On the spaces of bounded and compact multiplicative Hankel operators

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Abstract. A multiplicative Hankel operator is an operator with matrix representation $M(\alpha) = \{\alpha(nm)\}_{n,m=1}^{\infty}$, where $\alpha$ is the generating sequence of $M(\alpha)$. Let $\mathcal{M}$ and $\mathcal{M}_0$ denote the spaces of bounded and compact multiplicative Hankel operators, respectively. In this note it is shown that the distance from an operator $M(\alpha) \in \mathcal{M}$ to the compact operators is minimized by a nonunique compact multiplicative Hankel operator $N(\beta) \in \mathcal{M}_0$. Intimately connected with this result, it is then proven that the bidual of $\mathcal{M}_0$ is isometrically isomorphic to $\mathcal{M}$, $\mathcal{M}_0^{\ast\ast} \cong \mathcal{M}$. It follows that $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$. The dual space $\mathcal{M}_0^\ast$ is isometrically isomorphic to a projective tensor product with respect to Dirichlet convolution. The stated results are also valid for small Hankel operators on the Hardy space $H^2(\mathbb{D}^d)$ of a finite polydisk.

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1. Introduction

Given a sequence $\alpha: \mathbb{N} \to \mathbb{C}$, we consider the corresponding multiplicative Hankel operator $m = M(\alpha): \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined by

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \sum_{n,m=1}^{\infty} a(n)\overline{b(m)}\alpha(nm), \quad a, b \in \ell^2(\mathbb{N}).$$

Initially, we consider this equality only for finite sequences $a$ and $b$. It defines a bounded operator $M(\alpha): \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, with matrix representation $\{\alpha(nm)\}_{n,m=1}^{\infty}$ in the standard basis of $\ell^2(\mathbb{N})$, if and only if there is a constant $C > 0$ such that

$$|\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})}| \leq C\|a\|_{\ell^2(\mathbb{N})}\|b\|_{\ell^2(\mathbb{N})}, \quad a, b \text{ finite sequences.}$$
Multiplicative Hankel operators are also known as Helson matrices, having been introduced by Helson in [14, 15].

There are two common alternative interpretations. One is in terms of Dirichlet series. Let $\mathcal{H}^2$ be the Hardy space of Dirichlet series, the Hilbert space with $(n^{-s})_{n=1}^\infty$ as a basis. Elements $f \in \mathcal{H}^2$ are holomorphic functions in the half-plane $\{ s \in \mathbb{C} : \text{Re } s > 1/2 \}$. If

$$f(s) = \sum_{n=1}^\infty a(n)n^{-s}, \quad g(s) = \sum_{n=1}^\infty b(n)n^{-s}, \quad \rho(s) = \sum_{n=1}^\infty \alpha(n)n^{-s},$$

then

$$\langle M(\alpha)a, b \rangle_{\ell^2(N)} = \langle fg, \rho \rangle_{\mathcal{H}^2}.$$ 

Hence there is an isometric correspondence between Helson matrices and Hankel operators on $\mathcal{H}^2$, since the forms associated with the latter are precisely of the type $(f, g) \mapsto \langle fg, \rho \rangle_{\mathcal{H}^2}$.

The second interpretation is in terms of the Hardy space of the infinite polytorus $H^2(\mathbb{T}^\infty)$, the Hilbert space with basis $(z^\kappa)_\kappa$, where $z = (z_1, z_2, \ldots)$, and $\kappa = (\kappa_1, \kappa_2, \ldots)$ runs through the countably infinite, but finitely supported, multi-indices. Identify each integer $n$ with a multi-index $\kappa$ of this type through the factorization of $n$ into the primes $p_1, p_2, \ldots$,

$$n \leftrightarrow \kappa \text{ if and only if } n = \prod_{j=1}^\infty p_j^{\kappa_j}. $$

Under this equivalence, multiplicative Hankel operators correspond to additive Hankel operators on a countably infinite number of variables,

$$\langle M(\alpha)a, b \rangle_{\ell^2(N)} = \sum_{\kappa, \kappa'} a(\kappa)b(\kappa')\alpha(\kappa + \kappa').$$

Hence the multiplicative Hankel operators correspond isometrically to small Hankel operators on $H^2(\mathbb{T}^\infty)$, since the matrix representations of the latter are of the form $\{\alpha(\kappa + \kappa')\}_{\kappa, \kappa'}$. See [14, 15] for details.

In particular, the Helson matrices generalize the small Hankel operators on the Hardy space of any finite polytorus $H^2(\mathbb{T}^d)$, $d < \infty$. In fact, the results in this note have analogous statements for small Hankel operators on $H^2(\mathbb{T}^d)$; every proof given remains valid verbatim after restricting the number of prime factors, that is, the number of variables.

The first result is the following. We denote by $\mathcal{B}(\ell^2(N))$ and $\mathcal{K}(\ell^2(N))$, respectively, the spaces of bounded and compact operators on $\ell^2(N)$.

