FOUR VARIATIONS ON A THEME OF S. B. PREŠIĆ CONCERNING SEMIGROUP FUNCTIONAL EQUATIONS

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1. Introduction. Let $S_1$ and $S_2$ be nonempty sets and suppose that:
   (i) $g_1, \ldots, g_n$ map $S_1$ into $S_1$;
   (ii) $G$ is the minimal semigroup generated by $g_1, \ldots, g_n$;
   (iii) $J$ maps $S_1 \times S_2^n \{\top, \bot\}$;
   (iv) $f \in S_2^n$.

In a number of papers ([1]–[7]) S. B. Prešić considered various instances of the equation in $f$:

$$(1.1) \quad J(x, f(g_1 x), f(g_2 x), \ldots, f(g_n x)) = \top \quad (g_i x = g_i(x))$$

where $g_1$ is the identity mapping, and under certain conditions determined its general solution. We briefly sketch Prešić results.

In [1] he constructed the general solution of the equation

$$f(x) = f(gx),$$

under the condition that $g$ is a bijection, i.e. that $G$ is a group. In papers [3], [4], [6] the general solution of the equation

$$(1.2) \quad a_1(x)f(x) + a_2(x)f(g_2 x) + \cdots + a_n(x)f(g_n x) = F(x)$$

is determined under the condition that $G$ is a group of order $n$. (Of course, in the case of equation (1.2) it is necessary to assume that $S_2$ is a field, and that $a_k: S_1 \to S_2, F: S_1 \to S_2$ are given).

In [5] Prešić determined the general solution of the equation

$$J(f(x_1, x_2, \ldots, x_n), f(x_2, x_3, \ldots, x_1), \ldots, f(x_n, x_1, \ldots, x_{n-1})) = \top$$
where $S_1 = A^n$, $x_i \in A$. Finally, in [7] Prešič constructed the general solution of the equation (1.1) under the condition that $G$ is a group.

Hence, if $G$ is a group, the problem of solving (1.1) is completely solved. However, Prešič also tried to solve (1.1) without additional conditions on the semigroup $G$. One such attempt is presented in [2]. Naturally, if the condition “$G$ is a group” is suppressed, one has to introduce other conditions — in fact, a request on $J$, i.e. on the form of the equation (1.1). One such condition is given in [2], theorem 1. Also, paper [4], theorem 2, where the equation (1.2) with $F(x) = 0$ is considered, a condition for the coefficients $a_i$ and on the semigroup $G$ is imposed which makes it possible to construct the general solution of that equation.

In this note we shall consider some special cases of the equation (1.1); in other words, we shall develop some variations on this theme of S. B. Prešič.

2. First variation. Consider the equation

$$J(f(g_1x), f(g_2x), \ldots, f(g_nx)) = T$$

and suppose that $G = \{g_1, g_2, \ldots, g_n\}$ is a semigroup. Denote $g_ig_j$ by $g_{ij}$, and let $M$ be the matrix of the table of $G$, i.e. $M = \|g_{ij}\|_{n \times n}$. Denote the i-th column of $M$ by $c_i(M)$.

If the following conditions are satisfied:

(i) there exists $i$ ($1 \leq i \leq n$) such that $c_i(M) = \|g_1g_2\ldots g_n\|_i$; in other words, the semigroup $G$ has a right unit element;

(ii) for every $i$ ($1 \leq i \leq n$) we have

$$\{c_1(M), c_2(M), \ldots, c_n(M)\} = \{c_1(Mg_i), c_2(Mg_i), \ldots, c_n(Mg_i)\}$$

then every possible equation (2.1) has the general solution of the form

$$f(x) = F(\Pi(x), \Pi(g_1x), \Pi(g_2x), \ldots, \Pi(g_nx))$$

where $\Pi: S_1 \rightarrow S_2$ is arbitrary and $F: S_n^{n+1} \rightarrow S_2$ is constructed by the method described by Prešič [7].

