Dynamics and Julia sets of iterated elliptic functions

Jane Hawkins and Mónica Moreno Rocha

Abstract. We study the parametrized family of elliptic functions of the form $F_{\Lambda, b}(z) = \wp_\Lambda(z) + b$ for $b \in \mathbb{C}$, $\Lambda$ a lattice, and $\wp_\Lambda$ the Weierstrass elliptic function with period lattice $\Lambda$. We show that the dynamics depend on $b$ as $b$ varies within one fundamental region of $\mathbb{C}/\Lambda$, and on the lattice $\Lambda$. We analyze properties of the Julia sets, and bifurcations of $F_{\Lambda, b}$, focusing on real rectangular lattices; in particular the dynamical properties are more diverse than those coming from the family $\wp_\Lambda$ with $\Lambda$ varying.

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1. Introduction

In this paper we show that, when iterating meromorphic functions, the connectivity of the Julia set changes when a constant is added to the Weierstrass elliptic $\wp$ function, without changing the period lattice. Given a lattice $\Lambda$, we consider maps: $F_{\Lambda,b}(z) = \wp_{\Lambda}(z) + b$, $b \in \mathbb{C}$, and show for example that Cantor Julia sets occur when a constant is added to $\wp_{\Lambda}$, even when $J(\wp_{\Lambda})$ is connected. Iterated elliptic functions have been the subject of study for some time starting with [18] and [10], and now there is a significant literature on the topic (see for example [9] – [13], [15] – [19], and [25]). It is known for example that for any square lattice $\Lambda$, the Julia set of $\wp_{\Lambda}$ is always connected [12, 4]. The connected Julia sets vary quite a bit and depend on a classical invariant called $g_2$, or equivalently on the generators of the period lattice $\Lambda$. We focus on real rectangular lattices in this paper, though many statements are proved more generally. We study bifurcations that occur in parameter space paying special attention to real parameters and parameters that lie on the horizontal half lattice lines, emphasizing that the resulting dynamics are quite different from each other. For example, for $\Lambda$ square, since 0 is a critical value and a pole of $\wp_{\Lambda}$, $b = 0$ is a very unstable parameter; every neighborhood of 0 contains $b$’s that can move the pole to either an attracting or repelling cycle. However there is much more stability when $b$ lies on the half lattice line as there are no poles near the critical values of $F_{\Lambda,b}$. Adding a constant to $\wp_{\Lambda}^n$, $n \geq 1$ was also studied in [15], from a different perspective.

In Section 2 we give preliminary definitions and background for iterated elliptic functions proving some new results relating critical values to the lattice, which are used to parametrize the dynamics in this paper. In Section 3 we introduce the parametrized family of mappings $F_{\Lambda,b}$ studied in the paper and prove some general properties of these maps. The main result in Section 3 is Theorem 3.2, which shows that for any lattice except possibly a triangular lattice (which has additional symmetries), one fundamental period for the lattice $\Lambda$ provides a parameter space in which we have a representative of each conformal equivalence class of maps $F_{\Lambda,b}$. In Section 4 we study the dynamical properties of maps with real parameters $b$. We show that for every real rectangular lattice there are constants $b$ such that the Julia set of $F_{\Lambda,b}$ is the whole sphere, and that same $b$ is also an accumulation point for parameters where $F_b$ has a super-attracting fixed point (Theorem 4.21). In Section 5 we look at a different part of parameter space along horizontal half lattice lines, and discuss bifurcations that can occur. We turn to some results about Cantor Julia sets for $F_{\Lambda,b}$ for real rectangular lattices, including square ones, in Section 6. We show that for some square lattices, whenever $b$ lies on a horizontal half lattice line, the Julia set is a Cantor set; we also show that toral bands can occur in the Fatou set of $F_{\Lambda,b}$. 
2. Preliminary definitions and notation

By \( \Lambda = [\lambda_1, \lambda_2] \) we denote the group \( \Lambda = \{ m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z} \} \subset \mathbb{C} \). If \( \lambda_1, \lambda_2 \in \mathbb{C} \) are non-zero and linearly independent over \( \mathbb{R} \), \( \Lambda \) is a lattice. Lattices determine double periods for elliptic functions; \( z + \Lambda \) denotes a coset of \( \mathbb{C}/\Lambda \) containing \( z \). A closed, connected subset \( Q \) of \( \mathbb{C} \) is a fundamental region for \( \Lambda \) if for each \( z \in \mathbb{C} \), \( Q \) contains at least one point in the same \( \Lambda \)-orbit as \( z \) and no two points in the interior of \( Q \) are in the same \( \Lambda \)-orbit. If \( Q \) is a parallelogram it is called a period parallelogram for \( \Lambda \).

**Definition 2.1.** Let \( \mathbb{C}_\infty = \mathbb{C} \cup \{ \infty \} \) denote the Riemann sphere. An elliptic function \( f: \mathbb{C} \to \mathbb{C}_\infty \) is a meromorphic function in \( \mathbb{C} \) which is periodic with respect to a lattice \( \Lambda \).

The **Weierstrass elliptic function** is defined by

\[
\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right),
\]

\( z \in \mathbb{C} \). The map \( \wp_\Lambda \) is an even elliptic function with poles of order 2. The derivative of the Weierstrass elliptic function is an odd elliptic function which is periodic with respect to \( \Lambda \). The Weierstrass elliptic function and its derivative are related by the differential equation

\[
\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3,
\]

where \( g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-4} \) and \( g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-6} \).

The numbers \( g_2(\Lambda) \) and \( g_3(\Lambda) \) are invariants of the lattice \( \Lambda \) in the following sense: if \( g_2(\Lambda) = g_2(\Lambda') \) and \( g_3(\Lambda) = g_3(\Lambda') \), then \( \Lambda = \Lambda' \). Furthermore given any \( g_2 \) and \( g_3 \) such that \( g_2^3 - 27g_3^2 \neq 0 \) there exists a lattice \( \Lambda \) having \( g_2 = g_2(\Lambda) \) and \( g_3 = g_3(\Lambda) \) as its invariants [8]. For \( \Lambda = [1, \tau] \), the functions \( g_i(\tau) = g_i(1, \tau) \), \( i = 2, 3 \), are analytic functions of \( \tau \) in the open upper half plane \( \text{Im}(\tau) > 0 \) ([8], Theorem 3.2). We have the following homogeneity in the invariants \( g_2 \) and \( g_3 \) [11].

**Lemma 2.2.** For lattices \( \Lambda \) and \( \Lambda' \), \( \Lambda' = k\Lambda \iff g_2(\Lambda') = k^{-4}g_2(\Lambda) \) and \( g_3(\Lambda') = k^{-6}g_3(\Lambda) \).

A lattice \( \Lambda \) is said to be **real** if \( \Lambda = \overline{\Lambda} := \{ \overline{\lambda} : \lambda \in \Lambda \} \), where \( \overline{\lambda} \) denotes the complex conjugate of \( \lambda \in \mathbb{C} \).

**Proposition 2.3.** [14] The following are equivalent:

1. \( \Lambda \) is a real lattice;
2. \( \wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)} \);
3. \( g_2, g_3 \in \mathbb{R} \).

Given any \( \Lambda \), for \( k \in \mathbb{C}\setminus\{0\} \), the following homogeneity property holds:

\[
\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u).
\]
2.1. Real rectangular period lattices for \( \wp_\Lambda \). For most of this paper we assume that \( \Lambda = [\lambda_1, \lambda_2] \), with \( \lambda_1 > 0 \) and \( \lambda_2 \) purely imaginary and lying in the upper half plane. Since a fundamental region \( Q \) can be chosen to be a rectangle with two sides parallel to the real axis and two sides parallel to the imaginary axis, \( \Lambda \) is called a real rectangular lattice.

**Remark 2.4.** 1. For any lattice \( \Lambda \), \( \wp_\Lambda \) has infinitely many simple critical points, one at each half lattice point, and we denote them by \( \{\omega_1, \omega_2, \omega_3\} + \Lambda \), where

\[
\omega_1 = \frac{\lambda_1}{2}, \quad \omega_2 = \frac{\lambda_2}{2}, \quad \omega_3 = \omega_1 + \omega_2.
\]

We denote the set of all critical points by \( \text{Crit}(\wp_\Lambda) \).

2. \( \wp_\Lambda \) has three critical values \( e_j = \wp_\Lambda(\omega_j) \) satisfying, when \( \Lambda \) is real rectangular, \( e_1 > 0 \). Also, one of these hold: \( e_2 < e_3 < 0 \) (if \( g_3 > 0 \)), \( e_2 < 0 < e_3 < e_1 \) (if \( g_3 < 0 \)), or \( e_3 = 0 \) (if \( g_3 = 0 \)). In the third case, \( e_2 = -e_1 \) and the lattice is called rectangular square.

3. Since for any lattice \( \Lambda \), \( e_1, e_2, e_3 \) are the distinct zeros of Equation (2.1), we have these critical value relations:

\[
\wp_\Lambda'(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3).
\]

Equating like terms in Equations (2.1) and (2.3), we obtain

\[
e_1 + e_2 + e_3 = 0, \quad e_1 e_3 + e_2 e_3 + e_1 e_2 = -\frac{g_2}{4}, \quad e_1 e_2 e_3 = \frac{g_3}{4}.
\]

From Equation (2.1), we write

\[
p(x) = 4x^3 - g_2 x - g_3.
\]

A cubic polynomial of the form (2.5) has discriminant:

\[
\Delta (g_2, g_3) = g_2^3 - 27g_3^2.
\]

4. The lattice \( \Lambda \) is real rectangular if and only if \( \Delta(g_2, g_3) > 0 \) and \( g_2 > 0 \).

Equivalently, \( \Lambda := \Lambda(g_2, g_3) \) is real rectangular if and only if \( (g_2, g_3) \) lies in the region: \( \mathcal{R} = \{(g_2, g_3) \in \mathbb{R}^2 : g_2^3 - 27g_3^2 > 0\} \).

5. \( \Lambda \) is real rectangular square if and only if the roots of \( p \) are \( 0, \pm \sqrt[3]{g_2}/2 \), and then we have: \( e_3 = 0 \) and \( e_1 = \sqrt[3]{g_2}/2 = -e_2 > 0 \).

2.1.1. Real rectangular lattice critical values. We can parametrize real rectangular lattices by their critical values \( \{e_1, e_2, e_3\} \) under \( \wp_\Lambda \); the invariants \( (g_2, g_3) \) they determine can be described explicitly.

**Proposition 2.5.** For any values \( e_1 > 0 \), and \( e_2 < 0 \) satisfying \( |e_2| < 2 e_1 \), if we set

\[
(g_2, g_3) = (4(e_1^2 + e_1 e_2 + e_2^2), -4(e_1^2 e_2 + e_1 e_2^2)),
\]

the corresponding map \( \wp_\Lambda \) has critical values \( \wp_\Lambda(\omega_j) = e_j, \ j = 1, 2, 3 \) with \( e_3 = -e_1 - e_2 \). The critical value \( e_3 \) satisfies \( e_2 < e_3 < e_1 \). Moreover the lattice \( \Lambda = \Lambda(g_2, g_3) \) is real rectangular, and all real rectangular lattices have \( (g_2, g_3) \) satisfying Equation (2.7).
<table>
<thead>
<tr>
<th>Prescribed Parameter</th>
<th>{{e_1, e_2, e_3}</th>
<th>(g_2, g_3)</th>
<th>\Lambda-generator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard square</td>
<td>{1, -1, 0}</td>
<td>(4, 0)</td>
<td>\gamma</td>
</tr>
<tr>
<td>Center lattice</td>
<td>{\omega_1, -\omega_1, 0}</td>
<td>(4\omega_1^2, 0)</td>
<td>2\omega_1</td>
</tr>
<tr>
<td>e_1</td>
<td>{e_1, -e_1, 0}</td>
<td>(4e_1^2, 0)</td>
<td>\sqrt{\frac{\gamma}{\omega_1}}</td>
</tr>
<tr>
<td>g_2</td>
<td>{\sqrt{g_2}, -\sqrt{g_2}, 0}</td>
<td>(g_2, 0)</td>
<td>\gamma \sqrt{2g_2^{-1/4}}</td>
</tr>
<tr>
<td>\Lambda-generator</td>
<td>{\sqrt{g_2}, \omega_2, 0}</td>
<td>(\frac{1}{\sqrt{g_2}}, 0)</td>
<td>k\gamma</td>
</tr>
</tbody>
</table>

Table 1. Parameter relationships for \(\wp_\Lambda\) on a rectangular square lattice, where \(\gamma \approx 2.62206\) denotes the lemniscate constant.