**Theorem 1.1.** Let $M(\alpha)$ be a bounded multiplicative Hankel operator. Then there exists a compact multiplicative Hankel operator $N(\beta)$ such that

$$\| M(\alpha) - N(\beta) \|_{\mathcal{B}(\ell^2(N))} = \inf \left\{ \| M(\alpha) - K \|_{\mathcal{B}(\ell^2(N))} : K \in \mathcal{K}(\ell^2(N)) \right\}. \quad (1)$$

The minimizer $N(\beta)$ is never unique, unless $M(\alpha)$ is compact.
The quantity on the right-hand side of (1) is known as the essential norm of $M(\alpha)$. For classical Hankel operators on $H^2(\mathbb{T})$, this result was proven by Axler, Berg, Jewell, and Shields in [6], and can be viewed as a limiting case of the theory of Adamjan, Arov, and Krein [1]. The demonstration of Theorem 1.1 requires only a minor modification of the arguments in [6], the main point being that a characterization of the class of bounded multiplicative Hankel operators is not necessary for the proof.

On $H^2(\mathbb{T})$, Nehari’s theorem [21] states that the class of bounded Hankel operators can be isometrically identified with $L^\infty(\mathbb{T})/H^\infty(\mathbb{T})$, where $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ denote the spaces of bounded and bounded analytic functions on $\mathbb{T}$, respectively. By Hartman’s theorem [13], the class of compact Hankel operators is isometrically isomorphic to $(H^\infty(\mathbb{T}) + C(\mathbb{T}))/H^\infty(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of continuous functions on $\mathbb{T}$. Note that the spaces $L^\infty$, $H^\infty$, and $H^\infty + C$ are all algebras, as proven by Sarason [26].

Luecking [20] observed, through a very illustrative argument relying on function algebra techniques, that the compact Hankel operators form an $M$-ideal in the space of bounded Hankel operators. The concept of an $M$-ideal will be defined shortly, but let us note for now that $M$-ideality implies proximinality; the distance from a bounded Hankel operator to the compact Hankel operators has a minimizer. Thus Luecking reproved some of the results of [6]. Since

$$((H^\infty + C)/H^\infty)^{**} \simeq L^\infty/H^\infty,$$

it follows that the bidual of the space of compact Hankel operators is isometrically isomorphic to the space of bounded Hankel operators. Spaces which are $M$-ideals in their biduals are said to be $M$-embedded.

The multiplicative Hankel operators, on the other hand, have thus far resisted all attempts to characterize their boundedness. It has been shown that a Nehari-type theorem cannot exist [22], and positive results only exist in special cases [14, 24]. In spite of this, the main theorem shows that Luecking’s result holds for multiplicative Hankel operators.

Let

$$\mathcal{M}_0 = \{ m = M(\alpha) : M(\alpha) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ compact} \}$$

and

$$\mathcal{M} = \{ m = M(\alpha) : M(\alpha) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ bounded} \}.$$

Equipped with the operator norm, $\mathcal{M}_0$ and $\mathcal{M}$ are closed subspaces of $K(\ell^2(\mathbb{N}))$ and $B(\ell^2(\mathbb{N}))$, respectively. For a Banach space $Y$, we denote by $\iota_Y$ the canonical embedding $\iota_Y : Y \to Y^{**}$,

$$\iota_Y y(y^*) = y^*(y), \quad y \in Y, \ y^* \in Y^*.$$

**Theorem 1.2.** There is a unique isometric isomorphism $U : \mathcal{M}_0^{**} \to \mathcal{M}$ such that $U_{|\mathcal{M}_0} m = m$ for every $m \in \mathcal{M}_0$. Furthermore, $\mathcal{M}_0$ is an $M$-ideal in $\mathcal{M}$. 
Remark. As pointed out earlier, Theorem 1.2 is also true when stated for small Hankel operators on $H^2(T^d)$, $d < \infty$. The biduality has in this case been demonstrated isomorphically in [18], with an argument based on the non-isometric Nehari-type theorems proven in [10, 17].

The M-ideal property means the following: there is an (onto) projection $L: \mathcal{M}^* \to \mathcal{M}^*_0$ such that

$$\|m^*\|_{\mathcal{M}^*} = \|Lm^*\|_{\mathcal{M}^*} + \|m^* - Lm^*\|_{\mathcal{M}^*}, \quad m^* \in \mathcal{M}^*,$$

where $\mathcal{M}^*_0$ denotes the space of functionals $m^* \in \mathcal{M}^*$ which annihilate $\mathcal{M}_0$. M-ideals were introduced by Alfsen and Effros [3] as a Banach space analogue of closed two-sided ideals in $C^*$-algebras. Very loosely speaking, the fact that $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$ implies that the norm of $\mathcal{M}$ resembles a maximum norm and, in this analogy, that $\mathcal{M}_0$ is the subspace of elements vanishing at infinity. The book [12] comprehensively treats M-structure theory and its applications.

We will make use of the following consequences of Theorem 1.2. Proximity of $\mathcal{M}_0$ in $\mathcal{M}$ was already mentioned, but the M-ideal property also implies that the minimizer is never unique [16]. It also ensures that $\mathcal{M}^*_0$ is a strongly unique predual of $\mathcal{M}$ [12, Proposition III.2.10]. This means that every isometric isomorphism of $\mathcal{M}$ onto $Y^*$, $Y$ a Banach space, is weak*-weak* continuous, that is, arises as the adjoint of an isometric isomorphism of $Y$ onto $\mathcal{M}^*_0$. On the other hand, $\mathcal{M}^*_0$ has infinitely many different preduals [11, Theorem 27].