**Remark 2.1.** If $G$ is a group, conditions (i) and (ii) are satisfied.

**Remark 2.2.** For any semigroup $G$ the inclusion

$$\{c_1(Mg_i), c_2(Mg_i), \ldots, c_n(Mg_i)\} \subset \{c_1(M), c_2(M), \ldots, c_n(M)\}$$

is valid for all $i$. Condition (ii) requests that $\subset$ is replaced by $=.$

**Remark 2.3.** If $G$ is a monoid (i.e. a semigroup with identity) satisfying (ii), then $G$ is a group.

**Remark 2.4.** There exists a semigroup satisfying (i) and (ii) which is not a group. An example is provided by $G = \{g_1, g_2, \ldots, g_n\}$ defined by $g_kg_i = g_k$ for
all $i$, $k = 1, 2, \ldots, n$. It is a question whether this is the only semigroup satisfying (i) and (ii) which is not a group.

**Example 2.1.** Let $G$ be the semigroup defined in Remark 2.4. and consider the equation

$$a_1 f(g_1 x) + a_2 f(g_2 x) + \cdots + a_n f(g_n x) = 0. \quad (2.2)$$

(Here we suppose that $S_2$ is a field and that $a_k \in S_2$). The general solution of the equation (2.2) is given by

$$f(x) = F(\Pi(x), \ (\Pi(g_1 x), \ \Pi(g_2 x), \ldots, \Pi(g_n x)), \quad \text{where } \Pi: S_1 \to S_2 \text{ is arbitrary, and } F \text{ is defined by}$$

$$F(u_1, u_2, \ldots, u_{n+1}) = u_1 \text{ if } a_1 u_2 + a_2 u_3 + \cdots + a_n u_{n+1} = 0$$

$$= 0 \text{ if } a_1 u_2 + a_2 u_3 + \cdots + a_n u_{n+1} \neq 0$$

In particular, if $a_1 + a_2 + \cdots + a_n \neq 0$, then the general solution of (2.2) can be written as

$$(f(x) = \Pi(x) - \frac{1}{a_1 + \cdots + a_n} (a_1 \Pi(g_1 x) + a_2 \Pi(g_2 x) + \cdots + a_n \Pi(g_n x)).$$

**3. Second variation.** If instead of the equation (2.1), we consider the more general equation (1.1), it can be shown that the condition “$G$ is a group” cannot be replaced by the weaker condition “$G$ is a semigroup satisfying (i) and (ii)” if we want to preserve the form of the general solution as given by Prešić [7].

**4. Third variation.** Since the condition “$G$ is a group” cannot be satisfactorily weakened (the weaker condition given in Section 2 is not very useful) it is natural to look for some suitable condition which can be placed on the function $J$ which will ensure solvability of the considered equation. Before we give some examples, we introduce some notations.

If $J: S_2^n \to \{\top, \bot\}$, $g_i: S_1 \to S_2$ and if the considered equation is

$$J(f(g_1 x), f(g_2 x), \ldots, f(g_n x)) = \top \quad (g_1 \text{ is right unit})$$

then replacing $x$ by $g_1 x, g_2 x, \ldots, g_n x$, we obtain the system

$$J(f(g_1 x), f(g_2 x), \ldots, f(g_n x)) = \top \quad (k = 1, 2, \ldots, n) \quad (4.1)$$

where $g_i x = g_k(g_j (x))$. Instead of the system (4.1) it is more convenient to operate with the “algebraic” system

$$J(u_{1k}, u_{2k}, \ldots, u_{nk}) = \top \quad (k = 1, 2, \ldots, n) \quad (4.2)$$
where \( u_{ij} \) is corresponded to \( g_{ij}(x) \).

We give two illustrations of the possibilities which arise by specifying \( J \) and/or \( G \).

**ILLUSTRATION 1.** Suppose that \( S_2 \) is a field, \( a_i \in S_2 \), and consider the equation

\[
(4.3) \quad a_1 f(g_1 x) + a_2 f(g_2 x) + \cdots + a_n f(g_n x) = 0
\]

Suppose, further, that the corresponding system (4.2), which in this case reads

\[
a_1 u_{1k} + a_2 u_{2k} + \cdots + a_n u_{nk} = 0 \quad (k = 1, 2, \ldots, n)
\]

reduces to one equation only. In other words, we suppose that there exist \( \alpha_k (k = 2, 3, \ldots, n) \) such that

\[
a_1 u_{1k} + a_2 u_{2k} + \cdots + a_n u_{nk} = \alpha_k (a_1 u_{11} + a_2 u_{21} + \cdots + a_n u_{n1}) \quad (k = 2, 3, \ldots, n)
\]

where some (or all) \( \alpha_k \) may be 0.

