Compact intertwining relations for composition operators on $H^\infty$ and the Bloch space

Ce-Zhong Tong, Cheng Yuan and Ze-Hua Zhou

Abstract. We study how certain Volterra type operators compactly intertwine with composition operators on the space of bounded analytic functions and composition operators on the Bloch space of the unit disk.

1. Introduction

If $X$ and $Y$ are two Banach spaces, the symbol $\mathcal{B}(X,Y)$ denotes the collection of all bounded linear operators from $X$ to $Y$. Let $\mathcal{K}(X,Y)$ be the collection of all compact elements of $\mathcal{B}(X,Y)$, and let $\mathcal{Q}(X,Y)$ be the quotient set $\mathcal{B}(X,Y)/\mathcal{K}(X,Y)$.

For linear operators $A \in \mathcal{B}(X,X)$, $B \in \mathcal{B}(Y,Y)$ and $T \in \mathcal{B}(X,Y)$, the phrase “$T$ intertwines $A$ and $B$ in $\mathcal{Q}(X,Y)$” (or “$T$ intertwines $A$ and $B$ compactly”) means that

$$TA = BT \mod \mathcal{K}(X,Y) \text{ with } T \neq 0. \quad (1.1)$$

The notation $A \propto_K B \ (T)$ represents the relation in equation $(1.1)$. In fact, if $T$ is an invertible operator on $X$, then the relation $\propto_K$ is symmetric.
Recall that the essential norm of a bounded linear operator $T$ is the distance from $T$ to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}.$$ 

Notice that $\|T\|_e = 0$ if and only if $T$ is compact. So estimates on $\|T\|_e$ lead to conditions for $T$ to be compact.

Let $\mathbb{D}$ be the unit disk in the complex plane. Denote by $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the collection of all the holomorphic self-mappings of $\mathbb{D}$. Every $\varphi \in S(\mathbb{D})$ induces a composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$.

Let $g \in H(\mathbb{D})$. The Volterra operator $J_g$ is defined by

$$J_g f(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

Another integral operator $I_g$ is defined by

$$I_g f(z) = \int_0^z f'(\zeta)g(\zeta)\,d\zeta, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

The notation $H^\infty(\mathbb{D})$ represents the algebra of bounded holomorphic functions with $\| \cdot \|_\infty$ as its supreme norm. For $\alpha > 0, \beta \in \mathbb{R}$, the Bloch type space $B_{\alpha, \log^\beta}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_* := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \log^\beta \frac{2}{1 - |z|^2} |f'(z)| < \infty.$$ 

Then $\| \cdot \|_*$ is a complete semi-norm on $B_{\alpha, \log^\beta}$. We denote the Banach space associated to $B_{\alpha, \log^\beta}$ by $\widetilde{B}_{\alpha, \log^\beta}$, where the norm is given by the formula

$$\|f\|_{\widetilde{B}_{\alpha, \log^\beta}} = |f(0)| + \|f\|_*.$$ 

The little Bloch space, denoted by $B_{\alpha, \log^\beta, 0}$, consists of $f \in B_{\alpha, \log^\beta}$ for which

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha \log^\beta \frac{2}{1 - |z|^2} |f'(z)| = 0.$$ 

In this paper, we abbreviate $B = B_{1, \log^0}$ and $B_{\log} = B_{1, \log^1}$.

Composition operators were studied intensively in the past few decades. A lot of efforts have been made on characterizing bounded and compact composition operators on spaces of analytic functions, for example, [Sha87] for Hardy space and [MM95] for Bloch spaces. Interested readers may refer to books [CM95, Sha93, Zhu05] and some recent papers [ZC08, ZS02, ZZ12] to learn more details on this subject.

The discussion of $J_g$ first arose in connection with semigroups of composition operators, and readers may refer to [SZ98] for the background. Recently, the boundedness and compactness of $J_g$ and $I_g$ on various spaces of analytic functions has attracted considerable attention. For example, the boundedness of $J_g$ on Hardy spaces, Bergman spaces, BMOA space, Bloch space and
The compact intertwining relations are characterized in \[AC01, AS97, SZ98, Xia01, Xia04, Xia08\], respectively. The same problems for the product of composition and Volterra operators on some function spaces in \(\mathbb{D}\) have also been discussed. See \[LS08c, LS08j, LS09\] for example.

