CERTAIN CONVEXITY THEOREMS FOR UNIVALENT ANALYTIC FUNCTIONS

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1. Let $m$ and $M$ be arbitrary fixed real number which satisfy the relations $(m, M) \in R$ where $R = \{(m, M) \mid m > \frac{1}{2}, (m - 1) < M < m\}$. Also, let $P$ denote the class of functions $F(z) = 1 + c_0z + c_1z^2 + \cdots$ which are regular and satisfy $\Re\{F(z)\} > \alpha; 0 \leq \alpha < 1$ and $|F(z) - m| < M$. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in the unit disc $D = \{z \mid |z| < 1\}$ and write $G(z) = zf'(z)/f(z)$ and $H(z) = 1 + zf'(z)/f(z)$. Then we denote the class of functions $G(z)$ for which $G(z) \in P$ by $S(m, M)$ while the class of functions $H(z) \in P$ by $K(m, M)$. We further assume that

$$a = \frac{M^2 - m^2 + m}{M} \quad \text{and} \quad b = \frac{m - 1}{M}.$$ 

Then, it follows that

$$f \in S(m, M) \Leftrightarrow \frac{zf'(z)}{f(z)} = \frac{1 + aw(z)}{1 - bw(z)}$$

where $w(z)$ is regular in $D$ and satisfy $w(0) = 0, |w(z)| < 1$. Similarly,

$$f \in K(m, M) \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + aW(z)}{1 - bW(z)}$$

where $W(0) = 0, |W(z)| < 1$ and is regular in $D$. If we write $a = \{\alpha - 2N\alpha + N\}/N$ and $b = (N - 1)/N$ and make $N \to \infty$ then, it is equivalent to say that $b \to 1$ and $a \to 1 - 2\alpha$. In this case define

$$S^*(\alpha) = \lim_{\substack{a \to 1 - 2\alpha \\ b \to 0}} S(m, M); \quad 0 \leq \alpha < 1$$

and

$$K(\alpha) = \lim_{\substack{a \to 1 - 2\alpha \\ b \to 0}} K(m, M); \quad 0 \leq \alpha < 1.$$
The functions in $S^*(\alpha)$ and $K(\alpha)$ are usual functions of starlike univalent functions of order $\alpha$ and convex functions of order $\alpha$.

Also, if we let $S_0$ be the class of regular functions $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$ in $D_0 = \{z \mid |0 < |z| < 1\}$ and $Q$ denote the class of functions $F$ regular and satisfy $|F(z) + m| < M$, then define the class of functions:

$$\Gamma(m, M) = \left\{ f \mid f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \in S_0 \text{ and } zf'(z)/f(z) \in Q \right\}$$

and

$$\sum(m, M) = \left\{ f \mid f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \in S_0 \text{ and } 1 + zf'(z)/f(z) \in Q \right\}.$$

As before, we have

$$f \in \Gamma(m, M) \iff \frac{zf'(z)}{f(z)} = \frac{1 + aw_1(z)}{1 - bw_1(z)}$$

and

$$f \in \sum(m, M) \iff 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + aw_2(z)}{1 - bw_2(z)}$$

where $w_i; i = 1, 2$ are regular in $D$ and satisfy $w_i(0) = 0, |w_i(z)| < 1$. Also it follows that

$$\Gamma^*(\alpha) = \lim_{\substack{a \to 1-2\alpha \\ b \to 0 \\ \text{ as } a \to 1-2\alpha \\ b \to 0}} \Gamma(m, M), \quad 0 \leq \alpha < 1,$$

and

$$\sum^*(\alpha) = \lim_{\substack{a \to 1-2\alpha \\ b \to 0 \\ \text{ as } a \to 1-2\alpha \\ b \to 0}} \sum(m, M), \quad 0 \leq \alpha < 1.$$

Then $\Gamma^*(\alpha)$ and $\sum^*(\alpha)$ denote the usual class of starlike and convex functions in $D_0$. In this paper, we shall prove the following theorems which in particular include the results proved in [1–3] or else obtained as a $\lim_{a \to 1 - 2\alpha}$ and $b \to 0$.