If \( a_1 + a_2 a_2 + \cdots + a_n a_n \neq 0 \), the general solution of (4.3) is given by

\[
f(x) = \Pi(x) - \frac{1}{a_1 + a_2 a_2 + \cdots + a_n a_n} (a_1 \Pi(g_1 x) + a_2 \Pi(g_2 x) + \cdots + a_n \Pi(g_n x))
\]

where \( \Pi : S_1 \to S_2 \) is arbitrary.

**REMARK 4.1.** The semigroup \( G \), defined in Remark 2.4, is such that its corresponding “algebraic” system always reduces to one equation only.

**EXAMPLE 4.1.** Consider the real equation

\[
(4.4) \quad af(x, y) + bf(y, x) + cf(x, x) + df(y, y) = 0
\]

with \( a + b + c + d = 0 \), i.e., the equation

\[
(4.5) \quad af(x, y) + bf(y, x) + cf(x, x) - (a + b + c)f(y, y) = 0.
\]

The equation (4.5) leads to the system

\[
\begin{align*}
a u + b v + c w - (a + b + c) z &= 0 \\
b u + a v - (a + b + c) w + c z &= 0 \\
0 &= 0 \\
0 &= 0
\end{align*}
\]

If \( a + b + c + d \neq 0 \), the equation (4.4) is easily reduced to the (cyclic) group equation

\( af(x, y) + bf(y, x) = 0 \).
and it will reduce one equation only, if

\[
\frac{a}{b} = \frac{b}{a} = -\frac{c}{a + b + c} = -\frac{a + b + c}{c}.
\]

The system (4.6) yields two possibilities:

(i) \( a = b, \ c = -a \);
(ii) \( a + b = 0, \ c \) arbitrary.

In case (i) the equation (4.5) becomes

\[
f(x, y) + f(y, x) - f(x, x) - f(y, y) = 0
\]

and has general solution

\[
f(x, y) = \Pi(x, y) - \frac{1}{2}(\Pi(x, y) + \Pi(y, x) - \Pi(x, x) - \Pi(y, y)),
\]

while in case (ii) we obtain the equation

\[
a f(x, y) - a f(y, x) + c f(x, x) - c f(y, y) = 0
\]

with the general solution

\[
f(x, y) = \Pi(x, y) - \frac{1}{2a}(a \Pi(x, y) - a \Pi(y, x) + c \Pi(x, x) - c \Pi(y, y)),
\]

where in both cases \( \Pi \) is an arbitrary function.

**Illustration 2.** Suppose that \( G = \{g_1, g_2, \ldots, g_m, g_{m+1}, \ldots, g_n\} \) is a monoid satisfying

\[
g_i g_j = g_j \quad \text{for all} \ i(1 \leq i \leq n) \quad \text{and} \ j = m + 1, \ldots, n,
\]

and consider the equation

\[
J_1(f(x), f(g_2x), \ldots, f(g_nx)) = J_2(f(x), f(g_2x), \ldots, f(g_nx))
\]

where \( J_1(a, a, \ldots, a) = J_2(a, a, \ldots, a) \) for all \( a \).

If the system

\[
J_1(u_{1k}, u_{2k}, \ldots, u_{nk}) = J_2(u_{1k}, u_{2k}, \ldots, u_{nk}) \quad (k = 1, 2, \ldots, n)
\]

implies

\[
u_1 = F(u_{m+1}, \ldots, u_n) \quad (u_1 \text{corresponds to } f(g_kx))
\]

where

\[
F(F(a_1, \ldots, a_1), F(a_2, \ldots, a_2), \ldots, F(a_{n-m}, \ldots, a_{n-m})) = F(a_1, \ldots, a_{n-m}),
\]
then the general solution of (4.7) is given by

$$f(x) = F(\Pi(g_{m+1}x), \ldots, \Pi(g_nx)),$$

where $\Pi$ is an arbitrary function.