The predual of $\mathcal{M}$ is well known to have an almost tautological characterization as a projective tensor product with respect to Dirichlet convolution,

$$\mathcal{X} = \ell^2(N) \ast \ell^2(N).$$

The space $\mathcal{X}$ is also referred to as a weak product space. We defer the precise definition to the next section – after establishing the main theorems, we essentially show, following [25], that all reasonable definitions of $\mathcal{X}$ coincide.

**Theorem 1.3.** There is an isometric isomorphism $L: \mathcal{X} \to \mathcal{M}^*_0$ such that $L^*U^{-1}: \mathcal{M} \to \mathcal{X}^*$ is the canonical isometric isomorphism of $\mathcal{M}$ onto $\mathcal{X}^*$, where $U: \mathcal{M}^*_0 \to \mathcal{M}$ is the isometric isomorphism of Theorem 1.2.

Informally stated, $\mathcal{M}^*_0 \simeq \mathcal{X}$ and $\mathcal{X}^* \simeq \mathcal{M}$. Theorem 1.3 follows at once from Theorem 1.2 and the uniqueness of the predual of $\mathcal{M}$, but we also supply a direct proof. While the duality $\mathcal{X}^* \simeq \mathcal{M}$ is a rephrasing of the definition of $\mathcal{M}$, it is difficult to identify a common approach to dualities of the type $\mathcal{M}^*_0 \simeq \mathcal{X}$ in the existing literature. Often, the latter duality is deduced (isomorphically) via a concrete description of $\mathcal{M}$. For a small selection of relevant examples, see [4, 8, 12, 18, 19, 23, 28].

The idea behind this note is that the direct view of $\mathcal{M}$ as a subspace of $\mathcal{B}(\ell^2(N))$ already provides sufficient information to prove Theorems 1.1, 1.2, and 1.3. In this direction, Wu [28] worked with an embedding into the space
of bounded operators to deduce duality results for certain Hankel-type forms on Dirichlet spaces.

The proofs of the results only have two main ingredients. The first is a device to approximate elements of $\mathcal{M}$ by elements of $\mathcal{M}_0$ (Lemma 2.1). Such an approximation property is necessary, because if $\mathcal{M}_0^{**} \simeq \mathcal{M}$, then the unit ball of $\mathcal{M}_0$ is weak* dense in the unit ball of $\mathcal{M}$. The second ingredient is an inclusion of $\mathcal{M}$ into a reflexive space; in our case, $\ell^2(\mathbb{N})$. Analogous theorems could be proven for many other linear spaces of bounded and compact operators using the same technique.

2. Results

For a sequence $a$ and $0 < r < 1$, let

$$D_r a(n) = r^{\sum_{j=1}^{\infty} j^{\kappa_j}} a(n), \quad n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

Note that

$$\sum_{\kappa} r^{2 \sum_{j=1}^{\infty} j^{\kappa_j}} = \prod_{j=1}^{\infty} \frac{1}{1 - r^{2j}} < \infty.$$

Hence it follows by the dominated convergence theorem that $D_r : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is a compact operator. Furthermore, $D_r$ is self-adjoint and contractive, $\|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq 1$. The dominated convergence theorem also implies that $D_r \to \text{id}_{\ell^2(\mathbb{N})}$ in the strong operator topology (SOT) as $r \to 1$, that is, $\lim_{r \to 1} D_r a = a$ in $\ell^2(\mathbb{N})$, for every $a \in \ell^2(\mathbb{N})$. A study of the operators $D_r$ in the context of Hardy spaces of the infinite polytorus can be found in [2].

The Dirichlet convolution of two sequences $a$ and $b$ is the new sequence $a \ast b$ given by

$$(a \ast b)(n) = \sum_{k|n} a(k)b(n/k), \quad n \in \mathbb{N}.$$ 

If $a$ and $b$ are two finite sequences, then

$$\langle M(\alpha) a, b \rangle_{\ell^2(\mathbb{N})} = (\alpha, a \ast b), \quad (2)$$

where $(a,b) = \sum_{n=1}^{\infty} a(n)b(n)$ denotes the bilinear pairing between $a,b \in \ell^2(\mathbb{N})$. Note also that, for $0 < r < 1$,

$$D_r (a \ast b) = D_r a \ast D_r b. \quad (3)$$

The following simple lemma is key.

Lemma 2.1. Let $M(\alpha)$ be a bounded multiplicative Hankel operator, $M(\alpha) \in \mathcal{M}$. For $0 < r < 1$, let $\alpha_r = D_r \alpha$. Then $M_{\alpha_r} \in \mathcal{M}_0$,

$$\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_{\alpha}\|_{\mathcal{B}(\ell^2(\mathbb{N}))},$$

and $M_{\alpha_r} \to M_{\alpha}$ and $M_{\alpha_r}^* \to M_{\alpha}^*$ SOT as $r \to 1$. 


Proof. By (2) and (3), it holds for finite sequences \( a \) and \( b \) that
\[
\langle M(\alpha_r) a, b \rangle_{\ell^2(\mathbb{N})} = \langle M_\alpha D_r a, D_r b \rangle_{\ell^2(\mathbb{N})}.
\]
Hence \( M_{\alpha_r} = D_r M_\alpha D_r \). We conclude that \( M_{\alpha_r} \) is compact, \( \|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \), and \( M_{\alpha_r} \to M_\alpha \) SOT as \( r \to 1 \). Similarly, \( M^*_{\alpha_r} = M^*_\alpha \to M^*_\alpha \) SOT as \( r \to 1 \).