2. We have:

**Theorem 1:** Let $f \in S(m, M)$ and

$$F(z) = \left( \frac{c + 1}{z^c} \right) \int_0^z \left( \frac{1}{1 - t} \right)^{-1} f(t) dt, \quad c > -\frac{1 - a}{1 + b}$$

where $a$, $b$ are defined by

$$a = \frac{M^2 - m^2 + m}{M} \quad \text{and} \quad b = \frac{m - 1}{b}; \quad (m, M) \in R.$$
and \( r(a, b) \) be the unique positive root of the equation
\[
(a + 2b + d) - 2(ad + bd + b + d)r - 2(b^2 - d^2) + (a + d) + 2b(1 - d^2) \\
- d(ad + b^2)r^2 - 2d((a + b) + b(b + d)r^3) \\
- d(ad + 2bd + b^2)r^4 = 0.
\]
then, \( f(z) \) is starlike of order \( \beta \) for \( |z| < r_0 \), where \( r_0 \) is the smallest positive root of the equation
\[
(1 - \beta) - \{\beta(b - d) + a + b + 2d\}r + d(a + b\beta)r^2 = 0
\]
if \( r_0 \leq r(a - b) \), otherwise \( r_0 \) is the smallest positive root of the equation
\[
0 = \sqrt{(1 - d)}\{(1 - d) + (1 + d)x\}\{(1 + 2a + 4b + b^2) + (1 - b^2)x\} + \\
(\sqrt{E - 1 + bd}) - (1 + bd)x
\]
where
\[
x = \frac{1 + r^2}{1 - r^2}, \quad E = -\beta(b + d) + 2d - (a + b) \quad \text{and} \quad d = \frac{a - bc}{c + 1}.
\]
This result is sharp.

PROOF. Since \( F \in S(m, M) \) there exists a regular function \( w(z) \) with \( w(0) = 0 \), \( |w(z)| < 1 \) and
\[
\frac{zF'(z)}{F(z)} = \frac{1 + aw(z)}{1 + bw(z)}.
\]
From (2.7) and (2.1) we get
\[
\frac{f(z)}{F(z)} = \frac{1 + \frac{a - bc}{c + 1}w(z)}{1 - bw(z)} = \frac{1 + dw(z)}{1 - bw(z)}.
\]
Differentiating (2.8) logarithmically with respect to \( z \) and using (2.7), we get,
\[
\Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq -\beta + \Re\left\{\frac{1 + aw(z)}{1 - bw(z)}\right\} + \\
+ (b + d)\Re\left\{\frac{w(z)}{(1 - bw(z))(1 + dw(z))}\right\} - \frac{(b + d)(r^2 - |w(z)|^2)}{(1 - r^2)|1 - bw(z)||1 + dw(z)|}.
\]
Here we have used the well known inequality
\[
|zw'(z) - w(z)| \leq \frac{r^2 - |w(z)|^2}{1 - r^2}.
\]
If we take

$$p(z) = \frac{1 + dw(z)}{1 - bw(z)} \tag{2.10}$$

then

$$|p(z) - A| \leq B \tag{2.11}$$

where

$$A = \frac{1 + bdr^2}{1 - b^2r^2} \tag{2.12}$$

and

$$B = \frac{(b + d)r}{1 - b^2r^2}. \tag{2.14}$$

Substituting value of $w(z)$ from (2.10) in (2.9) we get

$$\Re \left\{ \frac{zf'(z)}{f(z)} - \beta \right\} \geq \frac{1}{b + d} \left[ E - d\Re \left\{ \frac{1}{p(z)} \right\} + (a - 2b)\Re \{p(z)\} - \frac{r^2|bp(z) + d|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|} \right]. \tag{2.14}$$

If we take $p(z) = A + u + iv$, $|p(z)| = R$ and use (2.12) and (2.13) in (2.14) we get

$$\Re \left\{ \frac{zf'(z)}{f(z)} - \beta \right\} \geq \frac{1}{b + d} \left[ E - \frac{d(A + u)}{R^2} + (a + 2b)(A + u) - \frac{B^2 - u^2 - v^2}{R} \left( \frac{1 - b^2r^2}{1 - r^2} \right) \right] = \frac{1}{b + d} \cdot P(u, v). \tag{2.15}$$

Differentiating $P(u, v)$ partially with respect to $v$ we get

$$\frac{\partial P(u, v)}{\partial v} = \frac{v}{R} \left[ \frac{2d(A + u)}{R^3} + \left\{ 2 + \frac{B^2 - u^2 - v^2}{R} \right\} \left( \frac{1 - b^2r^2}{1 - r^2} \right) \right]. \tag{2.16}$$