**Example 4.2.** Consider the real functional equation

$$f(x,y,z)^2 + f(y,y,z)^2 + f(z,z,z)^2 =$$

$$= f(x,y,z)f(y,y,z) + f(y,y,z)f(z,z,z) + f(z,z,z)f(x,y,z).$$

(4.8)

If $g_1(x,y,z) = (x,y,z)$, $g_2(x,y,z) = (y,y,z)$, $g_3(x,y,z) = (z,z,z)$, then clearly $g_1g_2g_3 = g_3$ $(i = 1, 2, 3)$. The corresponding “algebraic” system

$$u^2 + v^2 + w^2 - uv - vw - wu = 0$$

$$v^2 + w^2 - 2vw = 0$$

$$0 = 0$$

yields $u = v = w$, and hence the general solution of (4.8) is

$$f(x,y,z) = \Pi(z,z,z)$$

where $\Pi$ is an arbitrary function, or equivalently,

$$f(x,y,z) = \Phi(z)$$

where $\Phi$ is an arbitrary function.

**Example 4.3.** Let $g:S_1 \to S_2$ where $S_2$ is a field and let \( \{i, g, \ldots, g^{n-1}\} \) be a cyclic group of order $n$. Furthermore, let $h:S_1 \to S_2$ be such that $gh = h$, $h^2 = h$. Then, clearly $g^k h = k$ for all $k$, and the mappings $g, h$ generate the semigroup $G = \{i, g, \ldots, g^{n-1}, h, hg, \ldots, hg^{n-1}\}$. Consider the equation

$$a_1f(x) + a_2f(gx) + \cdots + a_nf(g^{n-1}x)$$

$$+ b_1f(hx) + b_2f(hgx) + \cdots + b_nf(hg^{n-1}x) = 0$$

(4.9)

where $\sum_{v=1}^{n} a_v + \sum_{v=1}^{n} b_v = 0$.

If

$$D = \begin{vmatrix}
    a_1 & a_2 & \cdots & a_n \\
    a_n & a_1 & \cdots & a_{n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_2 & a_3 & \cdots & a_1
\end{vmatrix} \neq 0,$$

then the general solution of the equation (4.9) is

$$f(x) = \frac{1}{D}(D_1\Pi(hx) + D_2\Pi(hgx) + \cdots + D_n\Pi(hg^{n-1}x))$$
where $D_1, D_2, \ldots D_n$ are the determinants obtained from $D$ by replacing the first column of $D$ by $(-b_1, -b_n, \ldots, b_2), (-b_2, -b_1, \ldots, -b_3), \ldots, (-b_n, -b_{n-1}, \ldots, -b_1)$ respectively.

**Remark 4.2.** A useful case of the equation (4.9) is provided by

$$
S_1 = R^n, \quad S_2 = R, \quad g(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_1), \\
\quad h(x_1, x_2, \ldots, x_n) = (x_1, x_1, \ldots, x_1).
$$

5. **A special functional equation.** We shall now apply some of the previous results to the real equation

$$
(5.1) \quad af(x, y) + bf(y, x) + cf(x, x) + df(y, y) = 0.
$$

We distinguish between several cases (excluding the trivial case $a = b = c = d = 0$).

(I) $a^2 = b^2 > 0$.

(I.i) $a + b + c + d \neq 0$. Then (5.1) implies $f(x, x) = 0$, and it reduces to the cyclic equation

$$
af(x, y) + bf(y, x) = 0
$$

which has the following general solution

(I.i.a) $f(x, y) = 0 \quad (a \neq b^2)$

(I.i.b) $f(x, y) = \Pi(x, y) - \Pi(y, x) \quad (a = b)$

(I.i.c) $f(x, y) = \Pi(x, y) + \Pi(y, x) \quad (a + b = 0)$.

(I.ii) $d + b + c + d = 0$.

(I.ii.a) If $a^2 \neq b^2$, we arrive at a special case of Example 4.2, and the general solution of (5.1) is

$$
f(x, y) = \frac{bd - ac}{a^2 - b^2} \Pi(x, x) + \frac{bc - ad}{a^2 - b^2} \Pi(y, y).
$$

(I.ii.b) If $a + b = 0$, this is a special case of Example 4.1, and the general solution of (5.1) is

$$
f(x, y) = \Pi(x, y) - \frac{1}{2a}(a\Pi(x, y) + b\Pi(y, x) + c\Pi(x, x) + d\Pi(y, y)).
$$

(I.ii.c) If $a = b$, we distinguish between:

(I.ii.c.1) $a + c = 0$. This is again a special case of Example 4.1; the equation becomes

$$
f(x, y) + f(y, x) - f(x, x) - f(y, y) = 0
$$
and its general solution is
\[ f(x, y) = \Pi(x, y) - \frac{1}{2}(\Pi(x, y) + \Pi(y, x) - \Pi(x, x) - \Pi(y, y)). \]