The following is a recognizable consequence, cf. [27, Theorem 1]. Note that if \( S_n \) and \( T_n \) are operators such that \( S_n \to S \) and \( T_n \to T \) SOT, and if \( C \) is a compact operator, then \( S_n CT_n^* \to SCT^* \) in operator norm.

**Proposition 2.2.** Let \( M(\alpha) \in \mathcal{M} \). Then \( M(\alpha) \in \mathcal{M}_0 \) if and only if
\[
\lim_{r \to 1} \|M(\alpha_r) - M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = 0. \tag{4}
\]

**Proof.** If (4) holds, then \( M(\alpha) \in \mathcal{M}_0 \), since \( M(\alpha_r) \) is compact for every \( 0 < r < 1 \). If \( M(\alpha) \in \mathcal{M}_0 \), then (4) holds, since \( M(\alpha_r) = D_r M(\alpha) D_r = D_r M(\alpha) D_r^* \) and \( D_r \to \text{id}_{\ell^2(\mathbb{N})} \) SOT as \( r \to 1 \). \( \square \)

Recall next the main tool from [6].

**Theorem 2.3 ([6]).** Let \( T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) be a non-compact operator and \( (T_n) \) a sequence of compact operators such that \( T_n \to T \) SOT and \( T_n^* \to T^* \) SOT. Then there exists a sequence \( (c_n) \) of non-negative real numbers such that \( \sum_n c_n = 1 \) for which the compact operator
\[
J = \sum_n c_n T_n
\]
satisfies
\[
\|T - J\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf \{ \|T - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} : K \in \mathcal{K}(\ell^2(\mathbb{N})) \}
\]

Lemma 2.1 and Theorem 2.3 immediately yield the existence part of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( M(\alpha) \) be a bounded multiplicative Hankel operator and let \( (r_k) \) be a sequence such that \( 0 < r_k < 1 \) and \( r_k \to 1 \). Then \( M(\alpha) \) has a best compact approximant of the form
\[
N = \sum_k c_k M(\alpha_{r_k}).
\]
But then \( N = N(\beta) \) is a multiplicative Hankel operator, \( \beta = \sum_k c_k \alpha_{r_k} \).

The non-uniqueness of \( N(\beta) \) follows immediately once we have established Theorem 1.2, by general M-ideal results [16]. In fact, if \( M(\alpha) \notin \mathcal{M}_0 \), then the set of minimizers \( N(\beta) \) is so large that it spans \( \mathcal{M}_0 \). \( \square \)

Note that
\[
\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \geq \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N |\alpha(n)|^2 = \|\alpha\|_{\ell^2(\mathbb{N})}.
\]
Therefore the inclusion $I: M_0 \to \ell^2(\mathbb{N})$ is a contractive operator, $Im = I(M(\alpha)) = \alpha$. We can state Theorem 1.2 slightly more precisely in terms of $I$.

**Theorem 1.2.** Consider the bitranspose $U = I^*: M_0^{**} \to \ell^2(\mathbb{N})$. Then $U M_0^{**} = M$, viewing $M$ as a (non-closed) subspace of $\ell^2(\mathbb{N})$. Furthermore, $U I_{M_0} m = m, \ m \in M_0$,

and

$$\|Um^{**}\|_{B(\ell^2(\mathbb{N}))} = \|m^{**}\|_{M_0^{**}}, \ m^{**} \in M_0^{**}. $$

If $V: M_0^{**} \to M$ is another isometric isomorphism such that $V I_{M_0} m = m$ for all $m \in M_0$, then $V = U$. Furthermore, $M_0^{**}$ is an $M$-ideal in $M$.

**Proof.** We identify $(\ell^2(\mathbb{N}))^* \simeq \ell^2(\mathbb{N})$ linearly through the pairing $(a, b) = \sum_{n=1}^{\infty} a(n)b(n)$ between $a, b \in \ell^2(\mathbb{N})$. With this convention, $I^*: \ell^2(\mathbb{N}) \to M_0^{**}$ is also contractive, and

$$I^*a(m) = (\alpha, a), \ a \in \ell^2(\mathbb{N}), \ m = M(\alpha) \in M_0.$$

Since $I$ is injective, $I^*$ has dense range. In particular, $M_0^{**}$ is separable. Furthermore, $I^{**}: M_0^{**} \to \ell^2(\mathbb{N})$ is injective. By the reflexivity of $\ell^2(\mathbb{N})$, we have that $I^{**} I_{M_0} = I$, since

$$(I^{**} I_{M_0} m, a) = I_{M_0} m (I^* a) = (\alpha, a) = (Im, a)$$

for every $m = M(\alpha) \in M_0$ and $a \in \ell^2(\mathbb{N})$. The interpretation, viewing $M$ as a non-closed subspace of $\ell^2(\mathbb{N})$, is that $I^{**} I_{M_0} m = m$, for all $m \in M_0$.

Consider any $m^{**} \in M_0^{**}$, and let $\alpha = I^{**} m^{**} \in \ell^2(\mathbb{N})$. Since $M_0^{**}$ is separable, the weak* topology of the unit ball $B_{M_0^{**}}$ of $M_0^{**}$ is metrizable.