If $d \geq 0$, quantity in the square brackets is positive. If $d < 0$ we see that

$$\frac{1 - b^2r^2}{1 - r^2} + \frac{d(A + u)}{R^3} \geq 1 + \frac{d(1 + br)^2}{(1 - dr)^2} \geq 0$$

and therefore the quantity in the square brackets in (2.16) is positive.
So \( \frac{\delta P(u, v)}{\delta v} \geq 0 \) if \( v \geq 0 \) and \( \frac{\delta P(u, v)}{\delta v} < 0 \) if \( v < 0 \) therefore

\[
(2.17) \quad \min_v P(u, v) = P(u, 0) =
\]

\[
eq E - \frac{d}{R} + (a + 2b)R - \frac{B^2 - (R - A)^2}{R} \left( \frac{1 - b^2 r^2}{1 - r^2} \right) \equiv P(R)
\]

where \( R = A + u \).

\( P'(R) \) is an increasing function of \( R \) and \( P'(R_0) = 0 \) where

\[
(2.18) \quad R_0 = \left[ \frac{(1 - d)(1 + dr^2)}{(a + 2b + 1) - (a + 2b + b^2)r^2} \right]^{1/2}.
\]

Again we see that \( P'(A + B) > 0 \) therefore \( R_0 \leq A + B \). Since \( P'(R) \) is increasing function of \( R \) and \( A - B \leq R \leq A + B \) we have

\[
(2.19) \quad \min_R P(R) = \begin{cases} P(A - B) & \text{if } 0 \leq R_0 \leq A - B \\ P(R_0) & \text{if } A - B \leq R_0 \leq A + B. \end{cases}
\]

\[
= \begin{cases} 
\frac{(b + d)[(1 - \beta) - \beta(b - d) + a + b + 2d)r + d(a + b\beta)r^2]}{(1 - d)(1 + br)} & \text{if } R_0 \leq A - B \\
(E - 1 + bd) - (1 + bd) x + \sqrt{(1 - d)((1 - d) + (1 + d)x)((1 + 2a + 4b + b^2) + (1 - b^2)x)} & \text{if } R_0 \geq A - B
\end{cases}
\]

where \( x = \frac{1 + r^2}{1 - r^2} \).

Let us take

\[
(2.20) \quad Q(r) = (A - B)^2 - R_0^2 = \left( \frac{1 - dr}{1 + br} \right)^2 - \frac{(1 - d)(1 + dr^2)}{(a + 2b + 1) - (a + 2b + b^2)r^2}.
\]

Therefore \( Q(r) \) is a decreasing function of \( r \) and

\[
Q(0) = \frac{(a + b) + (b + d)}{(a + b) + (1 + b)} \geq 0 \quad \text{and} \quad Q(1) = -\frac{2(1 - d)(b + d)}{(1 + b)(1 - b^2)} \leq 0.
\]

Therefore \( Q(r) \) has unique root in \((0, 1)\).

Let it be \( r(a, b) \). Hence if \( r \leq r(a, b) \), \( Q(r) \geq 0 \) i.e. \( A - B \geq R_0 \) and if \( r \geq r(a, b) \), \( Q(r) \leq 0 \) i.e. \( A - B \leq R_0 \). So from (2.19) and (2.20) the result follows.

The equality in (2.4) is attained for the function \( F(z) = z(1 - bz)^{-\frac{a + b}{ab}} \) and that in (2.5) for the function

\[
F(z) = z(1 - 2kbz + b^2 z^2)^{-\frac{a + b}{2ab}}
\]
where \( k \) is given by
\[
\frac{1 + k(a-b)r - br^2}{1 - 2kbr + b^2r^2} = \left\{ \frac{(1-d)(1+dr^2)}{(a+2b+1) - (a+2b+b^2)r^2} \right\}^{1/2}.
\]

Similarly by using the method of theorem 1 following theorems follow.

**Theorem 2.** If \( f(z) \) is regular in \( D \) and satisfy
\[
F(z) = \left( \frac{c+2}{z^{c+1}} \right) \int_0^z t^{c-1} f(t)g(t)dt, \quad c \geq 0
\]
where \( F \in S^*(\beta) \) and \( g \in S(m,M) \) then \( f(z) \) is univalent and starlike of order \( \beta \) in \( |z| < r_0 \) where \( r_0 \) is the smallest positive root of the equation
\[
\begin{align*}
(1 - \beta)(c+2) - \{(c+2)(a+2b-b\beta) + 2(1-\beta)(2-\beta)r + (2b(1-\beta)(2-\beta) - \\
- (1-\beta)(c+2b) - 2(c+1+\beta)(a+b)r^2 - (c+2\beta)(a+b)r^3 = 0.
\end{align*}
\]
This result is sharp.