**Proof.** Setting \(e_1, e_2,\) and \(e_3\) as in the hypotheses, by construction we have \(\sum_{j=1}^3 e_j = 0\) and \(e_2 < e_3 < e_1\). The proposed value \(g_3\) satisfies

\[
\frac{g_3}{4} = -(e_1^2 e_2 + e_1 e_2^2) = (e_1 e_2) \cdot (-e_1 - e_2) = e_1 e_2 e_3,
\]

and similarly

\[
-\frac{g_2}{4} = -(e_1^2 + e_1 e_2 + e_2^2) = e_1 \cdot e_2 + e_1 \cdot (-e_1 - e_2) + e_2 \cdot (-e_1 - e_2) = e_1 e_2 + e_1 e_3 + e_2 e_3.
\]

Using Equations (2.1) and (2.4), by uniqueness of the roots of (2.5), the result follows. The condition \(|e_2| < 2e_1\) ensures that \(e_3 < e_1,\) so \(e_1 = \wp_\Lambda(\omega_1)\) as claimed. Real lattices are characterized by having \((g_2, g_3)\) that satisfy \(\Delta(g_2, g_3) \neq 0,\) and among real lattices, real rectangular are precisely those with distinct real critical values \({e_1, e_2, e_3}\) satisfying the properties of Equation (2.4), so the result is proved. \(\square\)

Starting from the standard square lattice in row 2, all other entries of Table 1 follow from the homogeneity equation (2.2) for \(\wp_\Lambda,\) and the table shows how the various invariants for \(\wp_\Lambda\) interact with each other. By definition the center lattice (shown in row 3 of Table 1) is the lattice (and corresponding value of \(g_2\)) for which the associated Weierstrass \(\wp\) function \(\wp_\Lambda\) has a super-attracting fixed point at \(\omega_1.\) It follows that \(\omega_1 = (2/\gamma)^{-2/3} \approx 1.19787.\)

For real rectangular lattices, we use the Arithmetic Geometric Mean of two nonnegative numbers \(A\) and \(B\) (this is discussed in various sources, e.g., [1]).

**Definition 2.6.** Given \(A, B > 0,\) we first set \(A_0 = A\) and \(B_0 = B.\) We then define two sequences \({A_n}, \{B_n\}, n = 0, 1, \ldots\) by

\[
A_{n+1} = \frac{1}{2}(A_n + B_n), \quad B_{n+1} = \sqrt{A_n B_n},
\]
where for $B_{n+1}$ we always choose the positive square root. The Arithmetic Geometric Mean (AGM) of Gauss, is the common limit of the two sequences, and is written $\mathcal{M}(A, B)$.

Since we restrict to real rectangular lattices here, we always assume that $e_1 > 0$. Therefore the expression under the radical sign of the following is always positive, so we define:

$$AG_1(e_1, e_2) := \mathcal{M}(\sqrt{e_1 - e_2}, \sqrt{e_1 - e_3})$$

and

$$AG_2(e_1, e_2) := \mathcal{M}(\sqrt{e_1 - e_2}, \sqrt{e_3 - e_2}).$$

For $(g_2, g_3)$ as in Proposition 2.5 we have $\Lambda = \left[ \frac{\pi}{AG_1(e_1, e_2)}, \frac{\pi i}{AG_2(e_1, e_2)} \right]$, (see [1]).

**Lemma 2.7.** If $\Lambda = [\lambda_1, \lambda_2]$ is a real rectangular lattice, with $\lambda_1$ real and $\lambda_2$ purely imaginary, and the corresponding critical values are $e_j, j = 1, 2, 3$, then:

$$\frac{\pi}{\sqrt{e_1 - e_2}} \leq |\lambda_1| \leq \frac{\pi}{\sqrt{e_1 - e_3}},$$

$$\frac{2\pi}{\sqrt{e_1 - e_2} + \sqrt{e_3 - e_2}} \leq |\lambda_2| \leq \frac{\pi}{[(e_1 - e_2)(e_3 - e_2)]^{\frac{1}{2}}}.$$  

Another important identity we use throughout is the following.

**Theorem 2.8.** [8] Let $\Lambda$ be any lattice and $u \in \mathbb{C}$. Then for each $i \in \{1, 2, 3\}$,

$$\varphi^{n}(u) = \varphi_{\Lambda}(u) \omega_{i} = \varphi_{\Lambda}(u + \omega_{i}).$$

The next definition appears in different forms; we use the definition from [7].

**Definition 2.9.** [7] By $\varphi^{n}$ we denote the $n$-fold composition of $\varphi$ with itself; we define the postcritical set of $\varphi_{\Lambda}$ by $\mathcal{P}(\varphi_{\Lambda}) = \bigcup_{n>0} \varphi_{\Lambda}^{n}(\text{Crit}(\varphi_{\Lambda})).$

**Lemma 2.10.** [10]. If $\Lambda$ is real rectangular, $\mathcal{P}(\varphi_{\Lambda}) \subset \mathbb{R} \cup \{\infty\}$.

2.2. Fatou and Julia sets for elliptic functions. Background definitions and properties for meromorphic functions appear in ([2] – [5]) and [6]; if $S \subset \mathbb{C}$ is a set, then $cl(S)$ denotes the topological closure of $S$.

Let $f: \mathbb{C} \to \mathbb{C}$ be a meromorphic function with at least two distinct poles. The Fatou set $F(f)$ is the set of points $z \in \mathbb{C}$ such that $\{f^{n} : n \in \mathbb{N}\}$ is defined and normal in some neighborhood of $z$. The Julia set is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C} \setminus F(f)$. Montel’s theorem implies that $J(f) = cl\left(\bigcup_{n \geq 0} f^{-n}(\infty)\right)$. 
A meromorphic function is *Class S* if $f$ has only finitely many critical and asymptotic values; for each lattice $\Lambda$, every elliptic function with period lattice $\Lambda$ is of *Class S*. Therefore the basic dynamics are similar to those of rational maps with the exception of the poles. The first result holds for all *Class S* functions as was shown in ([3], Corollary 4 and Theorem 12).

**Theorem 2.11.** For any lattice $\Lambda$, the Fatou set of an elliptic function $f_\Lambda$ with period lattice $\Lambda$ has no wandering domains and no Baker domains.

In particular, all Fatou components of $f_\Lambda$ are preperiodic, and because there are only finitely many critical values, we have a bound on the number of attracting periodic points that can occur.

We define the family of elliptic functions of interest in this paper. Let $\Lambda$ be a lattice.

(Main Family) \[ F_{\Lambda, b}(z) = \wp_{\Lambda}(z) + b, \quad \text{for} \quad b \in \mathbb{C}. \]

Theorem 2.12. For any lattice $\Lambda$, $F_{\Lambda, b}$ has no cycle of Herman rings.

Since $\wp_{\Lambda}$ has three distinct critical values, so does $F_{\Lambda, b}$; this limits the number of disjoint forward invariant Fatou cycles to at most three. Each of these cycles is one of four types, summarized by the following result.

**Theorem 2.13.** For any lattice $\Lambda$, and any $b \in \mathbb{C}$, each periodic Fatou component of $F_{\Lambda, b}$ contains one of these:

1. a linearizing neighborhood of an attracting periodic point;
2. a Böttcher neighborhood of a super-attracting periodic point;
3. an attracting Leau petal for a periodic parabolic point. The periodic point is in $J(F_{\Lambda, b})$;
4. a periodic Siegel disk containing an irrationally neutral periodic point.

The proof of Lemma 2.14 is given for $\wp_{\Lambda}$ in [11] but remains the same for any elliptic function.

**Lemma 2.14.** If $\Lambda$ is any lattice and $f_\Lambda$ is an elliptic function with period lattice $\Lambda$, then $J(f_\Lambda) + \Lambda = J(f_\Lambda)$, and $F(f_\Lambda) + \Lambda = F(f_\Lambda)$.

**Definition 2.15.** Given two elliptic functions $f = f_\Lambda$ and $g = g_{\Lambda'}$ over period lattices $\Lambda$ and $\Lambda'$ respectively, we say $f$ is *conformally conjugate* to $g$ if there exists a map $\phi(z) = \alpha z + \beta, \alpha \neq 0$ such that $f \circ \phi = \phi \circ g$.

3. The parametrized family of elliptic functions $F_b$, $b \in \mathbb{C}$

For each fixed lattice $\Lambda$, we study the dynamical and parametric planes of the one-parameter family of elliptic functions

(Main Family) \[ F_{\Lambda, b}(z) := \wp_{\Lambda}(z) + b, \quad \text{for} \quad b \in \mathbb{C}, \]
which we will denote by \( F_b \) when the lattice is fixed. Clearly \( \text{Crit}(F_{\Lambda,b}) = \text{Crit}(\varphi_\Lambda) \); the critical values of \( F_{\Lambda,b} \) are \( \{v_i = e_i + b : i = 1, 2, 3\} \), and the critical relations from (2.4) are:

\[
(3.1) \quad \sum_{i=1}^{3} v_i = 3b, \quad \sum_{i \neq j} v_i v_j = 3b^2 - \frac{g_2}{4}, \quad v_1 v_2 v_3 = b^3 - b \frac{g_2}{4} + \frac{g_3}{4}.
\]

For each fixed lattice \( \Lambda \) we say that the holomorphic family of meromorphic maps \( F_b \) parametrized over \( b \in A \subset C \) is reduced if for all \( b \neq b' \) in \( A \), \( F_b \) and \( F'_b \) are not conformally conjugate.

We show that you need look no further than one period parallelogram \( Q \) for the constant \( b \) for a reduced family of maps \( F_b \).

**Proposition 3.1.** Given a fixed lattice \( \Lambda \), if \( F_b = \varphi_\Lambda + b \), then for any \( \lambda \in \Lambda \), \( F_b \) is conformally conjugate to \( F_{b+\lambda} \).

**Proof.** For \( \lambda \in \Lambda \), a straightforward computation shows that the map \( \phi(z) = z - \lambda \), conjugates \( F_b \) and \( F_{b+\lambda} \). \( \square \)

One can ask if there are conformally conjugate maps within a fundamental period.

**Theorem 3.2.** Suppose we have a lattice \( \Lambda = [\lambda_1, \lambda_2] \), which is not triangular. If \( F_b = \varphi_\Lambda + b \), and if \( b \) and \( b' \) are in the interior of a fundamental region \( Q \), then \( F_b \) is not conformally conjugate to \( F'_{b'} \).

**Proof.** Suppose that \( F_b \circ \phi(z) = \phi \circ F'_{b'}(z) \) for all \( z \in \mathbb{C} \). The conformal conjugacy has to fix \( \infty \) so \( \phi \) must be of the form \( \phi(z) = \alpha z + \beta \). Moreover, since 0 is a pole of \( F'_{b'} \), \( \phi(0) = \beta \) must be a pole of \( F_b \), so \( \beta = \lambda_0 \in \Lambda \). Moreover, \( \phi \) maps all poles to poles injectively, so we must have \( \phi(\Lambda) = \alpha \Lambda + \lambda_0 = \Lambda \), or equivalently \( \alpha \Lambda = \Lambda \) and since \( \phi^{-1} \Lambda = \Lambda \), we have \( \alpha \Lambda = \alpha^{-1} \Lambda = \lambda = \alpha^k \Lambda \), for all \( k \in \mathbb{Z} \), so \( |\alpha| = 1 \) and \( \alpha = e^{2\pi i/p} \) for some \( p \in \mathbb{N} \).

Therefore \( e^{2\pi i/p} \Lambda = \Lambda \), and if \( \alpha \neq 1 \), by [23] (and other classical sources), \( p = 2, 3, 4 \) or 6.

The critical values of \( F_b \) are \( e_1 + b, e_2 + b, e_3 + b \), and of \( F'_{b'} \) are \( e_1 + b', e_2 + b', e_3 + b' \). Since \( \phi \) must map the critical values of \( F'_{b'} \) to the critical values of \( F_b \), for \( j = 1, 2, 3 \), \( \phi(e_j + b') = e_k + b \) for some \( k = 1, 2, 3 \). We then have, using (2.4) and (3.1):

\[
3b = (e_1 + b) + (e_2 + b) + (e_3 + b)
\]

\[
= \phi(e_1 + b') + \phi(e_2 + b') + \phi(e_3 + b')
\]

\[
= (\alpha e_1 + \alpha b' + \lambda_0) + (\alpha e_2 + \alpha b' + \lambda_0) + (\alpha e_3 + \alpha b' + \lambda_0)
\]

\[
= 3(\alpha b' + \lambda_0),
\]

so \( b = \alpha b' + \lambda_0 \). Now from (3.2) it follows that

\[
F_b(\phi(z)) = \varphi_\Lambda(\alpha z + \lambda_0) + b \neq \varphi_\Lambda(\alpha z) + \alpha b' + \lambda_0
\]
and for all \( z \), this should equal:
\[
\phi(\varphi_{\Lambda}(z) + b') = \alpha \varphi_{\Lambda}(z) + \alpha b' + \lambda_0.
\]
Therefore for all \( z \in \mathbb{C} \), \( \varphi_{\Lambda}(\alpha z) = \alpha \varphi_{\Lambda}(z) \). By Equation (2.2) and the fact that \( \alpha^{-1} \Lambda = \Lambda \), this implies that \( \varphi_{\Lambda}(\alpha z) = \alpha^{-2} \varphi_{\Lambda}(z) = \alpha \varphi_{\Lambda}(z) \) for all \( z \).

