ON SOME SOLUTION OF EQUATION
\[ \varphi(x) + \varphi(f(x)) = F(x) \]
UNDER THE CONDITION THAT \( F \) SATISFIES \( F(f^p(x)) = F(x) \)

Mirjana Malenica

The results of this paper are connected with solutions \( \varphi(x) \) of the functional equation

\[ \varphi(x) + \varphi(f(x)) = F(x) \]

where \( f(x) \) and \( f(x) \) are given functions. Many mathematicians have been concerned with equation (1). Some special cases of this equation have been considered by H. Steinhaus [8], G. Hardy [4], R. Rad\'si [7] and others. Equations of more general type have been examined by M. Ghermanencu [3] and Kitamura [5].

Some papers of M. Kuczma and M. Barjaktarevi\'c are of special interest to this paper. In the paper [6] M. Kuczma proves that under some natural assumptions equation (1) possesses infinitely many solutions which continuous for every \( x \) this is not a root of the equation

\[ f(x) = x. \]

In the same paper, under assumption that the solution is continuous for \( x = x_0 \), satisfying equation (\( \ast \)), the author proves the existence of a most such solution. Further, he proves that with addition assumptions such a solution exists and it is given by an explicit formula.

M. Barjaktarevi\'c in his paper [1] proves four theorems. Two of them, using different regular methods of summability for the series

\[ \frac{1}{2}F(h) + \sum_{k=0}^{\infty} (-1)^k \left( F(f^k(x)) - F(h) \right), \]

\[ \text{\textsuperscript{1}} \text{The research presented in this paper was supported by zajednica za nau\'cki rad SR BiH.} \]
give sufficient conditions for the existence of a solution of equation (1). The other
two theorems give sufficient conditions for existence of a unique continuous solution
of (1) in \((a, b)\).

The author of this paper, using the same method, examined some solutions
of the equation (1) in the papers [9], [10] and [11]. In [9] and [10], using different
regular methods of the summability of the series (2), sufficient i. e. necessary and
sufficient conditions for the existence of a solution of the equation (1) are given. Al-
so, sufficient and necessary conditions are given, under which the considered regular
methods are translatable from the right on a set

\[
\mathcal{U} = \{ \{ s_n(x) \}_{n=0}^{+\infty} \}_{x \in [a, b]},
\]

in the sense of the definition of translation given in [2], page 21, where \(s_n(x)\) is a
partial sum of range \(n\) of the series (2).

In the [11] equation (1) is considered under the condition that the function \(F\)
has a special form for which the series (2), in general, diverges. Also conditions, are
given the series (2) to be \(T\)-sumable \((T = (a_{kn})\) a regular matrix transformation)
to \(\varphi\) so that the following relations are valid

\[
T s_n(x) \rightarrow \varphi(x) \\
\varphi(f(x)) + \varphi(x) = F(x).
\]

Connected with this, in [11] theorem I is proved.

In this paper we consider a similar cases depending on \(F\) as in [11], but with
a more general function \(F\).

Theorem I of [11] is related to \(p = 1\) (see paragraph II) and the results of this paper
are related to the case \(p > 1\). There are some specific differences between these two
cases.

In theorem I of [11] and throughout this paper it is assumed that

1. \(f(x)\) is a continuous strongly monotonic function on \([a, b]\) and that

\[
f(a) = a, \quad f(b) = b; \quad f(x) > x, \quad x \in (a, b);
\]

2. \(f^0(x) = x, f^{v+1}(x) = f(f^v(x)), \quad v \in \{0, \pm 1, \pm 2, \ldots \}\).

II

Let \(p \in \{2, 3, \ldots \}, p\) fixed and let the function \(F(x)\) have the properties

\[
\begin{align*}
F(f^p(x)) &= F(x) \quad x \in [a, b] \\
F(f^i(x)) &\neq F(x) \quad x \in [a, b], \quad i = 1, 2, \ldots , p - 1, p > 1 \\
F(x) &= g(x) \quad x \in [x_0, f(x_0)), x_0 \in (a, b)
\end{align*}
\]

where \(g(x)\) is an arbitrary chosen functions with domain \([x_0, f(x_0))\).
Further, sometimes use the following condition (5) on \( F(x) \):

\[
\Delta = \begin{cases} 
\text{there are points } x_l \in \{0, 1, \ldots, p\}, \text{ in } (a, b) \text{ so that} \\
1F(x_0) \ F(f(x_0)) \ F(f^2(x_0)) \ldots \ F(f^{p-1}(x_0)) \\
1F(x_1) \ F(f(x_1)) \ F(f^2(x_1)) \ldots \ F(f^{p-1}(x_1)) \\
1F(x_2) \ F(f(x_2)) \ F(f^2(x_2)) \ldots \ F(f^{p-1}(x_2)) \\
\ldots \\
1F(x_p) \ F(f(x_p)) \ F(f^2(x_p)) \ldots \ F(f^{p-1}(x_p)) 
\end{cases} \neq 0.
\]

We will be concerned with finding solution \( \varphi(x) \) on \([a, b]\) of equation (1) that have form

\[
\varphi(x) = a + b_0 F(x) + b_1 F(f(x)) + \cdots + b_{p-1}(x) F(f^{p-1}(x)).
\]

In connection with this the following theorem is valid.