As is the case for every Banach space, $I_{M_0}(B_{M_0})$ is weak* dense in $B_{M_0^{**}}$. Hence there is a sequence $(m_n)_{n=1}^{\infty}$ in $M_0$ such that $I_{M_0} m_n \to m^{**}$ weak* and $\|m_n\|_{B(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{M_0^{**}}$. Suppose that $m_n = M(\alpha_n)$ and let $a, b \in \ell^2(\mathbb{N})$ be two finite sequences. Then, since $I_{M_0} m_n \to m^{**}$ weak*,

$$\langle M(\alpha_n) a, b \rangle_{\ell^2(\mathbb{N})} = \langle \alpha_n, a \ast b \rangle = I^*(a \ast b)(m_n) \to m^{**}(I^*(a \ast b)) = \langle \alpha, a \ast b \rangle,$$

as $n \to \infty$. It follows that

$$\|\langle M(\alpha) a, b \rangle_{\ell^2(\mathbb{N})}\| = \|\langle \alpha, a \ast b \rangle\| \leq \lim_{n \to \infty} \|m_n\|_{B(\ell^2(\mathbb{N}))} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}$$

$$\leq \|m^{**}\|_{M_0^{**}} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}.$$

Since $a, b$ were arbitrary finite sequences, it follows that $M(\alpha) \in M$ and

$$\|M(\alpha)\|_{B(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{M_0^{**}}.$$

Since $\alpha = I^{**} m^{**}$ this proves that $I^{**}$ maps $M_0^{**}$ contractively into $M$.

Conversely, suppose that $m = M(\alpha) \in M$. By Lemma 2.1, for $0 < r < 1$, $M(\alpha_r) \in M_0$, $\|M(\alpha_r)\| \leq \|M(\alpha)\|$, and $\alpha_r \to \alpha$ in $\ell^2(\mathbb{N})$ as $r \to 1$. Define $m^{**} \in M_0^{**}$ by

$$m^{**}(I^* a) := \langle \alpha, a \rangle = \lim_{r \to 1} \langle \alpha_r, a \rangle = \lim_{r \to 1} I^* a(M(\alpha_r)), \ a \in \ell^2(\mathbb{N}). \quad (5)$$
This specifies an element \( m^{**} \in M_0^{**} \) since \( I^* \) has dense range in \( M_0^* \) and 
\[
|m^{**}(I^*a)| \leq \lim_{r \to 1} \|M(\alpha r)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|I^*a\|_{M_0^*} \leq \|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|I^*a\|_{M_0^*}.
\]
From this inequality we also see that 
\[
\|m^{**}\|_{M_0^{**}} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}.
\] (6)
Furthermore, since \((I^{**}m^{**}, a) = m^{**}(I^*a) = (\alpha, a), \ a \in \ell^2(\mathbb{N}),\)
we have that \(I^{**}m^{**} = \alpha\). Hence \(I^{**}\) maps \( M_0^{**} \) bijectively and contractively onto \( M \). By (6), \( I^{**} : M_0^{**} \to M \) is also expansive, and hence it is an isometric isomorphism.

Recall that \( \mathcal{K}(\ell^2(\mathbb{N})) \) is an M-ideal in \( \mathcal{B}(\ell^2(\mathbb{N})) \) [9] – indeed, \( \mathcal{K}(\ell^2(\mathbb{N})) \) is a two-sided closed ideal in \( \mathcal{B}(\ell^2(\mathbb{N})) \). It is well known that there is an isometric isomorphism \( E : \mathcal{K}(\ell^2(\mathbb{N}))^{**} \to \mathcal{B}(\ell^2(\mathbb{N})) \) such that \( E|_{\mathcal{K}(\ell^2(\mathbb{N}))} K = K \) for all \( K \in \mathcal{K}(\ell^2(\mathbb{N})) \). Thus \( \mathcal{K}(\ell^2(\mathbb{N})) \) is M-embedded. Since \( M_0 \) is a closed subspace of \( \mathcal{K}(\ell^2(\mathbb{N})) \), \( M_0 \) is also M-embedded [12, Theorem III.1.6]. Hence, since we have shown that \( I^{**} : M_0^{**} \to M \) is an isometric isomorphism for which \( I^{**} \iota_{M_0} m = m \) for all \( m \in M_0 \), it follows that \( M_0 \) is an M-ideal in \( M \).

Finally, if \( V : M_0^{**} \to M \) is another isometric isomorphism such that \( V \iota_{M_0} m = m, m \in M_0 \), then \( F = V^{-1}I^{**} : M_0^{**} \to M_0^{**} \) is an isometric isomorphism such that \( F|_{M_0} = \iota_{M_0} \). However, since \( M_0 \) is M-embedded, \( F \) must be obtained as the bitranspose, \( F = G^{**} \), of an isometric isomorphism \( G : M_0 \to M_0 \) [12, Proposition III.2.2]. But then \( G = \text{id}_{M_0} \), since 
\[
m^*(Gm) = G^*m^*(m) = F\iota_{M_0} m(m^*) = m^*(m), \ m \in M_0, \ m^* \in M_0^*.
\]
Hence \( F = \text{id}_{M_0} \), and so \( V = I^{**} \). \( \square \)

The predual of a space of Hankel operators usually has an abstract description as a projective tensor product [5, 7, 10]. In the present context, let 
\[
X = \left\{ c : c = \sum_{\text{finite}} a_k \ast b_k, \ a_k, b_k \ \text{finite sequences} \right\},
\]
and equip \( X \) with the norm 
\[
\|c\|_X = \inf \sum_{\text{finite}} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})},
\]
where the infimum is taken over all finite representations of \( c \). By writing \( c = c \ast (1, 0, 0, \ldots) \) it is clear that \( \|c\|_X \leq \|c\|_{\ell^2(\mathbb{N})} \) for \( c \in X \).

