Nodal curves on K3 surfaces

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Abstract. In this paper, we study the Severi variety $V_{L,g}$ of genus $g$ curves in $|L|$ on a general polarized K3 surface $(X, L)$. We show that the closure of every component of $V_{L,g}$ contains a component of $V_{L,g-1}$. As a consequence, we see that the general members of every component of $V_{L,g}$ are nodal.

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1. Introduction

It was proved that every complete linear system on a very general polarized K3 surface $(X, L)$ contains a nodal rational curve [C1] and furthermore every rational curve in $|L|$ is nodal, i.e., has only nodes $xy = 0$ as singularities [C2]. The purpose of this note is to prove an analogous result on singular curves in $|L|$ of geometric genus $g > 0$.

For a line bundle $A$ on a projective surface $X$, we use the notation $V_{A,g}$ to denote the Severi varieties of integral curves of geometric genus $g$ in the complete linear series $|A| = \mathbb{P}H^0(A)$. For a K3 surface $X$, it is well known that every component of $V_{A,g}$ has the expected dimension $g$. Furthermore, using theory of deformation of maps, one can show that $\nu : \widehat{C} \to X$ is an immersion for $\nu$ the normalization of a general member $[C] \in V_{A,g}$ if $g > 0$ [HM, Chap. 3, Sec. B].

It was claimed that a general member of $V_{A,g}$ is nodal on every projective K3 surface $X$ and every $A \in \text{Pic}(X)$ as long as $g > 0$ in [C1, Lemma 3.1]. However, as kindly pointed out to the author by Edoardo Sernesi [DS, Sec. 3.3], the proof there is wrong. So this note provides a partial fix for this
problem, albeit only for singular curves in the primitive class $|L|$ on a general polarized K3 surface $(X, L)$. Our main theorem is

**Theorem 1.1.** For a general polarized K3 surface $(X, L)$, every (irreducible) component of $\overline{V}_{L,g}$ contains a component of $V_{L,g-1}$ for all $1 \leq g \leq p_a(L)$, where $\overline{V}_{L,g}$ is the closure of $V_{L,g}$ in $|L|$ and $p_a(L) = L^2/2+1$ is the arithmetic genus of $L$.

Clearly, the above theorem, combining with the fact that every rational curve in $|L|$ is nodal [C2], implies the following corollary by induction:

**Corollary 1.2.** For a general polarized K3 surface $(X, L)$, the general members of every component of $V_{L,g}$ are nodal for all $0 \leq g \leq p_a(L)$.

It was proved in [KLM, Theorem 1.3, 5.3 and Remark 5.6] that the general members of every component of $V_{L,g}$ are not trigonal for $g \geq 5$. Combining with [DS, Theorem B.4], it shows that the corollary holds for $5 \leq g \leq p_a(L)$. Of course, we have settled it for all genus $g$ here. As an application, it shows that the genus $g$ Gromov-Witten invariant computed in [BL] is the same as the number of genus $g$ curves in $|L|$ passing through $g$ general points.

A comprehensive treatment for $V_{m,L,g}$ is planned in a future paper.

As another potential application of Theorem 1.1, we want to mention the conjecture of the irreducibility of universal Severi variety $V_{L,g}$ on K3 surfaces:

**Conjecture 1.3.** Let $\mathcal{K}_p$ be the moduli space of polarized K3 surfaces $(X, L)$ of genus $p = p_a(L)$ and let

$$V_{L,g} = \{(X, L, C) : (X, L) \in \mathcal{K}_p, C \in V_{L,g}\} \quad (1.1)$$

be the universal Severi variety of genus $g$ curves in $|L|$ over $\mathcal{K}_p$. Then $V_{L,g}$ is irreducible.

If we approach the conjecture along the line of argument of J. Harris for the irreducibility of Severi variety of plane curves [H], we need to establish two facts:

- Every component of $\overline{V}_{L,g}$ contains a component of $V_{L,0}$.
- $V_{L,0}$ is irreducible and the monodromy action on the $p$ nodes of a rational curve $C \in V_{L,0}$ is the full symmetric group $\Sigma_p$ as $(X, L, C)$ moves in $V_{L,0}$.