**Theorem 1.** For the equation (1), in which function \( F(x) \) has the properties (4), to have a solution \( \varphi(x) \) on \([a, b]\) of the form (6), it is sufficient and, if \( F \) also satisfies condition (5), it is necessary that

\[
\begin{cases} 
p \text{ is odd number} \\
a = 0, \ b_i = \frac{1}{2}(-1)^i, \ i \in \{0, 1, \ldots, p-1\} 
\end{cases}
\]

i.e. that the function \( \varphi(x) \) has the form

\[
\varphi(x) = \sum_{i=0}^{p-1} (-1)^i F(f^i(x)).
\]

**Proof.** To prove that the conditions (7) are sufficient it is enough to substitute (8) into (1) in place of \( \varphi(x) \). Let us prove that the conditions (7) are necessary under the assumption that (4) and (5) are satisfied.

Now, according to (6), we get

\[
\varphi(x) + \varphi(f(x)) - F(x) = 2a + (b_{p-1} + b_0 - 1)F(x) + \sum_{i=1}^{p-1} (b_i + b_{i-1})F(f^i(x)).
\]

For \( \varphi(x) \) defined with (6), to be a solution of our equation it is necessary (and sufficient) that the right of (9) is identically equal to zero on \([a, b]\). Setting the right side of (9) equal to \( x_i(i = 0, 1, \ldots, p) \), we get \( p + 1 \) linear equation

\[
\beta_0 F(x_i) + \beta_1 F(f(x_i)) + \cdots + \beta_{p-1}(x_i) F(f^{p-1}(x_i)) + 2a = 0 \quad (i = 0, 1, \ldots, p),
\]
where for the sake of simplicity, we put
\[
\begin{align*}
\beta_0 &= b_{p-1} + b_0 - 1 \\
\beta_i &= b_i + b_{i-1} & (i = 1, \ldots, p - 1).
\end{align*}
\] (11)

System (10) is the system of \( p + 1 \) linear equation with \( p + 1 \) unknown elements
\[
2a, \beta_0, \beta_1, \ldots, \beta_{p-1}.
\]

Since the determinante \( \Delta \) of this system by assumption is different than zero, system (10) has only the trivial solution
\[
a = 0, \beta_i = 0 & (i = 0, 1, \ldots, p = 1).
\] (12)

System (12) reduce to a system of linear equation which for \( p \) odd number gives (7). For \( p \) an even number she system is contradictory.

Remark 1. We note that for the case \( p \geq 2 \) and \( p \) an even number \( \varphi(x) \) of the form (8) is not a solution of (1). Namely, we have
\[
\varphi(x) + \varphi(f(x)) = 0
\]

which contradics (5) in which \( \Delta \neq 0 \).

This proves the theorem.

We note the following: If \( F(x) \) has properties (4), then the series (2) obviously diverges since
\[
a_n(x) = (-1)^n \{ F f^n(x) - F(b) \}
\]
doesn't converge to zero as \( n \to \infty \).

Because of this, it is interest to ask if there exists a regular method of summability \( T = (a_{kk}) \) with which the series (2) wil be summable to \( T \)-sum \( \varphi(x) \), that is the solution of the considered equation, taking into account that \( F(x) \) has properties in (4). In the connection with that we have

**Theorem 2.** Let the following condition be satisfied:

a) function \( F \) has properties (4) and \( p > 2 \) is odd number,

b) the series (2) is \( T \)-sumable to sum \( \varphi(x) \), where \( T = (a_{kk}) \) is a regular matrix transformation

c) the limit
\[
\lim_{k \to \infty} B_r^k \text{ exists for each } r \in \{0, 1, \ldots, 2p - 2\},
\]

where
\[
B_r^k = \sum_{n=1}^{\infty} a_{k,n,2p+r}, \quad r \in \{0, 1, \ldots, 2p - 2\}.
\]
For the functions $\varphi(x)$ to be a solution of the equation (1) it is sufficient and, if $F$ in addition satisfies the conditions in (5) it is necessary that

\[(13) \quad a = 0, \quad b_i = \frac{1}{2}(-1)^i, \quad i \in \{0, 1, \ldots, p - 1\}\]

and solution $\varphi(x)$ is given by (6).

**Proof.** Because of theorem 1, it is sufficient to prove that, under the conditions of the present theorem, functions $\varphi(x)$ has the from (6).