We define the projective tensor product space \( X^* = \ell^2(\mathbb{N}) \ast \ell^2(\mathbb{N}) \) with respect to Dirichlet convolution as the Banach space completion of \( X \). It is essentially definition that \( X^* \simeq M \).
Lemma 2.4. For $m = M(\alpha) \in \mathcal{M}$, let

$$Jm(c) = (\alpha, c), \quad c \in X.$$  

Then $Jm$ extends to a bounded functional on $\mathcal{X}$ for every $m \in \mathcal{M}$, and $J: \mathcal{M} \to \mathcal{X}^*$ is an isometric isomorphism.

Proof. Let $m \in \mathcal{M}$. If $c \in X$ and $\varepsilon > 0$, choose a representation $c = \sum_{k=1}^{N} a_k \ast b_k$, where $a_k$ and $b_k$ are finite sequences for every $k$, and

$$\sum_{k=1}^{N} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_X + \varepsilon.$$  

Then

$$|Jm(c)| = \left| \sum_{k=1}^{N} \langle M(\alpha)a_k, b_k \rangle_{\ell^2(\mathbb{N})} \right| \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} (\|c\|_X + \varepsilon).$$

Hence $\|Jm\|_{\mathcal{X}^*} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$. Choosing finite sequences $a$ and $b$ such that $\|a\|_{\ell^2(\mathbb{N})} = \|b\|_{\ell^2(\mathbb{N})} = 1$ and $(M(\alpha)a, b)_{\ell^2(\mathbb{N})} > \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon$, and letting $c = a \ast b$ gives that

$$\|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon < \|Jm\|_{\mathcal{X}^*} \|c\|_X \leq \|Jm\|_{\mathcal{X}^*}.$$  

Hence $J$ is an isometry.

The inclusion of finite sequences into $X$ extends to a contractive map $E: \ell^2(\mathbb{N}) \to \mathcal{X}$. Let $\ell \in \mathcal{X}^*$ and let $c \in X$. Then $\ell(c) = (\alpha, c)$, where $\alpha = E^*\ell \in \ell^2(\mathbb{N})$. Then $m = M(\alpha) \in \mathcal{M}$, since $\ell \in \mathcal{X}^*$. Clearly $Jm = \ell$ and thus $J$ is onto. \qed

Theorem 1.3. For every $c \in X$, let

$$Lc(m) = (\alpha, c), \quad m = M(\alpha) \in \mathcal{M}_0.$$  

Then $L$ extends to an isometric isomorphism $L: \mathcal{X} \to \mathcal{M}_0^*$, and

$$L^*U^{-1} = J: \mathcal{M} \to \mathcal{X}^*$$

is the isometric isomorphism of Lemma 2.4. Here $U: \mathcal{M}_0^* \to \mathcal{M}$ is the isometric isomorphism of Theorem 1.2.

Proof. The quickest proof proceeds by noting that $\mathcal{M}_0^*$ is a strongly unique predual of $\mathcal{M}_0^*$, since $\mathcal{M}_0$ is $M$-embedded. This implies that the isometric isomorphism $JU: \mathcal{M}_0^* \to \mathcal{X}^*$ is the adjoint of an isometric isomorphism $E: \mathcal{X} \to \mathcal{M}_0^*$, $E^* = JU$. But then, for $c \in X$ and $m = M(\alpha) \in \mathcal{M}_0$,

$$Ec(m) = \iota_{\mathcal{M}_0} m(Ec) = E^* \iota_{\mathcal{M}_0} m(c) = JU \iota_{\mathcal{M}_0} m(c) \quad \text{(7)}$$

$$= Jm(c) = (\alpha, c) = Lc(m).$$

Hence $L = E$, and thus $L$ is an isometric isomorphism.

Alternatively, the weak$^\ast$-weak$^*$ continuity of $JU$ can be proven by hand. $L$ clearly extends to a contractive operator $L: \mathcal{X} \to \mathcal{M}_0^*$. The computation (7) shows that $JU \iota_{\mathcal{M}_0} = L^* \iota_{\mathcal{M}_0}$. Let $m^{**} \in \mathcal{M}_0^{**}$ and let $M(\alpha) = Um^{**}$.
From (5) we deduce that \( m_r^{**} = \iota_{M_0} M(\alpha_r) \rightarrow m_r^{**} \) weak* in \( M_0^{**} \). Hence \( L^* m_r^{**} \rightarrow L^* m_r^{**} \) weak* in \( \mathcal{X}^* \). On the other hand, for \( c \in X \),
\[
JU m_r^{**}(c) = (\alpha_r, c) = \lim_{r \to 1} \alpha_r, c = \lim_{r \to 1} JU m_r^{**}(c) = \lim_{r \to 1} L^* m_r^{**}(c) = L^* m_r^{**}(c).
\]
This shows that \( JU = L^* \), and hence \( L^* \) is an isometric isomorphism. \( \square \)

**Remark.** In the notation of Theorem 1.2, \( I^* c = Lc \) for \( c \in X \). Theorem 1.3 hence completes the picture of Theorem 1.2 by giving an interpretation of the operator \( I^* \).