The second fact comes easily for plane curves, while the establishment of the first fact is the focus of Harris’ proof (see also [HM, Chap. 6, Sec. E]).

The situation for $V_{L,g}$ is somewhat reversed at the moment: the first fact follows from our main theorem, while the difficulty lies in the second fact:

**Conjecture 1.4.** Let $V_{L,0}$ be the universal Severi variety of rational curves in $|L|$ over the moduli space $\mathcal{K}_p$ of polarized K3 surfaces $(X, L)$ of genus $p$.
and let

\[ W_{L,0} = \{(X, L, C, s_1, s_2, \ldots, s_p) : (X, L, C) \in V_{L,0}, \quad C_{\text{sing}} = \{s_1, s_2, \ldots, s_p\}\} \] (1.2)

Then \( W_{L,0} \) is irreducible.

Our above discussion shows that Conjecture 1.4 implies 1.3.

Conventions. We work exclusively over \( \mathbb{C} \). A K3 surface in this paper is always projective. A polarized K3 surface is a pair \((X, L)\), where \(X\) is a K3 surface and \(L\) is an indivisible ample line bundle on \(X\).

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2. Proof of Theorem 1.1

We start with the following observation:

Proposition 2.1. Let \( W \) be a component of \( V_{L,g} \) for a polarized K3 surface \((X, L)\) with \( \text{Pic}(X) = \mathbb{Z} \). The following are equivalent:

1. The closure \( \overline{W} \) of \( W \) in \( |L| \) contains a component of \( V_{L,g-1} \).
2. \( \dim(\overline{W} \setminus W) = g - 1 \).
3. For a set \( \sigma \) of \( g - 1 \) general points on \( X \), \( W \cap \Lambda_\sigma \) is not projective (i.e. complete), where \( \Lambda_\sigma \subset |L| \) is the locus of curves \( C \in |L| \) passing through \( \sigma \).

Proof. (1) \( \Rightarrow \) (2) is obvious. Since every curve in \( |L| \) is integral, we have

\[ \overline{W} \setminus W \subset \bigcup_{i \leq g} V_{L,i} \] (2.1)

And since \( \dim V_{L,i} \leq i \), we have (2) \( \Rightarrow \) (1).

Let \( \partial W = \overline{W} \setminus W \). Obviously, \( \dim(\partial W \cap \Lambda_\sigma) = \dim \partial W - (g - 1) \). Therefore, (2) \( \Rightarrow \) (3). On the other hand, if \( W \cap \Lambda_\sigma \) is not complete, then there exists \( C_\sigma \in \partial W \) passing through \( \sigma \). Then \( \dim \partial W \geq g - 1 \). So (3) \( \Rightarrow \) (2). \( \square \)

So it suffices to show that \( W \cap \Lambda_\sigma \) is not complete for every component \( W \) of \( V_{L,g} \). We prove this using a degeneration argument similar to the one in [C2]. A general K3 surface can be specialized to a Bryan-Leung (BL) K3 surface \( X_0 \), which is a K3 surface with Picard lattice

\[ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \] (2.2)

It can be polarized by the line bundle \( C + mF \), where \( C \) and \( F \) are the generators of \( \text{Pic}(X_0) \) satisfying \( C^2 = -2, CF = 1 \) and \( F^2 = 0 \). A general polarized K3 surface of genus \( m \) can be degenerated to \( (X_0, C + mF) \). Such \( X_0 \) has an elliptic fibration \( X_0 \to \mathbb{P}^1 \) with fibers in \( |F| \). For a general BL
K3 surface $X_0$, there are exactly 24 nodal fibers in $|F|$. A key fact here is that every member of $|C + mF|$ is “completely” reducible in the sense that it is a union of $C$ and $m$ fibers in $|F|$ (counted with multiplicities).