Therefore \( \alpha^3 = 1 \), so \( p = 1 \) or 3. In the first case, \( \alpha = 1 \) and \( \lambda_0 = 0 \) or \( b' \) and \( b \) are not both in \( Q \). Otherwise, \( p = 3 \), so the lattice must be triangular, and \( b = e^{2\pi i/3}b' + \lambda \). This proves the result.

\[\square\]

**Remark 3.3.** We often restrict to this parameter plane domain:
\[
Q = Q_{\Lambda} = \{ b \in \mathbb{C} : -\omega_1 < \Re(b) \leq \omega_1, -\Im(b) < \Im(b) \leq \Im(\omega_2) \}.
\]

We have some additional symmetries for the Julia sets of \( F_b \) that come from the analogous symmetry for \( \varphi_{\Lambda} \).

**Proposition 3.4.** For a fixed lattice \( \Lambda \), any \( b \in \mathbb{C} \), and any \( c \in \text{Crit}(\varphi_{\Lambda}) \),
\[
c + z \in J(F_b) \text{ if and only if } c - z \in J(F_b).
\]

**Proof.** Using Theorem 2.8 and \( \text{Crit}(F_{\Lambda,b}) = \text{Crit}(\varphi_{\Lambda}) \),
\[
F_b(c + z) = \varphi_{\Lambda}(c + z) + b = \varphi_{\Lambda}(c - z) + b = F_b(c - z).
\]

Define the horizontal half lattice line:
\[
L = \{ z \in \mathbb{C} : z = t + \omega_2, \, t \in \mathbb{R} \}.
\]

**Lemma 3.5.** Assume \( \Lambda \) is a real lattice and fix some \( b \in \mathbb{C} \).

1. Then, \( F_{\Lambda,b} \) is anticonformally conjugate to \( F_{\Lambda,b} \).
2. Moreover, for \( \Lambda \) rectangular, if for \( k \in \mathbb{Z} \), \( b_k \) denotes the reflection of \( b \) with respect to \( L + k\lambda_2 \), then \( F_{\Lambda,b} \) is anticonformally conjugate to \( F_{\Lambda,b_k} \).

**Proof.** Denote by \( \eta(z) = \bar{z} \), that is, the complex conjugate of \( z \); it is not hard to show that \( \eta \) is an anticonformal homeomorphism of the plane that implements the conjugacy.

The next result follows from Remarks 2.4 and Table 1 (cf. [10], Theorems 8.1, 8.2). Let \( \kappa = \Gamma(1/4)^2/(4\sqrt{\pi}) = \gamma/\sqrt{2} \).

**Lemma 3.6.** Let \( \Lambda \) be a real square lattice, so \( e_1 = \sqrt{g_2}/2 \) and \( \omega_1 = \kappa/g_2^{1/4} \) for any \( g_2 > 0 \). We then have:
1. \( e_1 = 2k\omega_1 \) for some \( k \in \mathbb{N} \), (and hence, the orbit of \( \omega_1 \) under \( \varphi_{\Lambda} \) lands on a pole after one iteration), if and only if
\[
g_2 = (4k\kappa)^{4/3}.
\]
2. The critical value \( e_1 = (2k + 1)\omega_1 \) for some \( k \in \mathbb{N}_0 \), (and thus \( (2k + 1)\omega_1 \) is a super-attracting fixed point for \( \varphi_{\Lambda} \) if and only if
\[
g_2 = (2(2k + 1)\kappa)^{4/3}.
\]
4. The maps \( F_b \) for real rectangular lattices and \( b \in \mathbb{R} \)

Throughout this section we assume \((g_2, g_3) \in \mathbb{R}, \) and \( \Lambda \) is the lattice associated to those invariants. We describe the dynamics for \( F_b \) for real parameters \( b \). As in (3.3), \( L \) is the horizontal half lattice line and \( V \) denotes the vertical half lattice line: \( V = \{ \omega_1 + iy : y \in \mathbb{R} \} \).

**Lemma 4.1.** For any real rectangular lattice \( \Lambda \), if \( b \in \mathbb{R} \), then \( F_b \) maps \( \mathbb{R} \), \( L, V \), and the imaginary axis into \( \mathbb{R} \). For all \( n > 0 \), \( F_b^n(t) \in [v_1, \infty) \) for all \( t \in \mathbb{R} \); the same is true for all \( z \in L \) and \( z \in V \) as long as \( n \geq 2 \).

**Proof.** Since \( \Lambda \) is real, \( e_2 < e_3 < e_1 \), with \( e_1 > 0 \) and \( e_2 < 0 \). For all \( t \in \mathbb{R}, \) \( \varphi_\Lambda(t) \in \mathbb{R} \) and \( \varphi_\Lambda(t) \geq e_1 \). Thus \( F_b^n(t) \in \mathbb{R} \) for all \( n > 0 \), and since \( F_b(t) \geq v_1 \) for all \( t \), then \( F_b^n(t) \geq v_1 \). Using Theorem 2.8, for any \( t \in \mathbb{R}, \ t + \omega_2 \in L \), so we have \( \varphi_\Lambda(t + \omega_2) \in \mathbb{R} \) and \( F_b(t + \omega_2) \in \mathbb{R} \). Similarly, if we show that the imaginary axis gets mapped into \( \mathbb{R} \), Theorem 2.8 will also show that points on \( V \), which are of the form \( u + \omega_1 \), with \( u \) purely imaginary map under \( \varphi_\Lambda \) into \( \mathbb{R} \). The result for purely imaginary numbers follows by Proposition 2.3(2), and the fact that \( \varphi_\Lambda \) is even; this implies purely imaginary numbers get mapped to real numbers for \( b \in \mathbb{R} \). \( \square \)

**Proposition 4.2.** Assume \( \Lambda = [2\omega_1, 2\omega_1 i] \) is a square lattice.

1. If \( b = \omega_1 \) (or an odd multiple of \( \omega_1 \)), then \( F_b^2(\omega_1) = F_b^2(\omega_2) = F_b^3(\omega_3) \); i.e., \( F_b \) has a single critical orbit.

2. If \( b \) is an odd multiple of \( \omega_1 \) define \( M(z) = e_1 \left( \frac{z + e_1}{z - e_1} \right) \). If \( t \in \mathbb{R}, \)

\[
F_b(t) = \varphi_\Lambda(t) + b \mapsto M \circ \varphi_\Lambda(\varphi_\Lambda(t)) + b \mapsto \ldots \mapsto (M \circ \varphi_\Lambda)^n(\varphi_\Lambda(t)) + b.
\]

Then \( M^{-1} = M, \ e_1 \mapsto \infty \mapsto e_1 \) and \( -e_1 \mapsto 0 \mapsto -e_1 \). Its fixed points are given by \( e_1 \pm \sqrt{2}e_1 \). Moreover, \( M \) sends the interval \((e_1, \infty)\) onto itself: this implies that \( M \) interchanges the intervals \([e_1, e_1 + \sqrt{2}e_1] \) with \([e_1 + \sqrt{2}e_1, \infty) \). \( M \) also sends the interval \((-\infty, e_1)\) onto itself, interchanges the intervals \((e_1 - \sqrt{2}e_1, e_1] \) with \((-\infty, e_1 - \sqrt{2}e_1) \) and flips the upper and lower half planes.

3. If \( \Lambda \) is the center square lattice, then for any \( b \in \mathbb{C}, \) \( F_b(v_1) = F_b(b + \omega_1) = F_b(b - \omega_1) = F_b(v_2) \), so the critical orbits of \( \omega_1 \) and \( \omega_2 \) coincide on the second iterate.

**Proof.** (1) follows from (3.1) and the symmetry of \( \varphi_\Lambda \) with respect to any critical point; a computation gives the result. (2) can be verified directly by writing \( b = (2j + 1)\omega_1 \) and using Theorem 2.8. To show (3), we have \( e_1 = \omega_1 = -e_2 \), and we apply Equation (2.9). \( \square \)

**4.1. The Schwarzian derivative.** The Schwarzian derivative plays an important role in the study of the dynamics of \( F_{\Lambda,b} \).
Definition 4.3. The Schwarzian derivative of a meromorphic function $f$ is given by

$$Sf(z) = \frac{f''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$ 

A few properties of $Sf$ are:
1. $f$ is a Möbius transformation if and only if $Sf = 0$.
2. The Schwarzian derivative of the composition of any two functions $f$ and $g$ is given by

$$S(f \circ g)(z) = Sf(g(z)) \cdot (g'(z))^2 + Sg(z).$$

From these properties we obtain the following result (cf. [9], [16]).

Proposition 4.4. If $\Lambda$ is a real rectangular lattice, and if $M$ is any Möbius map with real coefficients, then for all $t \in \mathbb{R}$, $t$ not a half lattice point,

$$S(M \circ \varphi_{\Lambda})(t) = S\varphi_{\Lambda}(t) < 0.$$ 

Proof. By Properties 1. and 2., it is enough to prove that if $\Lambda$ is real rectangular, then for $t \in \mathbb{R} \setminus \frac{1}{2} \Lambda$, $S\varphi_{\Lambda}(t) < 0$. We have already remarked that for these lattices, $\varphi_{\Lambda}(t) \geq e_1 > 0$ on $\mathbb{R}$. For any lattice $\Lambda$, we consider $\varphi = \varphi_{\Lambda}(z)$, for any $z \in \mathbb{C}$. From the differential equation in Equation (2.1) we have $2\varphi' \varphi'' = 12\varphi^2 \varphi' - g_2 \varphi'$. Then $\varphi'' = 6\varphi^2 - g_2/2$; and differentiating gives $\varphi''' = 12\varphi'\varphi'$, so

$$\frac{\varphi''}{\varphi'} = 12\varphi, \tag{4.1}$$

which is the first term in $S\varphi$. We now consider the duplication formula,

$$\varphi(2z) = \frac{1}{4} \left( \frac{\varphi''(z)}{\varphi'(z)} \right)^2 - 2\varphi(z).$$

This gives immediately that

$$\left( \frac{\varphi''(z)}{\varphi'(z)} \right)^2 = 4(\varphi(2z) + 2\varphi(z)),$$

and thus the second term becomes

$$-\frac{3}{2} \left( \frac{\varphi''(z)}{\varphi'(z)} \right)^2 = -6\varphi(2z) - 12\varphi(z). \tag{4.2}$$

Adding (4.1) and (4.2), we conclude that for any lattice $\Lambda$ and any $z \in \mathbb{C}$,

$$S\varphi(z) = -6\varphi(2z). \tag{4.3}$$

Therefore for real rectangular lattices $\Lambda$, when $z = t \in \mathbb{R}$, and $2t \notin \Lambda$, $S\varphi_{\Lambda}(t) < 0$ and $S(M \circ \varphi_{\Lambda})(t) < 0$, since $\varphi_{\Lambda}(2t) > 0$. 

□

The following corollary follows from Equation (4.3) and holds for an arbitrary lattice $\Lambda$. 


Corollary 4.5. For the Weierstrass elliptic function \( \wp \) over the lattice \( \Lambda \), \( S\wp \) is an even elliptic function over the lattice \( \frac{1}{2}\Lambda \).

Assume \( \Lambda \) is real rectangular and the map \( F_b, b \in \mathbb{R} \) has a non-repelling \( p \)-cycle. Write
\[
C = \{ t_0, F_b(t_0), \ldots, F_b^{p-1}(t_0) \} \subset \mathbb{R}.
\]
Then we consider its basin of attraction on \( \mathbb{R} \), namely
\[
B(C) = \{ x \in \mathbb{R} : F_b^k(x) \rightarrow C \text{ as } k \rightarrow \infty \}.
\]

We call the cycle \( C \) topologically attracting if \( B(C) \) contains an open interval \( U \); in this case we call \( B(C) \) the real attracting basin of \( C \). By \( B_0(C) \) we denote the union of components of \( B(C) \) in \( \mathbb{R} \) containing points from \( C \). \( B_0(C) \) is the real immediate (attracting) basin of \( C \). For \( \Lambda \) real rectangular, we have \( \text{cl}(P(F_b)) \subset \mathbb{R} \), so if \( C \) is non-repelling then \( C \subset [v_1, \infty) \) and \( B(C) \neq \emptyset \); i.e., \( C \) is topologically attracting on \( \mathbb{R} \) [10]. We extend a theorem of Singer on interval maps to this setting, (proved in [9] for rhombic square lattices):

Theorem 4.6. If \( \Lambda \) is real rectangular and \( b \in \mathbb{R} \), then:
1. the real immediate basin of a topologically attracting periodic orbit of \( F_{\Lambda,b} \) contains a real critical point.
2. If \( y \in \mathbb{R} \) is in a rationally neutral \( p \)-cycle for \( F_b \) then it is topologically attracting; i.e., there exists an open interval \( U \), with possibly \( y \in \partial U \), such that for every \( t \in U \), \( \lim_{n \rightarrow \infty} F_b^np(t) = y \).