Suppose that we had instead defined the projective tensor product space \( \ell^2(N) \hat{\otimes} \ell^2(N) \) as the sequence space
\[
\mathcal{Y} = \left\{ c : c = \sum_{k=1}^{\infty} a_k \hat{\otimes} b_k, a_k, b_k \in \ell^2(N), \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(N)} \|b_k\|_{\ell^2(N)} < \infty \right\},
\]
normed by
\[
\|c\|_\mathcal{Y} = \inf \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(N)} \|b_k\|_{\ell^2(N)},
\]
where the infimum is taken over all representations of \( c \). One would like to know that \( \mathcal{Y} = \mathcal{X} \). Indeed, it is not a priori clear that \( \mathcal{X} \) is a sequence space; or if \( \mathcal{X} \) is identifiable with a space of Dirichlet series, if considering multiplicative Hankel operators in that context. For \( \mathcal{Y} \) these properties are immediate.

**Lemma 2.5.** \( \mathcal{Y} \) is a Banach space.

**Proof.** Since \( \|(a \hat{\otimes} b)(n)\| \leq \|a\|_{\ell^2(N)} \|b\|_{\ell^2(N)} \) it is clear that
\[
e_n(c) = c(n), \quad c \in \mathcal{Y},
\]
defines an element \( e_n \in \mathcal{Y}^* \), for every \( n \in \mathbb{N} \). It follows that \( \|c\|_\mathcal{Y} = 0 \) if and only if \( c = 0 \).

Suppose that \( \sum_{k=1}^{\infty} c_k \) is an absolutely convergent series in \( \mathcal{Y} \). Then there are double sequences \( (a_{k,j}) \) and \( (b_{k,j}) \) such that \( c_k = \sum_{j=1}^{\infty} a_{k,j} \hat{\otimes} b_{k,j} \) for every \( k \)
\[
\sum_{k,j=1}^{\infty} \|a_{k,j}\|_{\ell^2(N)} \|b_{k,j}\|_{\ell^2(N)} < \infty.
\]
Then \( c = \sum_{k,j=1}^{\infty} a_{k,j} b_{k,j} \) is an element of \( \mathcal{Y} \) and
\[
\|c - \sum_{k=1}^{N} c_k\|_\mathcal{Y} \leq \sum_{k=N+1}^{\infty} \sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(N)} \|b_{k,j}\|_{\ell^2(N)} \rightarrow 0, \quad N \rightarrow \infty.
\]
Hence \( \sum_{k=1}^{\infty} c_k \) converges in \( \mathcal{Y} \) to \( c \). Thus \( \mathcal{Y} \) is complete. \( \square \)
We now prove that \( \mathcal{Y} = \mathcal{X} \). The details are similar to those of [25], where projective tensor products of spaces of holomorphic functions were considered. Note that \( \mathcal{X} \) is contractively contained in \( \mathcal{Y} \).

**Proposition 2.6.** The inclusion \( V: X \to \mathcal{Y} \) extends to an isometric isomorphism \( V: \mathcal{X} \to \mathcal{Y} \).

**Proof.** We make the following preliminary observation. Since for every \( 0 < r < 1 \),

\[
D_r(a \ast b) = D_r a \ast D_r b, \quad \|D_r\|_{\mathcal{B}(\ell^2(N))} \leq 1,
\]

\( D_r \) defines a bounded operator \( D_r: \mathcal{X} \to \mathcal{X} \),

\[
\|D_r\|_{\mathcal{B}(\mathcal{X})} \leq 1.
\]

Furthermore, since \( D_r \to \text{id}_{\ell^2(N)} \) SOT on \( \ell^2(N) \) as \( r \to 1 \), it follows that \( \|D_r c - c\|_{\mathcal{X}} \leq \|D_r c - c\|_{\ell^2(N)} \to 0 \) as \( r \to 1 \) for every \( c \in X \). Hence \( D_r \to \text{id}_{\mathcal{X}} \) SOT on \( \mathcal{X} \) as \( r \to 1 \).

As in Lemma 2.5, for each \( n \in \mathbb{N} \),

\[
e_n(c) = c(n), \quad c \in \mathcal{X},
\]

extends to a functional \( e_n \in \mathcal{X}^* \) with \( \|e_n\|_{\mathcal{X}^*} \leq 1 \). We show now that \( (e_n) \) is a complete sequence in \( \mathcal{X}^* \) with respect to the weak* topology. Suppose that \( c \in \mathcal{X} \) and that \( e_n(c) = 0 \) for all \( n \). Pick a sequence \( (c_k) \) in \( X \) such that \( c_k \to c \) in \( \mathcal{X} \). Then for fixed \( r < 1 \),

\[
\|D_r c\|_{\mathcal{X}} \leq \lim_{k \to \infty} (\|D_r(c - c_k)\|_{\mathcal{X}} + \|D_r c_k\|_{\mathcal{X}})
\]

\[
= \lim_{k \to \infty} \|D_r c_k\|_{\mathcal{X}} \leq \lim_{k \to \infty} \|D_r c_k\|_{\ell^2(N)}.
\]