Let $X$ be a family of K3 surfaces of genus $m$ over a smooth quasi-projective curve $T$ such that $X_0$ is a general BL K3 surface for a point $0 \in T$, $X_t$ are K3 surfaces of Pic($C$) = $\mathbb{Z}$ for $t \neq 0$ and $L$ is a line bundle on $X$ with $L_0 = C + mF$. After a base change, there exists $W \subset \mathcal{V}_{L,g}$ flat over $T$ such that $W_t$ is a component of $\mathcal{V}_{L_t,g}$ for all $t \neq 0$. Let $\sigma$ be a set of $g - 1$ general sections of $X/T$. It suffices to prove that $W_t \cap \Lambda_\sigma$ is not projective for $t$ general.

By stable reduction, there exists a family $f : Y \to X$ of genus $g$ stable maps over a smooth surface $S$ with the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi} & T
\end{array}
$$

(2.3)

where $S$ is flat and projective over $T$, $f_sY_s \in \overline{W}_t \cap \Lambda_\sigma$ on $X_t$ for all $s \in S_t$ and $t \in T$ and $S$ dominates $\overline{W} \cap \Lambda_\sigma$ via the map sending $s \to [f_sY_s]$. In other words, $f : Y \to X$ is the stable reduction of the universal family over $\overline{W}$ such that $f : Y_s \to X$ is the normalization of a general member $G \in W_t$ passing through the $g - 1$ points $\sigma(t)$ for $s \in S_t$ general and $t \neq 0$.

Let us consider the moduli map $\rho : S \to \overline{M}_g \times T$ sending $s \to ([Y_s], \pi(s))$, where $\overline{M}_g$ is the moduli space of stable curves of genus $g$ with $M_g$ its open subset parameterizing smooth curves. To show that $W_t \cap \Lambda_\sigma$ is not complete, it suffices to show that

$$
\rho^{-1}(\Delta \times T) \cap S_t \neq \emptyset
$$

(2.4)

for $t \neq 0$, where $\Delta = \overline{M}_g \setminus M_g$ is the boundary divisor of $\overline{M}_g$.

Let $F_1, F_2, \ldots, F_{g-1} \subset X_0$ be $g - 1$ fibers in $|F|$ passing through the $g - 1$ points $\sigma(0)$, respectively. Since $\sigma(0)$ are in general position, $F_1, F_2, \ldots, F_{g-1}$ are $g - 1$ general fibers in $|F|$ and $\sigma(0) \cap C = \emptyset$.

For every $s \in S_0$, $f_sY_s \in |C + mF|$ passes through $\sigma(0)$. Therefore, we must have

$$
f_sY_s = C + m_1F_1 + m_2F_2 + \ldots + m_{g-1}F_{g-1} + M_s
$$

(2.5)

for some $m_1, m_2, \ldots, m_{g-1} \in \mathbb{Z}^+$. Since the curves in $W_t \cap \Lambda_\sigma$ cover $X_t$ for $t \neq 0$, $f$ is surjective. Hence $f_sY_s$ covers $X_0$ as $s$ moves in $S_0$. Therefore, $M_s$ contains a moving fiber in $|F|$. More precisely, there exists a component $\Gamma$ of $S_0$ such that $\cup_{s \in \Gamma} M_s = X_0$.

For a general point $s \in \Gamma$, $M_s$ contains a general fiber $F_s$ in $|F|$. Therefore, $Y_s$ has components $\widehat{F}_{1,s}, \widehat{F}_{2,s}, \ldots, \widehat{F}_{g-1,s}, \widehat{F}_s$ dominating $F_1, F_2, \ldots, F_{g-1}, F_s$, respectively. And since $p_0(Y_s) = g$, $\widehat{F}_{1,s}, \widehat{F}_{2,s}, \ldots, \widehat{F}_{g-1,s}, \widehat{F}_s$ are all elliptic
curves. Indeed, it is very easy to see that its moduli \([Y_s]\) in \(\overline{M}_g\)
\[
[Y_s] = [\widehat{C}_s \cup \widehat{F}_{1,s} \cup \widehat{F}_{2,s} \cup ... \cup \widehat{F}_{g-1,s} \cup \widehat{F}_s]
\]  
(2.6)
is a smooth rational curve \(\widehat{C}_s\) with \(g\) elliptic “tails” \(\widehat{F}_{1,s}, \widehat{F}_{2,s}, ... , \widehat{F}_{g-1,s}, \widehat{F}_s\)
attached to it, where \(\widehat{C}_s\) is the component of \(Y_s\) dominating \(C\). Of course, when \(g \leq 2\), \(\widehat{C}_s\) is contracted under the moduli map.