The next two results appear in ([15] Proposition 2.8 and Proposition 3.8).

Lemma 4.7. For any \( \Lambda \) real rectangular and any \( b \in \mathbb{R} \), \( F_{\Lambda,b} \) has no cycles of Siegel disks.

Proposition 4.8. Let \( \Lambda \) be any real rectangular lattice, and \( b \in \mathbb{R} \). Then either \( J(F_b) = \mathbb{C}_\infty \), or there exists one real non-repelling cycle whose immediate basin of attraction contains a real critical point.

Since there are infinitely many real critical points, the following is sometimes more useful.

Corollary 4.9. Under the hypotheses of Prop 4.8, if \( F_b \) has a real non-repelling cycle, then its immediate basin of attraction contains \( v_1 = e_1 + b \).

Corollary 4.10. Suppose \( b \in \mathbb{R} \) is such that \( F_b \) has an attracting, super-attracting, or parabolic cycle \( C \) whose immediate basin contains \( (2k+1)\omega_1 \) for some \( k \in \mathbb{Z} \). Then \( F(F_b) \) coincides with the attracting basin of \( C \), and each critical orbit of \( F_b \) corresponding to \( \omega_2 \) and \( \omega_3 \) either eventually maps to the basin of \( C \) or belongs to the Julia set.

In order to study the behavior of maps associated to certain parameters \( b \) we develop some descriptive vocabulary.

Definition 4.11. Assume \( \Lambda \) is any real rectangular lattice, \( b \in \mathbb{R} \), and \( k \in \mathbb{N} \).
1. $b$ is an order $k$ prepole parameter for $\omega_1$ if $F^k_b(\omega_1) = j\lambda_1$, for some $j \in \mathbb{Z}$;
2. $b$ is an order $k$ precritical parameter for $\omega_1$ if $F^k_b(\omega_1) = (2j + 1)\omega_1$, for some $j \in \mathbb{Z}$, and $F^m_b(\omega_1) \notin \omega_1 + \Lambda$, for $0 < m < k$. If $k = 1$, we call $b$ precritical.
3. $b$ is a (period $k$) center parameter for $\omega_1$ if $F^k_b(\omega_1) = \omega_1$ (and $k$ is minimal).
4. We say $b$ is an order $k$ noncritical preperiodic parameter for $\omega_1$ if $F^k_b(\omega_1)$ is preperiodic but $b$ is none of the above.

These definitions lead to the next proposition.

**Proposition 4.12.** We assume $b \in \mathbb{R}$, $\Lambda$ is real rectangular, and all statements refer to the critical point $\omega_1$ unless otherwise specified.

1. Parameters for which $J(F_b) = \mathbb{C}_\infty$:
   (a) Every order $k$ prepole parameter $b$ gives Julia set the whole sphere for $F_b$.
   (b) If $g_2$ is chosen as in (3.4) and $g_3 = 0$, then $b = 0$ is an order 1 prepole.
   (c) If $b$ is a noncritical preperiodic parameter, then for some $k \in \mathbb{N}$, $F^k_b(\omega_1)$ is periodic of period $r > 1$, the cycle $C = \{F^k_b(\omega_1), F^{k+1}_b(\omega_1), \ldots, F^{k+r-1}_b(\omega_1)\}$ contains no critical point, and $J(F_b) = \mathbb{C}_\infty$.
   (d) If $\omega_1 \in J(F_b)$, then $J(F_b) = \mathbb{C}_\infty$.

2. Parameters for which $F_b$ has a super-attracting cycle:
   (a) If $b$ is a period $k$ center parameter, then the corresponding map $F_b$ has a super-attracting periodic orbit that contains $\omega_1$.
   (b) If $g_2$ is chosen as in (3.5), and $g_3 = 0$, then $b = 0$ is a center parameter of $F_0$.
   (c) Every order $k$ precritical parameter corresponds to a map $F_b$ with a super-attracting periodic orbit on $\mathbb{R}$ containing a real critical point of the form $(2j + 1)\omega_1$.
   (d) A precritical parameter $b$ satisfies $F_b(\omega_1) = (2j + 1)\omega_1$ for some nonzero integer $j$. The resulting critical orbit has the form:

$$\omega_1 \mapsto v_1 = (2j + 1)\omega_1 \Leftrightarrow$$

and $(2j + 1)\omega_1$ is a super-attracting fixed point.

**Proof.** We first prove 1(d); for $b \in \mathbb{R}$ by Definition 2.9 and Lemma 4.1, we consider the orbits of the non-real critical points $\omega_2$ and $\omega_3$ and have that $\text{cl}(\mathcal{P}(F_b)) \subset \{v_2, \infty\}$. If either $\omega_2$ or $\omega_3$ lands on a critical point in $[v_2, \infty)$, then the orbit lands on the same orbit as that of $\omega_1$ after one more iteration, so is in $J(F_b)$. By Theorem 4.6 every Fatou component that is not super-attracting must contain an infinite forward orbit of $\omega_1$ (no Siegel disk cycles or Herman rings occur under the hypotheses on $b$ and $\Lambda$). If there were a super-attracting cycle, by periodicity $\omega_1$ must land on that cycle so none exist, and any non-repelling cycle must have a basin containing $v_1$, which is impossible by hypothesis. Therefore $\omega_2$ and $\omega_3$ lie in the Julia set of $F_b$. 


along with $\omega_1$ and the result follows from Proposition 4.8. Parts (a), (b), and (c) all imply $\omega_1 \in J(F_b)$ from Definition 4.11, so follow immediately.

Properties 2(a), (c), and (d) follow directly from Definition 4.11, since all real critical points map to $v_1$, and 2(b) follows from the assumption on $b$.

\[ \square \]

**Remark 4.13.** (1) We can extend Definition 4.11 to define order $k$ properties with respect to any critical point. In particular, any order $k$ precritical parameter for $\omega_1$ is also a period $m$ center parameter for the critical point $c = (2j + 1)\omega_1$ if $F_b^k(\omega_1) = c$, and $F_b^m(c) = c$.

(2) When $b \notin \mathbb{R}$, and $b$ is noncritical preperiodic for $\omega_1$, then $F(F_b)$ need not be empty, as the orbits of $\omega_2$ and $\omega_3$ might lie in the basin of a non-repelling orbit.

\[ \] 4.2. Existence of parameters with prescribed dynamics. In Figure 1 we show a reduced region from Theorem 3.2 with the lattice outlined in green; the real axis seems to cut through a homeomorphic copy of the Mandelbrot set for some lattices but this is not always the case, as shown in Figure 2 for a rectangular lattice. (The origin is in the center of each figure.)

We see features of quadratic-like mappings in the parameter spaces, but the setting of elliptic functions allows us to prove the existence of prepole parameters for an arbitrary real rectangular lattice. Prepole parameters impacting the dynamics of $F_b$ also occur in parameter space. There are infinitely many order one prepole parameters; we find it useful to distinguish the two closest to the origin, and to identify the order 1 center parameter between them. When the parameter is real, the dynamics are driven by the real critical point, so we focus on $\omega_1$.

4.2.1. Prepole and precritical parameters for $\omega_1$ under $F_b$.

**Proposition 4.14** (Existence of order 1 prepole and precritical parameters for $\omega_1$). Let $(g_2, g_3) \in \mathcal{R}$ be given, and let $e_1$ be the critical value of the corresponding map $\wp_A$. Suppose $j \in \mathbb{N} \cup \{0\}$ satisfies either

\[
(4.4) \quad j\lambda_1 < e_1 < (2j + 1)\omega_1
\]

or

\[
(4.5) \quad (2j + 1)\omega_1 < e_1 < (j + 1)\lambda_1.
\]

Then for the map $F_b$ there exist order 1 prepole parameters $b_{pj}$ and $b_{pj+1}$ for $\omega_1$, exactly one of which is in $(-\omega_1, \omega_1]$; in addition there is exactly one order 1 precritical parameter, $b_c$ in $(-\omega_1, \omega_1]$. These parameters are arranged as follows.

1. $-\omega_1 < b_{pj} < 0 < b_c \leq \omega_1 < b_{pj+1}$ if (4.4) holds.
2. $b_{pj} < -\omega_1 < b_c < 0 < b_{pj+1} \leq \omega_1$ if (4.5) holds.
3. $|b_c - b_{pj}| = |b_c - b_{pj+1}| = \omega_1$. 

\[ \]
Figure 1. $b$-space for $F_b$ using $(g_2, g_3) \approx (5.7395, 0)$

Figure 2. $b$-space for $F_b$ using $(g_2, g_3) = (7, -3)$

**Proof.** Assume first $j\lambda_1 \leq e_1 < (2j+1)\omega_1$ for some integer $j \geq 0$ (Equation (4.4)). Then there exists an order 1 prepole parameter $b \in (-\omega_1, \omega_1]$ such that $F_b(\omega_1) = e_1 + b = q\lambda_1$ for some integer $q$ if and only if

(4.6) \hspace{1cm} b = q\lambda_1 - e_1;

from the assumption, we see that choosing $q = j$ gives

$b_{p_j} := j\lambda_1 - e_1,$
and \(-\omega_1 < b_{p_j} \leq 0\) as claimed. Any other choice of integer \(q\) would yield a pole parameter \(b\) outside the interval \((-\omega_1, 0]\). (Equation (4.4) implies that \(b_p = 0\) is possible.)

An order 1 precritical parameter \(b_c \in (-\omega_1, \omega_1]\) satisfies \(F_b(\omega_1) = e_1 + b_c = (2q + 1)\omega_1\) if and only if
\[
(4.7) \quad b_c := (2q + 1)\omega_1 - e_1 \leq \omega_1;
\]
again choosing \(q = j\) gives a unique \(b_c \in (-\omega_1, \omega_1]\) and \(b_c > 0\) by Equation (4.4). Clearly \(b_c - b_{p_j} = \omega_1^2\).

If \(b_{p_{j+1}} = (j + 1)\lambda_1 - e_1\), then clearly \(0 < b_c < \omega_1 < b_{p_{j+1}}\) and since \(\lambda_1 = 2\omega_1\), \(|b_c - b_{p_j}| = |b_c - b_{p_{j+1}}| = \omega_1\) as claimed.

The case when \((2j + 1)\omega_1 \leq e_1 < (j + 1)\lambda_1\), i.e., when Equation (4.5) holds is similar. In particular, we choose \(q = j + 1\) in Equation (4.6) so that \(b_{p_{j+1}} \in (0, \omega_1]\) and \(q = j\) in Equation (4.7) to obtain \(b_c \in (-\omega_1, 0]\) as claimed.

We denote by \(b_p\) the unique order 1 prepole parameter in \((-\omega_1, \omega_1]\) (from Proposition 4.14). We simplify the notation with the next definition.

**Definition 4.15.** For any real rectangular lattice \(\Lambda = [\lambda_1, \lambda_2]\), and for any \(j \in \mathbb{Z}\), we define \(p_j := b_{p_j} = j\lambda_1 - e_1\), where \(e_1\) is the positive real critical value of \(\varphi_\Lambda\) associated with \(\omega_1 > 0\).

We have the following consequence of the previous results.

**Corollary 4.16.** Assume \(\Lambda\) is real rectangular.
1. For any \(b = p_j, j \in \mathbb{Z}\) as in Definition 4.15, we have \(J(F_b) = \mathbb{C}_\infty\).
2. If \(b = b_c + q\lambda_1, q \in \mathbb{Z}\), then there is a super-attracting fixed point in \(F(F_b)\).

**Proof.** It suffices to consider \(F_{p_j} = \varphi_\Lambda(t) + p_j\), for \(p_j \in (-\omega_1, \omega_1]\) by Proposition 3.1. The result follows from Proposition 4.8 since there cannot be any non-repelling cycles. Similarly, \(F_{b_c}((2j + 1)\omega_1) = (2j + 1)\omega_1\) so we have a fixed critical point and Proposition 3.1 gives the result. \(\square\)

**4.2.2. Parabolic parameters for \(\omega_1\).** We turn to the existence of parameters which correspond to maps \(F_b\) with parabolic fixed points, which we call parabolic parameters. For any integer \(j\), set \(I_j = [j\lambda_1, (j + 1)\lambda_1]\).