Namely, under conditions of the present theorem we have

\[
s_n(x) = -\frac{1}{2}(-1)^n F(b) + \sum_{v=0}^{n} (-1)^v F_v \quad \text{with} \quad F_v = F(f^v(x)),
\]

\[
s'_k(x) = \frac{1}{2} F(b) \left\{ \sum_{n=0}^{\infty} a_{k,2n} - \sum_{n=0}^{\infty} a_{k,2n+1} \right\} + \sum_{n=0}^{\infty} a_{kn} \left( \sum_{v=0}^{n} (-1)^v F_v \right).
\]

If we put

\[
n = 2p \cdot m_n + r_n \quad (r_n = 0, 1, \ldots, 2p - 1; \quad m_n \in \{0, 1, 2, \ldots\}),
\]

it is easy to prove

\[
\sum_{v=0}^{n} (-1)^v F_v = \begin{cases} 
\sum_{v=0}^{p} (-1)^v F_v, & p \text{ is odd number } r_n \in \{0, 1, \ldots, 2p - 1\}.
\end{cases}
\]

\[
2m_n \sum_{v=1}^{p-1} (-1)^v F_v + \sum_{v=0}^{rn} (-1)^v F_v, \quad p \text{ is even number}
\]

Now for $p$ odd number

\[
s'_k(x) = -\frac{1}{2} F(b) \left\{ \sum_{n=0}^{\infty} a_{k,2n} - \sum_{n=0}^{\infty} a_{k,2n+1} \right\}
\]

\[
+ \sum_{n=0}^{\infty} a_{k,2pm_n + r_n} \left( \sum_{v=0}^{rn} (-1)^v F_v \right), \quad r_n \in \{0, 1, \ldots, 2p - 1\},
\]

\[
\varphi(x) = \lim_{k \to \infty} s'_k(x) = -\frac{1}{2} F(b) \{ B_0 + B_2 + \cdots + B_{2p-2} + B_1 - B_3 - \cdots - B_{2p-1} \}
\]

\[
+ \sum_{v=0}^{2p-1} B_r \left( \sum_{v=0}^{r} (-1)^v F_v \right),
\]

where

\[
B_r^k = \sum_{n=0}^{\infty} a_{k,n,2r+r}, \quad r \in \{0, 1, \ldots, 2p - 2\},
\]
so that
\[
\sum_{n=0}^{\infty} a_{k,2n} = B_0^k + B_2^k + \cdots + B_{2p-2}^k, \quad \sum_{n=0}^{\infty} a_{k,2n+1} = B_1^k + B_3^k + \cdots + B_{2p-1}^k,
\]
and
\[
B_r = \lim_{k \to \infty} B_r^k, \quad r \in \{0, 1, \ldots, 2p - 2\}
\]
and
\[
\lim_{k \to \infty} \sum_{n=0}^{\infty} a_{k,2n} = B_0 + B_2 + \cdots + B_{2p-2}, \quad \lim_{k \to \infty} \sum_{n=0}^{\infty} a_{k,2n+1} = B_1 + B_3 + \cdots + B_{2p-1}.
\]

Because of this, taking into account (4), after a long but straightforward calculation we get
\[
\varphi(x) = a + b_0 F_0 + b_1 F_1 + b_2 F_2 + \cdots + b_{p-1} F_{p-1},
\]
where the coefficients \(a, b_i (i = 0, 1, \ldots, p - 1)\) are given by
\[
a = \frac{1}{2} F(b)(B_1 + B_3 + \cdots + B_{2p-1} - B_0 - B_2 - \cdots - B_{2p-2})
\]
\[
b_j = (-1)^j \sum_{i=0}^{p-1} B_{i+j} \quad (j = 0, 1, 2, \ldots, p - 1).
\]

Therefore \(\varphi(x)\) is of the form (6).

**Remark 2.** We note that the case \(p = 1\), in a more general form, is included in theorems I of [11].

REFERENCES

[9] Mälenica M. *O rješanju funkcionalne jednačine \(\varphi(x) + \varphi(f(x)) = F(x)\)*, Akademija nauka i umjetnosti BiH, Radovi LXI/20, Sarajevo 1980. g. pp. 103–109.
[10] Mälenica M. *O rješavanju funkcionalne jednačine \(\varphi(x) + \varphi(f(x)) = F(x)\)* i neki specijalni primjeri regularnih transformacija, Akademija nauka i umjetnosti BiH, Radovi LXVIII/6, Sarajevo 1980. g. (u stampi).
On some solution of equation

[11] Maćenica M. O jednom rešenju jednačine \( \varphi(x) + \psi(f(x)) = F(x) \) kada funkcija \( F \) zadovoljava uslov \( F(f(x)) = F(x) \), Akademija nauka i umjetnosti B i H, Radovi LXIX/20, Sarajevo 1989. g. (u stampi).

Milena Maćenica
71000 Sarajevo
Ilja Engela 19
Jugoslavija