(iii.2) \( a + c \neq 0 \). Then (5.1) becomes
\[ (5.2) \quad f(x, y) + f(y, x) + \alpha f(x, x) - (2 + \alpha)f(y, y) = 0 \left( \alpha = \frac{C}{d} \right) \]
which together with
\[ f(y, x) + f(x, y) + \alpha f(y, y) - (2 + \alpha)f(x, x) = 0 \]
implies \( f(x, x) = f(y, y) \), and (5.2) reduces to
\[ (5.3) \quad f(x, y) + f(y, x) - 2f(x, x) = 0. \]
The general solution of (5.3) is given by [4]:
\[ f(x, y) = \Pi(x, y) - \Pi(y, x) + K \]
where \( K \) is an arbitrary constant, or, equivalently
\[ f(x, y) = \Pi(x, y) - \Pi(y, x) + 2\Pi(k, k). \]
where \( k \in R \) is fixed.

(ii) \( a^2 = b^2 = 0 \).

The equation (5.1) becomes
\[ (5.4) \quad cf(x, x) + df(y, y) = 0, \]
and we distinguish between two cases:

(iii.i) \( c + d \neq 0 \). Then (5.4) becomes
\[ (5.5) \quad f(x, x) = 0, \]
which is a special case of Example 4.1. Hence, the general solution of (5.5) is
\[ f(x, y) = \Pi(x, y) - \Pi(x, x). \]

(iii.ii) \( c + d = 0 \). The equation (5.4) becomes
\[ f(x, x) = f(y, y), \text{ i.e. } f(x, x) = f(k, k) \quad (k \text{ fixed}). \]
Its general solution is easily established to be
\[ f(x, y) = \Pi(x, y) - \Pi(x, x) + \Pi(k, k). \]
where \( k \in R \) is fixed.

Hence, in all cases the general solution of (5.1) can be expressed as a linear combination of \( \Pi(x, y), \Pi(y, x), \Pi(x, x), \Pi(y, y), \Pi(k, k) \), where \( \Pi \) is an arbitrary function, and \( k \in R \) is fixed.

6. Fourth variation. As we mentioned in the previous section, in \([4]\) Prešić considered, as an example, the real equation

\[
(6.1) \quad f(x, y) + f(y, x) - 2f(x, x) = 0
\]

and remarked that its general solution

\[
f(x, y) = \Pi(x, y) - \Pi(y, x) + K
\]

(\( \Pi \) arbitrary function, \( K \) arbitrary constant) cannot be expressed by means of the semigroup \( G = \{g_1, g_2, g_3, g_4\} \), where \( g_1(x, y) = (x, y), g_2(x, y) = (y, x), g_3(x, y) = (x, x), g_4(x, y) = (y, y) \).

Suppose that \( k \) is a fixed real number. It is easily shown that the general solution of (6.1) can be written in the form

\[
f(x, y) = \Pi(x, y) - \Pi(y, x) + 2\Pi(k, k),
\]

i.e. the equation (6.1) is solvable within the semigroup \( G' = \{g_1, g_2, g_3, g_4, g_5\} \)

where \( g_5(x, y) = (k, k) \).

This example shows that it may be possible to extend the initial semigroup \( G \) into a semigroup \( G_x \supset G \) which is such that it allows the equation to be solved within it. Indeed, B. Alimpić [8] showed that any equation of the form

\[
J(f(x, y), f(y, x), f(x, x)f(y, y)) = \top
\]

can be solved within semigroup \( G'' = \{g_1, \ldots, g_n\} \), where \( g_1, \ldots, g_n \) are defined as above, and \( g_6(x, y) = (x, k), g_7(x, y) = (k, x), g_8(x, y) = (y, k), g_9(x, y) = (k, y) \).

In the previous section we showed that any linear equation (5.1) can be solved within the semigroup \( G' \) (the wider semigroup \( G'' \) is not needed).

This suggests two questions:

(i) For a given equation (1.1) unsolvable within \( G \), does there exist an extended semigroup \( G_x \supset G \) such that the equation is solvable within \( G_x \).

(ii) If the answer to (i) is affirmative, is it possible to find the minimal extended semigroup \( G_m \) such that the equation is solvable within \( G_m \).

REFERENCES


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