**Lemma 4.17.** Suppose we have a real rectangular lattice \(\Lambda = [\lambda_1, \lambda_2]\) satisfying \(e_1 \in I_j\); then there exists a unique parameter value \(b_{+1} \in (-\omega_1, \omega_1]\) with the property that the map \(F_{b_{+1}} = \varphi_\Lambda + b_{+1}\) has a fixed point \(s_1 \in I_j\) such that \(F_{b_{+1}}'(s_1) = 1\).

**Proof.** The interval \(I_j\) is a fundamental period interval for \(\varphi_\Lambda|_\mathbb{R}\) and \(\varphi_\Lambda'|_\mathbb{R}\). We showed that there exists a parameter \(b_c = (2j + 1)\omega_1 - e_1 \in (-\omega_1, \omega_1]\) such that \(F_{b_c}(\omega_1)\) is a fixed critical point. Since \(\varphi_\Lambda'\) is monotone, real analytic, increasing on \((j\lambda_1, (j + 1)\lambda_1)\) for each \(j \in \mathbb{Z}\), and
\[
\varphi_\Lambda' : (j\lambda_1, (j + 1)\lambda_1) \to (-\infty, \infty),
\]
with \( \varphi'_\Lambda((2j + 1)\omega_1) = 0 \), there exists a unique \( s_1 \in ((2j + 1)\omega_1, (j + 1)\lambda_1) \) such that \( \varphi'_\Lambda(s_1) = 1 \). The corresponding parameter value \( b \), which makes \( s_1 \) a fixed point of \( F_b \), is then chosen to be \( b_{+1} = s_1 - \varphi_\Lambda(s_1) \).

It remains to show that \( b_{+1} \in (-\omega_1, \omega_1) \). We consider the function \( t - \varphi_\Lambda(t) \) on \((j\lambda, (j + 1)\lambda)\); its derivative \( 1 - \varphi'_\Lambda(t) \) is positive on \((j\lambda, s_1)\), 0 at \( s_1 \), and negative on \((s_1, (j + 1)\lambda)\). Therefore \( s_1 \) is a maximum point, with maximum value \( b_{+1} \), so

\[
-\omega_1 < b_e = (2j + 1)\omega_1 - \varphi_\Lambda(\omega_1) = (2j + 1)\omega_1 - \varphi_\Lambda((2j + 1)\omega_1) < b_{+1}.
\]

If \( b_p > 0 \), then \(-\omega_1 < b_e < b_{+1} < b_p \leq \omega_1\) (by definition of \( b_p \)) and the result is proved. Otherwise \( b_p \leq 0 \), and Equation (4.4) holds so \( j\lambda \leq e_1 < (2j + 1)\omega_1 \). Since \( \varphi_\Lambda \) has its minimum at \((2j + 1)\omega_1 \) on \((j\lambda_1, (j + 1)\lambda_1)\), then \( \varphi_\Lambda(s_1) > e_1 \). Moreover \((2j + 1)\omega_1 < s_1 < (j + 1)\lambda_1 \) by construction. These inequalities give:

\[
\begin{align*}
-\omega_1 &< b_{+1} = s_1 - \varphi_\Lambda(s_1) \\
&< (j + 1)\lambda_1 - e_1 \\
&< (j + 1)\lambda_1 - j\lambda_1 \\
&< \lambda_1.
\end{align*}
\]

In this case if \( b_{+1} > \omega_1 \), then we replace it by \( \tilde{b}_1 = b_{+1} - \lambda_1 \in (-\omega_1, \omega_1) \). Then \( F_{\tilde{b}_1}(s_1 - \lambda_1) = \varphi_\Lambda(s_1 - \lambda_1) + b_{+1} - \lambda_1 = \varphi_\Lambda(s_1) + s_1 - \varphi_\Lambda(s_1) - \lambda_1 = s_1 - \lambda_1 \); also \( F'_{\tilde{b}_1}(s_1 - \lambda_1) = 1 \) so the result is proved.

In the interval \((-\omega_1, \omega_1)\), we always find \( b_{+1} \) such that \( b_e < b_{+1} \) by Lemma 4.17. We have a similar lemma for the existence and placement of a parabolic parameter \( b_{-1} \).

**Lemma 4.18.** Suppose we have a real rectangular lattice \( \Lambda = [\lambda_1, \lambda_2] \) as above. Then there exists a unique parameter value \( b_{-1} \in (-\omega_1, \omega_1) \) with the property that the map \( F_{b_{-1}} \) has a fixed point \( s_{-1} \in (j\lambda_1, (j + 1)\lambda_1) \) such that \( F'_{b_{-1}}(s_{-1}) = -1 \).

**Proof.** The proof is essentially the same as the proof of Lemma 4.17 since there is a unique \( s_{-1} \in (j\lambda_1, (2j + 1)\omega_1) \) such that \( \varphi'_\Lambda(s_{-1}) = -1 \). It remains to show that \( b_{-1} \in (-\omega_1, \omega_1) \). Since \( s_{-1} < (2j + 1)\omega_1 < s_1 \) (by monotonicity of \( \varphi'_\Lambda \)) we established that \( t - \varphi_\Lambda(t) \) is increasing there, so \( b_{-1} < b_e \) follows.

By symmetry of both \( \varphi_\Lambda \) and \( \varphi'_\Lambda \) about critical points (and using \( \varphi'_\Lambda \) is an odd function while \( \varphi_\Lambda \) is even),

\[
s_1 - (2j + 1)\omega_1 = (2j + 1)\omega_1 - s_{-1}, \quad \text{and} \quad \varphi_\Lambda(s_1) = \varphi_\Lambda(s_{-1})
\]

and therefore

\[
b_{-1} = s_{-1} - \varphi_\Lambda(s_{-1}) = (2j + 1)\lambda_1 - s_1 - \varphi_\Lambda(s_1) > (2j + 1) - (2j + 1) - \omega_1 = -\omega_1,
\]

since \( \varphi_\Lambda(s_1) < s_1 + \omega_1 \) and \( s_1 < (j + 1)\lambda_1 \), so the result is proved. \( \square \)
4.2.3. Higher order precritical and prepole parameters for $\omega_1$. Recall that an order 2 precritical parameter is $b$ such that $F_b^2(\omega_1)$ is a critical point, and an order 2 prepole has $F_b^2(\omega_1)$ a lattice point. Equivalently, $F_b(v_1)$ is a critical or lattice point respectively.

For the next result we shift our focus to a fundamental region in parameter space on the interval: $U_j = (p_j, p_{j+1}]$, chosen such that $e_1 \in I_j$ (so $p_j = j\lambda_1 - e_1$). We set $v_b = F_b(\omega_1)$.

**Proposition 4.19.** For any real rectangular lattice $\Lambda$, there exists some $T_0 > 0$ dependent on $\Lambda$, such that if $t > T_0$, there is a $b_t \in U_j$ such that $F_{b_t}(v_{b_t}) = t$.

Before giving the proof we mention an important consequence of this result.

**Theorem 4.20** (Order 2 precritical and prepole parameters). If $t = \omega_1 + \lambda > T_0 > 0$, $\lambda \in \Lambda$ real, then

$$\omega_1 \mapsto v_{b_t} \mapsto t \mapsto v_{b_t}$$

so $F_{b_t}(v_{b_t})$ lies in a super-attracting period 2 orbit, making $b_t$ an order 2 precritical parameter for $\omega_1$. Moreover if $t = \lambda > T_0$, $\lambda \in \Lambda$, then

$$\omega_1 \mapsto v_{b_t} \mapsto t = \lambda \mapsto \infty$$

so $b_t$ is an order 2 prepole parameter.

**Proof.** (of Proposition 4.19) We showed in Lemma 4.18 that $p_j < b_{-1}$, and we note that if $b \in (p_j, b_{-1})$ then $F_b(\omega_1) < F_{b_{-1}}(\omega_1)$; i.e., $v_b < v_{b_{-1}}$, and $v_b$ decreases in $b$ to $j\lambda_1$ as $b$ decreases to $p_j$ from the right.

Given $t$, if we find a point $c_t > 0$ such that

$$t + e_1 = \varphi_\Lambda(c_t) + c_t$$

the result follows, because we then set $b_t = c_t - e_1$, so that the orbit of $\omega_1$ under $F_{b_t}$ is:

$$\omega_1 \mapsto e_1 + b_t = v_{b_t} = e_1 - e_1 + c_t \mapsto \varphi_\Lambda(c_t) + c_t - e_1 = t$$

as claimed.

To show Equation (4.9) has a solution, we note that $\varphi_\Lambda'$ is monotone increasing on every interval $I_j$, $j \in \mathbb{Z}$, and $\varphi_\Lambda' + 1 < 0$ on $(j\lambda_1, s_{-1}) \subset I_j$; therefore $\varphi_\Lambda(t) + t$ is monotone decreasing from $\infty$ to $\varphi_\Lambda(s_{-1}) + s_{-1}$ on $(j\lambda_1, s_{-1})$. So as long as $t > T_0 = \varphi_\Lambda(s_{-1}) + s_{-1} - e_1$, there exists a unique $c_t \in (j\lambda_1, s_{-1})$ on which $\varphi_\Lambda(c_t) + c_t = t + e_1$, which is Equation (4.9), so the result is shown.

We next show that higher order prepole and precritical parameters accumulate on the prepole parameters $p_j$. We give the proof for order 3 precritical parameters.
Theorem 4.21 (Order 3 precritical parameters for $\omega_1$). Given any real
rectangular lattice $\Lambda$, there are infinitely many order 3 precritical
parameters that have $b_p := p_j \in (-\omega_1, \omega_1]$ as a limit point. The parameter $b_p$ is a two-
sided accumulation point for these precritical parameters.

Proof. Given a lattice, $\Lambda$, consider $e_1 > 0$, and $\omega_1$ determined by $\Lambda$. We
define the map:

$$S_2(b) = F_6^2(v_1) = F_{\omega}^3(\omega_1).$$

If $S_2(b) = (2j + 1)\omega_1$ for some integer $j$, then the parameter $b$ is order 3
precritical for $\omega_1$ (because $F_{\omega}^3(\omega_1)$ is a critical point).

On $C_\infty$, for $R > 0$, let $B_R(\infty) = \{z : |z| > R\}$. Take a (planar) ball
of the form $B_\epsilon(b_p) \subset C$, then $S_2 : B_\epsilon(b_p) \setminus \{b_p\} \to C_\infty$; we have that $S_2$ is
meromorphic for $\epsilon$ small.

Therefore there exists some large $R$ such that $B_R(\infty) \setminus \{x\} \subset S_2(B_\epsilon(b_p))$.
We choose any $\gamma_j := (2j + 1)\omega_1 \in B_R(\infty) \cap \mathbb{R}^+$. Then there exists some real
parameter $b \in B_\epsilon(b_p)$ mapping to $\gamma_j$.

By choosing a sequence $\epsilon_m = 2^{-m}$, starting with $m$ large enough, we
obtain the result.

Essentially the same proof shows that order 3 prepole parameters accumu-
late on $b_p$ as well, by choosing $\gamma_j = (2j)\omega_1 = j\lambda_1 \in B_R(\infty) \cap \mathbb{R}^+$.

4.2.4. Noncritical preperiodic parameters for $\omega_1$. Our standing as-
sumption is that $(g_2, g_3) \in \mathcal{K}$; we find the nonnegative integer $j$ such that
$e_1 \in I_j$. The parameter $b$ is noncritical preperiodic for $\omega_1$ means by defini-
tion that $\omega_1$ is not periodic, but it terminates in a cycle not containing a
critical point. A noncritical preperiodic parameter implies that $J(F_0) = C_\infty$
by Proposition 4.12. We now turn to the existence of these parameters.

Lemma 4.22. For any $q \in \mathbb{N}$, there is a branch of the multi-valued function
$\varphi_\Lambda^{-1}(q\lambda_1 + e_1)$, with a value $\eta_q$ such that the parameter

$$b_q^+ = \eta_q - e_1 \in (0, \omega_1].$$

Similarly there is a branch of $\varphi_\Lambda^{-1}(q\lambda_1 + e_1)$, with a value denoted by $\gamma_q$ such that

$$b_q^- = \gamma_q - e_1 \in (-\omega_1, 0].$$

Proof. We first note that since $q\lambda_1 + e_1 > e_1$ for any $q \in \mathbb{N}$, there will
be exactly 2 real values of $\varphi_\Lambda^{-1}(q\lambda_1 + e_1)$ in each periodic interval $I_j = [j\lambda_1, (j + 1)\lambda_1)$, since $\varphi_\Lambda|_R : I_j \to [e_1, \infty)$ is two-to-one except at the critical
point $j\lambda_1 + \omega_1$. Since $\varphi_\Lambda$ maps each interval $I_j^+ = [j\lambda_1 + \omega_1, (j + 1)\lambda_1)$
and $I_j^- = (j\lambda_1, j\lambda_1 + \omega_1]$ injectively onto $[e_1, \infty)$, there is exactly one value
$t_{j,q} \in I_j^+$ such that $\varphi_\Lambda(t_{j,q}) = q\lambda_1 + e_1$; also there is one value $s_{j,q} \in I_j^-$ such that
$\varphi_\Lambda(s_{j,q}) = q\lambda_1 + e_1$. 