Note that \(\widehat{F}_{1,s}, \widehat{F}_{2,s}, ..., \widehat{F}_{g-1,s}, \widehat{F}_s\) are isogenous to \(F_1, F_2, ..., F_{g-1}, F_s\), respectively. As \(s\) moves on \(\Gamma\), \(F_s\) moves in \(|F|\). So \(\widehat{F}_s\) has varying moduli. This shows that \(\rho\) maps \(S\) generically finitely onto its image. That is,
\[
\dim \rho(S) = 2.
\]  
(2.7)
Furthermore, when \(F_s\) becomes one of 24 nodal fibers in \(|F|\), \(\widehat{F}_s\) becomes a union of rational curves. Therefore, there exists \(b \in \Gamma\) such that \(\widehat{F}_b\) is a connected union of rational curves with normal crossings and \(p_*(\widehat{F}_b) = 1\).
The moduli \([Y_b]\) of \(Y_b\) is thus a smooth rational curve with \(g - 1\) elliptic tails and one nodal rational curve attached to it. Consequently,
\[
\rho(b) \in \Delta_0 \times T
\]  
(2.8)
where \(\Delta_0\) is the component of \(\Delta\) whose general points parameterize curves of genus \(g - 1\) with one node. Combining (2.7), (2.8) and the fact that \(\Delta_0\) is \(\mathbb{Q}\)-Cartier, we conclude that
\[
\rho(S) \cap (\Delta_0 \times T) \neq \emptyset \text{ has pure dimension 1.}
\]  
(2.9)
Therefore, for every connected component \(G\) of \(\rho^{-1}(\Delta_0 \times T)\), we have
\[
\dim \rho(G) = 1.
\]  
(2.10)
If \(\rho^{-1}(\Delta_0 \times T) \cap S_t \neq \emptyset\) for \(t \neq 0\), then (2.4) follows and we are done. Otherwise,
\[
\rho^{-1}(\Delta_0 \times T) \subset S_0.
\]  
(2.11)
Let \(G\) be the connected component of \(\rho^{-1}(\Delta_0 \times T)\) containing the point \(b\). Then \(G \subset S_0\) and \(\dim \rho(G) = 1\).

Let \(B\) be an irreducible component of \(G\) passing through \(b\). For \(Y_b\), we have
\[
f_s Y_b = C + m_1 F_1 + m_2 F_2 + ... + m_{g-1} F_{g-1} + M_b
\]  
(2.12)
with \(M_b\) supported on the union \(F_s\) of 24 nodal rational curves in \(|F|\). Therefore, for \(s \in B\) general, \(M_s\) must also be supported on \(F_s\); otherwise, \(M_s\) contains a general member \(F_s\) of \(|F|\), the moduli \([Y_s]\) of \(Y_s\) is given by (2.6) and \([Y_s] \notin \Delta_0\). Consequently, \(M_s = M_b\) for all \(s \in B\) and \(\rho\) is constant on \(B\).

For a component \(Q\) of \(G\) with \(q \in B \cap Q \neq \emptyset\), the same argument shows that \(M_s = M_q\) is supported on \(F_s\) for all \(s \in Q\) and \(\rho\) is constant on \(Q\). And since \(G\) is connected, we can use this argument to show that \(\rho\) is constant on every component of \(G\), i.e., constant on \(G\). This contradicts (2.10).
References