Since \( c_k \to c \) in \( \mathcal{X} \) and \( e_n \in \mathcal{X}^* \), we have that \( \lim_{k \to \infty} c_k(n) = e_n(c) = 0 \) for every \( n \). Furthermore, \( |c_k(n)| \leq \|e_n\|_{\mathcal{X}^*} \|c_k\|_{\mathcal{X}} \leq \|c_k\|_{\mathcal{X}} \) is uniformly bounded in \( k \) and \( n \). Hence it follows by the dominated convergence theorem that \( \lim_{k \to \infty} \|D_r c_k\|_{\ell^2(N)} = 0 \) and thus that \( D_r c = 0 \). Since \( D_r c \to c \) in \( \mathcal{X} \) as \( r \to 1 \) we conclude that \( c = 0 \). Therefore \( (e_n) \) is complete.

Hence \( \mathcal{X} \) is a space of sequences. More precisely, since every evaluation \( e_n \) is a bounded functional on \( \mathcal{Y} \) as well, the extension \( V: \mathcal{X} \to \mathcal{Y} \) of the inclusion map is given by

\[
Vc = (e_n(c))_{n=1}^\infty, \quad c \in \mathcal{X}.
\]

The completeness of \( (e_n) \) implies that \( V \) is injective.

We next prove that \( V \) is onto. The argument is precisely as in [25], but we include it for completeness. For a sequence \( a \) and \( m \in \mathbb{N} \), let \( a^m = (a(1), \ldots, a(m), 0, \ldots) \). Given \( a \in \ell^2(N) \) and \( \delta > 0 \), choose a sequence \( (m_1, m_2, \ldots) \) such that \( \|a - a^{m_k}\|_{\ell^2(N)} \leq 2^{-k} \). Let \( a_k = a^{m_k+1} - a^{m_k} \). Then, for sufficiently large \( K \),

\[
a = a^{m_K} + \sum_{k=K}^{\infty} (a^{m_{k+1}} - a^{m_k}), \quad \sum_{k=K}^{\infty} \|a^{m_{k+1}} - a^{m_k}\|_{\ell^2(N)} < \delta.
\]
Hence we can write \( a = \sum_{j=1}^{\infty} a_j \), where each \( a_j \) is a finite sequence and 
\[ \sum_j \|a_j\|_{\ell^2(\mathbb{N})} < \|a\|_{\ell^2(\mathbb{N})} + \delta. \]

Given \( c \in \mathcal{Y} \) and \( \varepsilon > 0 \), choose \( (a_k)^\infty_{k=1} \) and \( (b_k)^\infty_{k=1} \) such that
\[ c = \sum_{k=1}^{\infty} a_k \star b_k, \quad \sum_k \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + \varepsilon. \]

For each \( k \), write, as in the preceding paragraph, \( a_k = \sum_{j=1}^{\infty} a_{k,j} \), \( b_k = \sum_{j=1}^{\infty} b_{k,j} \), where each \( a_{k,j} \) and \( b_{k,j} \) is a finite sequence and
\[ \sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} < \|a_k\|_{\ell^2(\mathbb{N})} + \delta_k, \quad \sum_{j=1}^{\infty} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \|b_k\|_{\ell^2(\mathbb{N})} + \delta_k. \]

Here the \( \delta_k \) are chosen so that
\[ \sum_{k=1}^{\infty} (\|a_k\|_{\ell^2(\mathbb{N})} + \delta_k)(\|b_k\|_{\ell^2(\mathbb{N})} + \delta_k) < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \varepsilon. \]

Then \( c = \sum_{k,j,l=1}^{\infty} a_{k,j} \star b_{k,l} \), and
\[ \sum_{k,j,l=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,l}\|_{\ell^2(\mathbb{N})} < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \varepsilon. \]

Relabeling, we have a representation \( c = \sum_{n=1}^{\infty} a_n \star b_n \) where \( a_n \) and \( b_n \) are finite sequences and \( \sum_n \|a_n\|_{\ell^2(\mathbb{N})} \|b_n\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + 2\varepsilon. \) Let \( c_N = \sum_{n=1}^{N} a_n \star b_n \). Then \( c_N \to c \) in \( \mathcal{Y} \), and furthermore \( (c_N) \) is a Cauchy sequence in \( X \), hence has a limit \( \tilde{c} \) in \( X \). By continuity of the functionals \( \varepsilon_n \) on both \( \mathcal{Y} \) and \( X \), we find in view of (8) that \( V\tilde{c} = c \). Hence \( V \) is onto.

Furthermore, since \( V \) is contractive,
\[ \|c\|_{\mathcal{Y}} \leq \|\tilde{c}\|_{X} = \lim_{N \to \infty} \|c_N\|_{X} < \|c\|_{\mathcal{Y}} + 2\varepsilon. \]

We already showed that \( V \) is injective, so that \( \tilde{c} \) is uniquely defined by \( c \). On the other hand, \( \varepsilon \) is arbitrary. We conclude that \( \|c\|_{\mathcal{Y}} = \|\tilde{c}\|_{X} \). It follows that \( V \) is an isometric isomorphism. \( \square \)

References


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