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If \( j_0 \lambda_1 \leq e_1 < j_0 \lambda_1 + \omega_1 \), we choose \( \eta_q = s_{j_0,q} \) and \( \gamma_q = t_{(j_0-1),q} \). Then setting \( b_q^+ = \eta_q - e_1 \) gives the first result and \( b_q^- = \gamma_q - e_1 \) gives the second.

We obtain a similar result if \( j_0 \lambda_1 + \omega_1 \leq e_1 < (j_0 + 1) \lambda_1 + \omega_1 \). \( \square \)

**Proposition 4.23.** (Noncritical preperiodic parameters accumulate on \( b_p \).) Let \( \Lambda \) be a fixed real rectangular lattice, and suppose \( e_1 \in I_j \). Denote by \( b_p \in (-\omega_1, \omega_1) \) either \( p_j \) or \( p_{j+1} \). Then for any large enough integer \( q > j \), we can choose a branch of \( \varphi_\Lambda^{-1}(q\lambda_1 + e_1) \) with value \( \eta_q \in I_j \) such that the parameter

\[
b_q = \eta_q - e_1 \in (0, \omega_1]
\]

gives rise to the map \( F_{b_q} \) with a preperiodic critical point for \( \omega_1 \). The orbit of \( \omega_1 \) under \( F_{b_q} \) is:

\[
\omega_1 \mapsto v_1 \mapsto \zeta_q,
\]

with \( \zeta_q \) a repelling fixed point for \( F_{b_q} \). Moreover, \( b_p \) is a limit point for the \( b_q \)'s, as \( q \to \infty \).

**Proof.** We assume that \( e_1 \) satisfies Equation (4.4); the proof when Equation (4.5) holds is similar. The hypotheses imply that \( b_p = j \lambda_1 - e_1 \in (-\omega_1, 0] \), and \( b_c > b_p \) (if \( b_p < 0 \); if \( b_p = 0 \) replace \( b_c \) by \( b_c + \lambda_1 \) in what follows.) We can write \( P_j^+ \) for the restriction of \( \varphi_\Lambda^{-1} \) to \( I_j^+ \), and \( P_j^- \) for the restriction of \( \varphi_\Lambda^{-1} \) to \( I_j^- \); we then choose the inverse \( \gamma_q = P_j^-(q\lambda_1 + e_1) \) that yields \( b_q^- \) from Lemma 4.22. Then \( b_c > b_q^- > b_p \), and

\[
|b_q^- - b_p| = |\gamma_q - j \lambda_1| \searrow 0
\]
as \( q \to \infty \). This follows since \( \varphi_\Lambda \) decreases monotonically from \( \infty \) to \( e_1 \) on \( (j \lambda_1, j \lambda_1 + \omega_1] \), so \( b_p \) is an accumulation point since \( b_q^- \searrow b_p \).

\[
F_{b_q^-}(\omega_1) = \varphi_\Lambda(\omega_1) + P_j^-(q\lambda_1 + e_1) - e_1 = P_j^-(q\lambda_1 + e_1),
\]

and

\[
F_{b_q^-}(P_j^-(q\lambda_1 + e_1)) = \varphi_\Lambda(P_j^-(q\lambda_1 + e_1)) + P_j^-(q\lambda_1 + e_1) - e_1 = q\lambda_1 + P_j^-(q\lambda_1 + e_1),
\]

and since \( q\lambda_1 \) is a lattice point, by periodicity we have

\[
F_{b_q^-}(q\lambda_1 + P_j^-(q\lambda_1 + e_1)) = q\lambda_1 + P_j^-(q\lambda_1 + e_1).
\]

Therefore the point \( \zeta_q = q\lambda_1 + P_j^-(q\lambda_1 + e_1) \) is a repelling fixed point since \( F_{b_q^-}' \) decreases to \( -\infty \) as \( t \searrow j \lambda_1 \), and \( \zeta_q \) gets closer to lattice points of the form \( (j + q)\lambda_1 \) as \( q \) increases. \( \square \)
4.3. Examples. We illustrate some of the preceding results with examples. For the first several examples we use the center square lattice with \( g_2 = (2\pi)^{4/3} \approx 5.7395 \), with \( \omega_1 = e_1 \) and \( \kappa = \Gamma(1/4)^2/(4\sqrt{\pi}) \). For \( F_b(z) = \varphi_A(z) + b \), we have a center parameter at \( b_c = 0 \); also \( b_p = \omega_1 \) since there is an order 1 prepole parameter at each endpoint of the interval \(( -\omega_1, \omega_1 ] \). For the map \( F_{b_c} = \varphi_A \), two critical points terminate at the same super-attracting fixed point while \( \omega_3 \) is a prepole. We use the notation for branches of inverses of \( \varphi_A \): \( P^+_j \) and \( P^-_j \), from Proposition 4.23.

1. A square lattice with all critical points terminating in repelling fixed points.
   Using \( b_M = P^+_1(\lambda_1) \approx -0.6642 \), we have the critical orbit:
   \[
   v_1 = \omega_1 + P^+_1(\lambda_1) \mapsto 3\omega_1 + P^+_1(\lambda_1) = p \mapsto p \approx 2.9294, 
   \]
   (using Theorem 2.8); \( p \) is a repelling fixed point and \( b_M \in (-\omega_1, b_{-1}) \).
   We know that \( F_{b_M}(v_2) = F_{b_M}(v_1) \) so \( \omega_2 \) terminates in a repelling orbit.
   More surprising is that \( \omega_3 \) is also preperiodic. In this example we have:
   \[
   v_3 = b_M \mapsto P^+_0(\lambda_1) \approx 1.7315, 
   \]
   a repelling fixed point.
   If a parameter has all critical points terminating in repelling cycles, we call it a Misiurewicz parameter. Even for a square lattice, in general one cannot expect \( \omega_3 \) to terminate in a repelling cycle when \( \omega_1 \) and \( \omega_2 \) do.

2. Using Theorem 4.20 and choosing \( t = 3\omega_1 \) we obtain an order 1 precritical parameter \( b_s \approx -0.7123 \) such that
   \[
   \omega_1 \mapsto e_1 + b_s = v_1 \mapsto 3\omega_1. 
   \]
   In parameter plane, \( b_s \) lies between \( b_p - 2\omega_1 \) and \( b_{-1} \) and is a center parameter for \( 3\omega_1 \). Thus \( b_s < 0 \) and \( 0 < v_1 < \omega_1 \), but \( F_{b_s}(v_1) = \varphi_A(v_1) + b_s = 3\omega_1 \).

3. \( J(F_b) \) is a Cantor set for a square lattice for some \( b \in \mathbb{R} \). We use the values \( (g_2, g_3) = (1, 0) \) and we set \( b = \omega_1 - e_1 = \kappa - 1/2 \), so there is a super-attracting fixed point at \( \omega_1 \). Using the approximations from Lemma 2.7, we have that \( 1/2 = e_1 < e_2 + b < 1 < b < e_1 + b = \omega_1 \). We know that there is an attracting basin for the fixed point at \( \omega_1 = \kappa \approx 1.854 \), and numerical estimates show that its immediate basin of attraction contains all the critical values. In particular, it is enough to show it contains \( \omega_2 = \omega_1 - 1 \approx .854 \), which is equivalent to showing that
   \[
   |F_b(\omega_1 - 1) - \omega_1| = |\varphi_A(\omega_1 - 1) - 1/2| < 1. 
   \]
   This can be shown using Theorem 2.8. Once we know that all critical values are in the immediate attracting basin of an attracting fixed point, \( J(F_b) \) is a Cantor set by [12], as shown on the left in Figure 5. When \( b = 0 \) it is known that \( J(F_0) \) is connected [4].
5. Dynamical properties of $F_b$ with $b$ on the half lattice line

In this section we show that for parameters $b$ lying on the half lattice line $L$, the dynamics vary from those on $\mathbb{R}$ as the parameter moves along $L$. We continue to assume that $\Lambda = [\lambda_1, \lambda_2]$, with $\lambda_1 > 0$ and $\lambda_2$ purely imaginary. We consider parameters from the principal horizontal half period line defined in Equation (3.3): $L = \{b \in \mathbb{C} : b = t + \omega_2, \ t \in \mathbb{R}\}$. The line $L$ contains all critical points of the form $\omega_2 + n\lambda_1$ and $\omega_3 + m\lambda_1$, $m, n \in \mathbb{Z}$.

**Lemma 5.1.** For any real rectangular lattice, and any parameter $b \in L$, the function $F_b$ maps $L$ into $L$.

**Proof.** Set $b = a + \omega_2$ for some $a \in \mathbb{R}$. Since $F_b(t + \omega_2) = \varphi_\Lambda(t + \omega_2) + a + \omega_2$, it is enough to show that $\varphi_\Lambda(t + \omega_2)$ is real for any $t \in \mathbb{R}$. This follows from Theorem 2.8 and the assumption that $\Lambda$ is real. \hfill \Box

**Lemma 5.2.** For any parameter $b \in L$, $F_b$ maps $\mathbb{R}$ into $L$ and the line $V = \{\omega_1 + iy : y \in \mathbb{R}\}$ and $-V$ into $L$.

**Proof.** $\varphi_\Lambda$ takes $\mathbb{R}$ and $L$ to $\mathbb{R}$, and $\varphi_\Lambda$ maps $V$ and $-V$ into $\mathbb{R}$ [8], so $F_b$ maps $\mathbb{R}, V,$ and $-V$ into $L$ when $b \in L$. \hfill \Box

**Remark 5.3.** 1. When $\Lambda$ is real square, it follows from Proposition 4.2(2), that:

\begin{equation}
\varphi_\Lambda(t + \omega_2) = e_2 \left( \frac{\varphi_\Lambda(t) + e_2}{\varphi_\Lambda(t) - e_2} \right), \ t \in \mathbb{R}.
\end{equation}

2. From Lemma 5.1, for $b \in L$, the map $F_b$ can be decomposed into its real and imaginary parts, with the imaginary part the constant value $\omega_2$: writing $b = (a, \omega_2)$ and $z = (t, \omega_2)$ we have

$F_b(z) = (\ell_a(t), \omega_2),$

where

$\ell_a(t) = \varphi_\Lambda(t + \omega_2) + a, \ t \in \mathbb{R}.$
Since the postcritical set determines the dynamics of $F_b$, the usefulness of looking at $\ell_a$ is shown in the next two lemmas. Several graphs of $\ell_a$ for different values of $a$ are shown in Figure 3.

**Lemma 5.4.** Given $F_b$ as above, with $b = a + \omega_2$ and $a \in \mathbb{R}$, $\text{cl}(\mathcal{P}(F_b)) \subset L$.

**Proof.** The points in the postcritical set coming from $\omega_2$ and $\omega_3$ clearly remain on $L$ under iteration. Moreover $\varphi_{\Lambda}$ maps $V, -V$, and $\mathbb{R}$ into $L$ by Lemma 5.2; since $\omega_1$ lies on $V \cap \mathbb{R}$, the result follows. □

**Lemma 5.5.** Given $F_b$ as above, $b = a + \omega_2$, and $a \in \mathbb{R}$, for $z_0 = t_o + \omega_2$, $t_o \in \mathbb{R}$, we have that $F_b(z_0) = z_o$ if and only if $\ell_a(t_o) = t_o$. Moreover, $\ell_a$ is periodic on $\mathbb{R}$ of period $\lambda_1$.

**Proof.** We have $\ell_a(t_o) = t_o = \varphi_{\Lambda}(t_o + \omega_2) + a$ if and only if $t_o + \omega_2 = \varphi_{\Lambda}(t_o + \omega_2) + a + \omega_2 = F_b(t_o + \omega_2)$ if and only if $F_b(t_o + \omega_2) = t_o + \omega_2$. Since $\varphi_{\Lambda}(t + \lambda_1) = \varphi_{\Lambda}(t)$, the second statement follows. □

### 5.1. Properties of the auxiliary map $\ell_a$.

Based on the discussion above, we shift our focus to the real numbers to study the dynamics when $b \in L$. We note that for $a = 0$, $\ell_0$ is just the map $\varphi_{\Lambda}|_L$, and by periodicity, restricting the map to a fundamental region,

$$\ell_0 : (-\omega_1, \omega_1] \to [e_2, e_3],$$

since $\ell_0(-\omega_1) = \varphi_{\Lambda}(-\omega_1 + \omega_2) = \ell_0(\omega_1) = e_3$. Since $\Lambda$ is real rectangular, $e_2 < 0$ and $e_3 > e_2$ so the maximum value occurs at the two endpoints of the interval. There is a critical point of $\ell_0$ at 0 which is a minimum, since $\ell_0(0) = \varphi_{\Lambda}(\omega_2) = e_2 < 0$. The maximum value $e_3$ will be positive, negative or zero depending on $g_3$ being negative, positive, or 0 respectively.

