alpha/2} \sum_{k=1}^{N} \rho_k; & 0 < \alpha < 1 \\ (\log N)^{-1} \sum_{k=1}^{N} \rho_k; & \alpha = 1 \end{cases} = o(1) \]

(9') \[ H_N^{(\alpha)} = N^{1-\alpha} \sum_{k=1}^{\infty} \rho_k^2 = o(1). \]

**Corollary 2.** If $f \in \{n^\alpha\} - BV$ \((0 \leq \alpha \leq 1)\), then we have
(i) \( f \in C \Rightarrow (5') \Rightarrow (6) \Rightarrow (7') \Rightarrow (8') \).

(ii) \((9') \Rightarrow (6')\).

**Theorem 2.** If \( f \in \{ p^\alpha \} - BV \) (0 \( \leq \alpha < 1/2 \)) and

\[
J_N^{(\alpha)} = N^{-(1+\alpha)} \sum_{1}^{[N^\alpha]} n^2 \rho_n^2 = o(1)
\]

for some \( \beta > (1 - \alpha/1 - 2\alpha) \), then we have \((6')\).

**Remark 1.** For \( f \in BV \), these results have been got by N. Wiener [13] and S. M. Lozinskii [4].

**Remark 2.** If \( f \) is \( r \)-th bounded variation, the similar results are given by B. I. Gohburg [2].

**Proof of Theorem 1**

(i) \( f \in C \Rightarrow (5) \); From (iii) of [IV], we have

\[
\sum_{1}^{2N} |f(I_n^*)|^2 = \sum_{1}^{2N} |f(I_n^*)|/\lambda_i \cdot |f(I_n^*)| < M \lambda_2 N \omega_f(\pi/N)
\]

where \( \omega_f(\cdot) \) is a modulus of continuity of \( f \). Then, from \( \lambda_2 N = O(\lambda_N) \), we get

\[
2N \int_{0}^{2\pi} |f(I_n^*)|^2 dx = O(\lambda_N \omega_f(\pi/N))
\]

where \( I^* = [x - \pi/N, x + \pi/N] \). By Parseval’s equality,

\[
I_N = (N/\lambda_N) \sum_{i}^{\infty} \rho_n^2 \sin^2(n\pi/2N) = O(\omega_f(\pi/N)) = o(1).
\]

(6) \( \Rightarrow (7) \); From Schwartz’s inequality and (6), we get

\[
T_N^2 = (N^2 \lambda_N)^{-1} \left( \sum_{1}^{n} n \rho_n \right)^2 < (N \lambda_N)^{-1} \sum_{1}^{N} n^2 \rho_n^2 = J_N = o(1).
\]

(7) \( \Rightarrow (8) \); Putting \( u_N = \sum_{1}^{N} n \rho_n \), then \( u_N = O(N \lambda_N^{1/2}) \) and

\[
\sum_{1}^{N} \rho_n = \sum_{1}^{N} \frac{1}{n} (u_n - u_{n-1})
\]

\[
= (u_N/N) + \sum_{1}^{N-1} (n+1)^{-1} (u_n/n)
\]

\[
= o(\lambda_N^{1/2}) + o(\lambda_N^{1/2}) \sum_{1}^{N-1} (1/n + 1)
\]

\[
= o(\lambda_N^{1/2} \cdot \log N).
\]
(ii) (9) \Rightarrow (6); Putting \( A_N = \sum_N^\infty \rho_n^2 \), we have \( A_N = o(\lambda_N/N) \) from (9). So,

\[
J_N = (N\lambda_N)^{-1} \sum_1^N n^2 \rho_n^2 \\
= (N\lambda_N)^{-1} \left\{ N^2 A_N - \sum_1^{N-1} (2n+1)A_n \right\} \\
= o(1) + o \left( 1/N \lambda_N \cdot \sum_1^{N-1} (2n+1)(\lambda_n/n) \right) = o(1).
\]

**Proof of Theorem 2.**

\[
I_N^{(a)} = N^{1-a} \sum_1^\infty n^2 \sin^2(n\pi/2N) \\
= N^{1-a} \left\{ \sum_1^{[N\pi]} \rho_n^2 \sin(n\pi/2N) + \sum_{[N\pi]+1}^\infty \rho_n^2 \sin^2(n\pi/2N) \right\} \\
= I_{N,1}^{(a)} + I_{N,2}^{(a)} \text{ for some } \varepsilon > 0.
\]

From Corollary 1, we have \( \rho_n = O(n^{\alpha-1}) \). So, accounting of \( 0 \leq \alpha < 1/2 \),

\[
I_{N,2}^{(a)} = O \left( N^{1-a} \sum_{[N\pi]+1}^\infty n^{2\alpha-2} \right) \\
O = \left( N^{1-a} \int_{N\pi}^\infty t^{2\alpha-2} dt \right) \\
O = (N^{1-a}(N\pi)^{2\alpha-1}) = O(N^\alpha \pi^{2\alpha-1}).
\]

Putting \( x = N^{\beta-1} \), then

\[
I_{N,2}^{(a)} = O(N^{(1-a)-\beta(1-2\alpha)}) = o(1).
\]

Further,

\[
I_{N,1}^{(a)} = N^{1-a} \sum_{n=1}^{[N\pi]} \rho_n^2 (n\pi/2n)^2 \\
= O \left( N^{-(1+\alpha)} \sum_{1}^{[N\pi]} \rho_n^2 n^2 \right) = O(J_N^{(a)}) = o(1).
\]
REFERENCES


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