**Theorem 3.** If \( f(z) \) is regular in \( D \) and satisfies
\[
F(z) = \left( \frac{c+2}{z^{c+1}} \right) \int_0^z t^{c-1} f(t)g(t)dt, \quad c \geq 0
\]
where \( F \in S^*(\beta) \) and \( g \in K(\alpha) \), then \( f(z) \) is starlike of order \( \beta \) for \( |z| < r_0 \), where \( r_0 \) is the smallest positive root of the equation
\[
\begin{align*}
(c+2)(2-\beta) + 2\{(c+\beta+1) - (1-\beta)(2-\beta)\}r + \beta(c+2\beta)r^2 - (1+r) \\
- \{(c+2) + (c+2\beta)r\}B(\alpha, r) = 0
\end{align*}
\]
where
\[
B(\alpha, r) = \begin{cases} 
\frac{(2\alpha-1)r}{(1-r)^2[1-\alpha][1-(1-r)^{2\alpha-1}]}, & \alpha \neq \frac{1}{2} \\
-\frac{r}{(1-r)\log(1-r)}, & \alpha = \frac{1}{2}
\end{cases}
\]
This result is sharp.

**Theorem 4.** If \( f(z) \) is regular in \( D \) and satisfy
\[
F(z) = \left( \frac{c+2}{z^{c+1}} \right) \int_0^z t^{c-1} f(t)g(t)dt, \quad c \geq 0
\]
where $F \in S^*(\beta)$ and $g(z)/z \in P(\alpha)$ then $f(z)$ is univalent and starlike of order $\beta$ in $|z| < r_0$ where $r_0$ is the smallest positive root of the equation

$$(c + 2)(1 - \beta) - 2\{(c + 2)(1 - \alpha\beta) + (1 - \beta)(2 - \beta)\} + 2\{(c + 2)(2\alpha - \alpha\beta - 1) - (2\alpha - 1) \}
$$

$$+(3 - 2\beta - 8\alpha + 6\alpha\beta - \beta^2 - 2\alpha\beta^2)\{r - 2\{c(3 - 4\alpha - \beta + \alpha\beta) + (3 + 2\beta - 8\alpha + 6\alpha\beta - \beta^2 - 2\alpha\beta^2)\}r^2 + 4\{(c + 2\beta)(2\alpha - \alpha\beta - 1) - (2\alpha - 1) \}
$$

$$(1 - \beta)(2 - \beta)\}r^3 - (2\alpha - 1) (1 - \beta)(c + 2\beta)r^4 = 0.$$

The result is sharp.

Theorem 5. Let $F \in \Gamma(m, M)$ and $f(z)$ be defined by

$$F(z) = \frac{c}{z^c+1} \int_0^z t^c f(t) dt, \quad c \geq 1$$

and $r(a, b)$ be the unique positive root of the equation

$$(a + d) + 2\{d(a + b) - (d - b)\}r + 2\{b^2 - d^2\} - (a + d) + d(ad + b^2)\}r^2 - 2d\{(a + b) + b(d - b)\}r^3 - d(ad + b^2)r^4 = 0$$

and $d \leq 0$ then $f(z)$ is meromorphic starlike of order $\beta$ for $|z| < r_0$, where $r_0$ is the smallest positive root of the equation

$$(1 - \beta) + \{(a + b + 2d) - (b + d)\}\beta + (ab + bd + d^2 - bd\beta)r^2 = 0$$

if $0 < r_0 \leq r(a, b)$, and that of the equation

$$(E - 1 + bd) - (1 + bd)x + \sqrt{(1 + d)\{(1 + d) + (1 - d)x\}\{(1 - 2a + b^2) + (1 - b^2)x\}}$$

if $r(a, b) \leq r_0$ where

$$x = \frac{1 + r^2}{1 - r^2}, \quad E = (a - b) - (d - b)\beta \quad \text{and} \quad d = \frac{a + b + c}{c}.$$

Equality is attained for the functions

$$F(z) = \frac{(1 + b)z^{2k}}{z}$$

$$F(z) = \frac{[(1 - b)z^{1+k} + (1 + b)z^{1-k}]^{2k}}{z}$$

where $k$ is determined from

$$\frac{1 - k(a + b)z + abz^2}{1 - b^2z^2} = \left\{ \frac{(1 + d)(1 - dr^2)}{(1 - a) + (a - b^2)r^2} \right\}^{1/2}.$$
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