For $\ell_a$, with $a \in \mathbb{R}$, the maxima, minima, and critical points occur at the same points in the interval, independent of $a$. We assume $a \in [-\omega_1, \omega_1]$, and set $I_a = [e_2 + a, e_3 + a]$, so $\ell_a : \mathbb{R} \to I_a$ or by periodicity, we can write: $\ell_a : [-\omega_1, \omega_1] \to I_a$.

The range of values for the derivative of $\ell_a$ can be easily computed. The map $\ell'_a$ can be written as $\ell'(t)$ since it does not depend on $a$.

**Proposition 5.6.** For $\Lambda$ real rectangular, $b \in L$, and $F_b$, $\ell_a$ as above, the function $\ell'(t) = \varphi'_{\Lambda}(t + \omega_2)$ is a real analytic, periodic, and odd function, which maps onto the interval $[-\sqrt{\eta}, \sqrt{\eta}]$ with $\eta = -g_3 + (g_2/3)^{3/2} > 0$.

**Proof.** This follows from classical identities in ([8], Chapter 2.23). □

The next result follows from Theorem 2.8 and is a generalization of Proposition 4.2 (2) to real rectangular lattices.

**Proposition 5.7.** For $\Lambda$ a real rectangular lattice, for all $b \in \mathbb{C}$, and $z \in L$, writing $z = t + \omega_2$, we have $F_b(z) = M \circ \varphi_{\Lambda}(t) + b$, where $M$ is the M"{o}bius
transformation defined by:

\[ M(z) = e_2 \left( \frac{z + e_2 + \frac{g_3}{4e_2}}{z - e_2} \right). \]

Moreover for \( a \in \mathbb{R} \), we have \( \ell_a(t) = M \circ \varphi_A(t) + a \).

**Proof.** The proofs of the two parts are almost identical so we prove the second statement. We use Theorem 2.8 and rewrite the numerator using the identities given in (2.4) to see that:

\[ \ell_a(t) = \frac{2e_2^2 + \frac{g_3}{4e_2^2}}{\varphi_A(t) - e_2} + e_2 + a = e_2 \left( \frac{\varphi_A(t) + e_2 + \frac{g_3}{4e_2^2}}{\varphi_A(t) - e_2} \right) + a = M \circ \varphi_A(t) + a, \tag{5.2} \]

\[ \square \]

The map \( M \) preserves the real line, interchanges the upper and lower half planes and permutes \( e_2 \) with \( \omega_2 \) and \( 0 \) with \( \frac{g_3}{4e_2^2} \).

It is of interest to determine when we obtain attracting cycles for \( F_b \). We have transformed the question into one for maps on the real line \((\ell_a)\), so we can use the Schwarzian derivative. Using Proposition 4.4 we prove the following result.

**Proposition 5.8.** If \( \Lambda \) is any real rectangular lattice, then for any \( b \in \mathbb{C} \), for all \( z \in L \), \( SF_b(z) < 0 \). Equivalently \( SL_a(t) < 0 \) for all \( t \in \mathbb{R} \).

**Proof.** Since \( e_1 > 0 \) for real rectangular lattices \( \Lambda \), for all real \( t \), \( \varphi_A(t) > 0 \), so we have that \( S \varphi_A(t) = -6 \varphi_A(2t) < 0 \). Then for \( z \in L \), writing \( z = t + \omega_2 \), we have that \( SF_b(z) = S(M \circ \varphi_A)(t) = S \varphi_A(t) = -6 \varphi_A(2t) < 0 \) by Proposition 4.4. \( \square \)

Using Proposition 5.8, we can apply the result from ([9], Theorem 4.1) to obtain the following result. Write \([z]\) for the coset \( z + \Lambda \).

**Theorem 5.9.** If \( \Lambda \) is a real rectangular lattice, and \( b = a + \omega_2 \), \( a \in \mathbb{R} \), then:

1. The immediate basin of an attracting periodic orbit of \( F_b \) on \( L \) contains an element of either \([\omega_2] \cap L \) or \([\omega_3] \cap L \) (or both).
2. If \( z_0 = t_0 + \omega_2 \), \((t_0 \text{ real})\) is in a rationally neutral \( p \)-cycle for \( F_b \) then it is topologically attracting in the sense that there is an open interval in \( L \) that is attracted to \( z_0 \), and a critical point in its immediate attracting basin that contains an element of either \([\omega_2] \cap L \) or \([\omega_3] \cap L \) (or both).
5.2. Center and parabolic parameters on $L$. In Definition 4.11 we defined order $k$ prepole and (period $m$) center parameters in terms of $\omega_1$; we carry those definitions over verbatim for $\omega_2$ and $\omega_3$. We first show that there are no prepole parameters along $L$.

Proposition 5.10. For $\Lambda$ real rectangular, there are no prepole parameters $b \in L$.

Proof. We show that if $b \in L$, then none of $\omega_1, \omega_2$, or $\omega_3$ land on a pole under $F_b$. For any $z \in \mathbb{R} \cup L$, $F_b(z) = u + a + \omega_2$, with $u = \wp_\Lambda(z) \in \mathbb{R}$, and $a \in \mathbb{R}$. Since $\omega_2$ is purely imaginary, $F_b(z) \in L$, which contains no poles. Therefore $F_b^n(z) \in L$ as well for each $n$, and hence $F_b^n(z) \notin \Lambda$. Since $\omega_1 \in \mathbb{R}, \omega_2, \omega_3 \in L$, there are no prepole parameters.

Many bifurcations in $b$-space depend on the lattice $\Lambda$, but the next result shows there exist center parameters $b \in L$ for any lattice $\Lambda$.

Theorem 5.11 (Fixed center parameters for $\omega_2$ and $\omega_3$). For any real rectangular lattice $\Lambda$ the parameters on $L$ given by $b_j = \omega_j - \varepsilon_j$ each give $F_{b_j}$ with a super-attracting fixed point at $\omega_j$, $j = 2, 3$.

Proof. The point $\omega_1$ is a super-attracting fixed point for $\ell_{(\omega_1-\varepsilon_3)}$, and 0 is a fixed critical point for $\ell_{-\varepsilon_2}$. The result follows from Lemma 5.5.

The connectivity of the resulting Julia sets in Theorem 5.11 depends on the lattice; we develop this idea further below.

5.2.1. Parabolic parameters on $L$. Whenever $\eta \geq 1$ from Proposition 5.6, we obtain parabolic behavior for some parameters on $L$.

Proposition 5.12 (Existence of parabolic parameters). If $(g_2, g_3) \in \mathcal{R}$ and $\Delta(g_2, g_3 + 1) > 0$, in the set $(-\omega_1, \omega_1] + \omega_2 \subset L$, there exist parameters $b_j \in L$, $j = 1, 2$ for which $F_{\Lambda, b_j}$ has a parabolic fixed point in each periodic interval on $L$ with multiplier $(-1)^j$. In particular in the interval $(-\omega_1, \omega_1]$, there are exactly 2 parameters $a_{1}^+, a_{2}^+$ for which $F_b$, $b = a_{1}^+ + \omega_2$, $j = 1, 2$ has a fixed point with derivative 1, and two parameters $a_{1}^-, a_{2}^-$ for which $F_b$, $b = a_{2}^- + \omega_2$, $j = 1, 2$ has a fixed point with derivative $-1$.

Proof. If $(g_2, g_3) \in \mathcal{R}$ then $\Delta(g_2, g_3) > 0$. If in addition $\Delta(g_2, g_3 + 1) > 0$, then by Proposition 5.6 we have $-g_3 + (g_2/3)^{3/2} = \eta > 1$. By periodicity on each line segment of $L$ of length $2\omega_1$, the maximum value of $\ell'$ is $\sqrt{\eta} > 1$, and the minimum value is $-\sqrt{\eta} < -1$. By the Intermediate Value Theorem there exists a point $t_0 \in (0, \omega_1)$ such that $\ell'(t_0) = 1 < \sqrt{\eta}$; since $\ell'$ is an odd function, we have $\ell'(-t_0) = -1$. Therefore setting $a_1 = t_0 - \ell_0(t_0)$, we have $\ell_{a_1}(t_0) = t_0$ with multiplier 1. Similarly $a_{-1} = -t_0 - \ell_0(-t_0) = -t_0 - \ell_0(t_0)$, we have $\ell_{a_{-1}}(-t_0) = -t_0$ with multiplier $-1$.

Since $\ell' = 0$ at the endpoints and midpoint of $(-\omega_1, \omega_1]$, the Intermediate Value Theorem guarantees the existence of two points $t_1^+ < t_2^+$ in $[0, \omega_1]$ such that $\ell' = 1$, and two points $t_1^- < t_2^-$ in $(-\omega_1, 0]$ such that $\ell' = -1$. Each of
these four points (or a translation by \( \pm \lambda_1 \)) becomes a parabolic fixed point of \( \ell_a \) for the appropriate choice of parameter \( a_j^\pm \in (-\omega_1, \omega_1) \). Specifically, we choose \( a_j^\pm = (t_j^+ + k\lambda_1) - \phi_A(t_j^+) \) for some \( k \in \mathbb{Z} \) such that \( a_j^\pm \in (-\omega_1, \omega_1) \). This is possible since there is a representative of \( (t_j^+ - \phi_A(t_j^+)) + \Lambda \) in every interval of \( L \) of length \( 2\omega_1 \).

**Remark 5.13.** When \( \triangle(g_2, g_3 + 1) = 0 \), every fundamental interval along \( L \) contains exactly two parabolic parameters, each corresponding to a map with fixed point; one with multiplier 1 and the other with multiplier \(-1\). Unlike the case when \( b \) is real, we next show that for some real rectangular lattices no parabolic parameters exist.

### 6. Maps \( F_b \) with Cantor Julia set

We recall that for any square lattice, \( J(\phi_L) \) is connected [11]. In addition for any example of a lattice \( \Lambda \) for which the connectivity of \( J(\phi_L) \) is known, it is connected. In this section we show that adding a constant changes the connectivity. As always we consider \( \Lambda \) to be a real rectangular lattice. The next result shows that for some lattices \( \Lambda \), every \( b \in L \) yields a map with a Cantor Julia set. Since we write parameters \( b \in L \) as \( b = a + \omega_2 \), a real, we denote a line segment along \( L \) by \([\alpha_1, \alpha_2] + \omega_2 \) where \( \alpha_1, \alpha_2 \in \mathbb{R} \).

**Theorem 6.1.** Let \( \Lambda = \Lambda(g_2, g_3) \) be any real rectangular lattice and suppose that \( (g_2, g_3) \in \mathcal{R} \) also satisfies: \( \triangle(g_2, g_3 + 1) < 0 \). Then for any \( b \in L \), \( J(F_b) \) is a Cantor set.

**Proof.** We write \( b = a + \omega_2 \). The conditions on the pair \((g_2, g_3)\) imply that \( |t_a'\cdot(t)| < 1 \) for all \( t \in \mathbb{R} \) by Proposition 5.6. Since \( \ell_a : \mathbb{R} \to [e_2 + a, e_3 + a] \), let \( p = \max\{|e_3 + a, e_2 + a + 2\omega_1|, I = [e_2 + a, p] \}, \) and consider \( \ell_a : I \to I \). By the Contraction Mapping Theorem, there exists a unique fixed point \( t_0 \in I \), and all points in \( I \) converge under iteration to \( t_0 \). Since \( I \) contains a fundamental period of \( \ell_a \), then all points \( t \in L \), and therefore all points in \( \pm V \) are attracted to the fixed point at \( t_0 \). The Fatou set is open, so by ([12], Corollary 3.11 and Theorem 3.12), we have a double toral band, and a Cantor Julia set (see Definition 6.9 below).

The region in \( \mathbb{R} \) where \((g_2, g_3)\) satisfies the hypotheses of Theorem 6.1 is shown in yellow on the left in Figure 4. A typical Julia set obtained by choosing \((g_2, g_3)\) satisfying the hypotheses of Theorem 6.1, with a generic value of \( b \) on \( L \) is shown in Figure 4. We can generalize some of these results to arbitrary lattices.

**Theorem 6.2.** If \( \Lambda \) is a real rectangular lattice and \( F_b \), \( b \in L \) has an attracting fixed point whose basin of attraction contains \([0, \omega_1] + \omega_2 \), then \( J(F_b) \) is a Cantor set.

**Proof.** Let \( z_0 \) satisfy \( F_b(z_0) = z_0 \) with \( |F_b'(z_0)| < 1 \). Then \( z_0 \in L \) by Lemma 5.4. Let \( \mathcal{A} \) denote the immediate attracting basin of \( z_0 \), and set \( \mathcal{A}_L = \mathcal{A} \cap L \).
By hypothesis, symmetry about 0, and periodicity, we have that $\mathcal{A}_F = L$, since the entire interval $[-\omega_1, \omega_1] + \omega_2 \subset F(F_b)$. By the proof of Lemma 5.4 we have that $\ell_a(\pm V) \subset \mathcal{A}$, if $b = a + \omega_2$, hence $\pm V \subset \mathcal{A}$ as well. The Fatou set is open, so by ([12], Corollary 3.11 and Theorem 3.12) we have a double toral band and a Cantor Julia set.

We generalize the conditions for $J(F_b)$ to be a Cantor set once more.

**Proposition 6.3.** Let $\Lambda$ be any real rectangular lattice. For any fixed $t_0 \in \mathbb{R}$, choosing $a = t_0 - \varphi_\Lambda(t_0 + \omega_2) \in \mathbb{R}$ gives $t_0$ as a fixed point of $\ell_a(t)$. When $t_0$ is attracting for $\ell_a$, setting $p = \max\{e_3 + a, e_2 + a + 2\omega_1\}$, and letting $U = [e_2 + a, p]$, if $\ell_a$ contains $U$ in its attracting basin, then for $b = a + \omega_2$, $J(F_b)$ is a Cantor set.

**Proof.** We have that $\ell_a(t_0) = \varphi_\Lambda(t_0 + \omega_2) + a = \varphi_\Lambda(t_0 + \omega_2) - \varphi_\Lambda(t_0 + \omega_2) + t_0 = t_0$. The derivative at the fixed point is exactly $\psi_\Lambda'(t_0 + \omega_2) \in \mathbb{R}$; assume it is attracting. Set $p = \max\{e_3 + a, e_2 + a + 2\omega_1\}$, and let $U = [e_2 + a, p]$; consider $\ell_a : U \rightarrow U$. If $U$ is in the basin of attraction of $t_0$, then all points $t \in L$ are as well. As above, it follows that all points in $\pm V$ are attracted to the fixed point at $t_0$, and $J(F_b)$ is a Cantor set.
We now turn to the existence of maps $F_b$ which satisfy the hypotheses of Theorem 6.2 but not those of Theorem 6.1.

**Example 6.4.** We start with a rectangular lattice $\Lambda$ determined by the invariants $(g_2, g_3) = (5, 1)$, so that $\Delta(g_2, g_3) = 98 > 0$ and $\Delta(g_2, g_3 + 1) = 17 > 0$; so Theorem 6.1 does not apply. One can check that $e_2 = -1$ for this lattice, and since $g_3 > 0$, we have that $e_3 < 0$. By choosing $a = 1$, Theorem 5.11 implies that 0 is a super-attracting fixed point for $\ell_a$.

By symmetry of the map $\ell_a$ about critical points, it is enough to check that on $[0, \omega_1]$, applying (5.2),

$$\ell_a(t) = \frac{2e_2^2 + \varphi(t)}{\varphi(1) - e_2} + e_2 + a = \frac{2 - 1/4}{\varphi(1) + 1} < t.$$  

This will imply that all points in $\mathcal{L}$ iterate to the fixed point at 0. Even though the maximum value of $\ell'_a > 1$, this is straightforward to check. We have:

$$\ell_a(t) = \frac{7}{4} \left(\varphi(t) + 1\right)^{-1} < \frac{7}{4} \left(1 + \frac{1}{t^2} + \frac{t^2}{4}\right)^{-1},$$

using the Laurent series expansion of $\varphi$ about 0, and truncating it after the first two terms (since $g_2, g_3 > 0$ this provides a lower bound for $\varphi(t)$ and an upper bound for $\ell_a(t)$).

Since

$$\left(1 + \frac{1}{t^2} + \frac{t^2}{4}\right)^{-1} = \left(\frac{4t^2 + 4 + t^4}{4t^2 + 4 + t^4}\right)^{-1} = \frac{4t^2}{4t^2 + 4 + t^4},$$

it suffices to show $7t^2 < t(t^2 + 2)^2$, for $t > 0$ and this can easily be shown.

**Remark 6.5.** For a real square lattice and $b$ of the form $b = a + \omega_2$, $a \in \mathbb{R}$, if we choose $a = e_1$, $\ell_a$, can be written as:

$$(6.1) \quad \ell_a(t) = \frac{2e_2^2}{e_1 + \varphi(t)}.$$  

Then a sufficient condition for $J(F_b)$ to be a Cantor set for a square lattice is given in Theorem 6.7.

**Proposition 6.6.** For a real rectangular square, the point 0 is a super-attracting fixed point of $\ell_{e_1}(t)$. When 0 is the only fixed point for $\ell_{e_1}$ on the interval $[0, \omega_1]$, then $J(F_{e_1 + \omega_2})$ is Cantor. It is necessary that $e_1 < (\gamma/2)^{2/3}$ for the condition to be satisfied.

**Proof.** By Theorem 5.11, $a = -e_2 = e_1$ will yield a super-attracting fixed point at 0. If 0 is the only fixed point of $\ell_a$, then we have $\ell_a(t) < t$ near 0, and therefore for all $t \in (0, \omega_1]$. Clearly if there is some $t$ such that $\ell_a(t) = t$ we have a fixed point, and if $\ell_a(t) > t$ for some $t \in (0, \omega_2)$, then by the Intermediate Value Theorem we have a fixed point in between 0 and $t$. If $\ell_a(t) < t$ on $(0, \omega_1]$, then $\ell_a(\omega_1) < \omega_1$. By Table 1, this means $e_1 < \frac{2}{\sqrt{61}}$, or $e_1 < (\gamma/2)^{2/3} \approx 1.19787.$
Figure 5. Cantor Julia sets, on the left using $b \in \mathbb{R}$ and $(g_2, g_3) = (1, 0)$, and on the right using $b \in L$ and $(g_2, g_3) = (3.24, 0)$. The attracting fixed point is marked in white and the period lattice is shown in green in both.

Since $\ell_a(t) < t$ on $[0, \omega_1]$, then $\ell_a^n(t) < \ell_a^{n-1}(t)$ for all $n \in \mathbb{N}$, and therefore the decreasing sequence $\{\ell_a^n(t)\}$ must converge to a fixed point which has to be $0$. Then all $t \in L$ and $V$ are attracted to $0$ and $J(F_b)$ is a Cantor set.

Using the idea for the proof above we obtain a continuum of examples with Cantor Julia set; the right picture of Figure 5 shows $J(F_b)$ from Theorem 6.7, using $e_1 = .9$.

**Theorem 6.7.** If $\varphi_\Lambda$ has a real square period lattice $\Lambda$ and $e_1 \leq 1$, then setting $b = e_1 + \omega_2$ we have that $J(F_b)$ is a Cantor set.

**Proof.** We show that Theorem 6.2 can be applied to yield the result. By Equation (6.1), $\ell_a(t) = \frac{2e_1^2}{e_1 + \varphi_\Lambda(t)}$ and has $0$ as a super-attracting fixed point. Since $\ell_a(t) < t$ for small $t > 0$, it suffices to show that $\ell_a(t) < t$ for all $t > 0$. As above we approximate $\varphi_\Lambda$ from below using its Laurent series expansion, all of whose nonzero coefficients are positive if $g_2 > 0$, and we have from Table 1, that $g_2 = 4e_1^2$. Therefore $\varphi_\Lambda(t) > \frac{1}{12} + \frac{e_1^2 t^2}{5}$, so it is enough to show that

$$10e_1 t^2 < \frac{10e_1 t^2 + 5e_1 t^2 + 5}{e_1^2 t^4 + 5e_1 t^2 + 5}.$$

Reducing Equation (6.2) to showing the quadratic polynomial $r(t) = 5e_1 t^2 - 10e_1 t + 5$ is positive as above, this holds when $e_1 \leq 1$ for all $t > 0$. □

This next result gives rise to a large number of examples, not satisfying the hypotheses of any of the previous results, of maps $F_b$ on square lattices with Cantor Julia sets.

**Proposition 6.8.** Assume $\Lambda$ is the center square lattice, and set $A = (-\omega_1, 0)$. Suppose $a \in A$ is such that there exists an attracting fixed point
to show there is an attracting fixed point $p$, and $t_0 \in A$. Then $t_0$ is the only fixed point in $A$; if $t_0$ is the only fixed point for $\ell_a$ on $I_a = [-\omega_1 + a, a]$, then $J(F_{a+\omega_2})$ is a Cantor set.

**Proof.** Since $t_0 \in A$, the interval $B = A \cap I_a = (-\omega_1, a) \neq \emptyset$. Since $\psi_A'(z) < 0$ on $A + \omega_2$, we have $\ell'(t_0) < 0$. Therefore $\ell_a$ is strictly decreasing on $A$ and therefore $t_0$ is the only fixed point of $\ell_a$ in $A$. By Theorem 4.6 there must be a critical point in the immediate basin of attraction of $t_0$, so since $-\omega_1$ is the only critical point in the domain and range of $\ell_a$, it iterates to $t_0$. But $\ell_a(-\omega_1) = a$, so $a$ is also in the immediate attracting basin and therefore the entire interval $B$ is.

For all $t \in I_a \setminus A$, we have $t \in (-2\omega_1, -\omega_1)$ and $\ell'(t) > 0$, so the sequence $\{\ell^n_a(t)\}$ increases until $\ell^n_a(t) \in B$ for some $n_o$. For $n > n_o$, $\{\ell^n_a(t)\}$ is attracted to $t_0$. Therefore the entire interval $I_a$ is in the attracting basin of $t_0$ and by Proposition 6.3, $J(F_b)$ is a Cantor set.

**Toral band Fatou components.** A fundamental region for an elliptic function can be identified with the torus $\mathbb{C}/\Lambda$; we consider Fatou components on a torus, and have the following definition from ([11], Definition 5.1 and Proposition 5.2).

**Definition 6.9.** A Fatou component $A_0$ of the map $F_b$ is a toral band if $A_0$ contains an open subset $U$ which is simply connected in $\mathbb{C}$, but $U$ projects to a topological band around the torus $\mathbb{C}/\Lambda$ containing a homotopically nontrivial curve. We say $A_0$ is a double toral band if $U \subset A_0$ contains a simple closed loop which forms the boundary of a fundamental region for $\Lambda$.

It is clear that when $J(F_b)$ is a Cantor set we have a double toral band but other types of toral bands can occur. We refer to a Fatou component $A_0$ as a single toral band if it is a toral band but not a double toral band.

**Proposition 6.10.** [11] For any lattice $\Lambda$, an elliptic function $f_\Lambda$ has a toral band if and only if there is a component of the Fatou set which is not completely contained in the interior of one fundamental region $Q$.

**Example 6.11.** We show the existence of a map $F_b$ with a toral band but such that $J(F_b)$ is not a Cantor set using numerical estimates from Lemma 2.7. For this example, $(g_2, g_3) = (7, -3)$. The critical values for the associated map $\varphi_\Lambda$ are $(e_1, e_2, e_3) = (1, -1.5, 5)$, by Proposition 2.5.

We choose $b \in L$ given by: $b = \omega_2 - e_2 = \omega_2 + 1.5$ so that we have a super-attracting fixed point at $\omega_2$. We have a second attracting fixed point on $L$; this is the fixed point contained in a toral band. If we denote by $(\eta_1, \eta_2, \eta_3)$, the real parts of $(v_1, v_2, v_3)$ which all lie on $L$, we have: $\eta_2 = 0 < \eta_2 = 2 < \eta_3 = 2.5$, and calculating the first few terms in the AGM sequence for $\lambda_1$, we see that $\eta_3 < \lambda_1$, and it then follows that $\omega_1 < \eta_2 < \eta_3 < \lambda_1$, and the function $\ell_a$ defined earlier, using $a = 1.5$, is concave down on the interval $(\omega_1, \lambda_1)$ and maps $L$ periodically onto $[0, 2]$. From here it is not too difficult to show there is an attracting fixed point $p = \alpha + \omega_2$, with $\omega_1 < \alpha < \eta_2$, and
with both \(v_2\) and \(v_3\) in its attracting basin. This gives us the existence of a toral band; the additional attracting fixed point at \(\omega_2\) implies that \(J(F_b)\) is not a Cantor set. The parameter space for the example is shown on the right in Figure 2.

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