Based on these results, we consider the composition operator \(C_\varphi: X \to X\), and the integral-type operator \(V_g (= J_g \text{ or } I_g): X \to \mathcal{B}\) where \(X\) represents \(H^\infty\) or \(\mathcal{B}\). We are interested in the compact intertwining relations

\[
(1.2) \quad C_\varphi |_{X^\sim \mathcal{K}} \quad C_\varphi |_{\mathcal{B}} \quad (V_g |_{X^\to \mathcal{B}}).
\]

To be more intuitive, we consider the following commutative diagram,

\[
\begin{array}{ccc}
X & \xrightarrow{C_\varphi} & X \\
\downarrow V_g & & \downarrow V_g \\
\mathcal{B} & \xrightarrow{C_\varphi} & \mathcal{B}
\end{array}
\]

mod \(\mathcal{K}(X, \mathcal{B})\).

If (1.2) holds for some \(\varphi \in S(\mathbb{D})\) and \(g \in H(\mathbb{D})\), we may also say that \(C_\varphi\) essentially commute with \(V_g\). In this paper, we make our effort to answer the following two main questions.

**(Q1):** What are the sufficient and necessary conditions on \(g\), so that

\[
C_\varphi |_{X^\sim \mathcal{K}} \quad C_\varphi |_{\mathcal{B}} \quad (V_g |_{X^\to \mathcal{B}})
\]

holds for every \(\varphi \in S(\mathbb{D})\)?

**(Q2):** What are the sufficient and necessary conditions on \(\varphi \in S(\mathbb{D})\), so that

\[
C_\varphi |_{X^\sim \mathcal{K}} \quad C_\varphi |_{\mathcal{B}} \quad (V_g |_{X^\to \mathcal{B}})
\]

holds for every bounded \(V_g\)?

It is of particular interest to consider the above two “global” questions. We omit the symbols, which induce the operators, to reflect the “global” essential commutativity. That is, we write \(C |_{X^\sim \mathcal{K}} \quad C |_{\mathcal{B}}\) to indicate that the operator \(V_g\) can commute with every composition operator \(C_\varphi\) essentially. Analogously, we write \(C_\varphi |_{X^\sim \mathcal{K}} \quad C_\varphi |_{\mathcal{B}}\) to indicate that the operator \(C_\varphi\) can commute with every bounded \(V_g: X \to \mathcal{B}\) essentially. By the way, the collections of \(g\) satisfying conditions similar as (Q1) was called the universal set of \(V_g\) by the authors in \[TZ14, TZ13\]. Our use of the term “universal set” should not be confused with the notion of “universal set” which appears in the dynamical theory of linear operators.

In the following discussion, we write \(A \lesssim B\) if there exists an absolute constant \(C > 0\) such that \(A \leq C \cdot B\), and \(A \approx B\) represents \(A \lesssim B\) and \(B \lesssim A\).
2. Preliminaries

Before the discussion of our main results, we need some preliminary propositions. For \( \varphi \in S(D) \), denote the Schwarz derivative of \( \varphi \) by

\[
\varphi^\#(z) := \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \varphi'(z).
\]

From the Schwarz Lemma we know that \( |\varphi^\#(z)| \leq 1 \) for any \( z \in D \), and the equality holds if and only if \( \varphi \) is an automorphism of the unit disk. The following lemma characterizes bounded and compact composition operators on \( B \) and \( H^\infty \) (see [MM95] and [CM95]).

**Lemma 2.1.** If \( \varphi \in S(D) \), then

1. Every \( \varphi \) induces a bounded composition operator on \( B \) and \( H^\infty \).
2. \( C_\varphi : B \to B \) is compact if and only if \( C_\varphi \) is bounded and
   \[
   \lim_{|\varphi(z)| \to 1} |\varphi^\#(z)| = 0.
   \]
3. \( C_\varphi : H^\infty \to H^\infty \) is compact if and only if \( \varphi(D) \subset D \).

The following criterion for compactness follows from standard arguments and its proof is similar to the method of Proposition 3.11 in [CM95]. Hence we omit the details.

**Lemma 2.2.** Suppose that \( \varphi \in S(D) \) and \( g \in H(D) \). Then \( C_\varphi V_g - V_g C_\varphi \) is compact from \( X \to B \) if and only if for any bounded sequence \( \{f_k\}, k = 1, 2, \ldots \) in \( X \) which converges to zero uniformly on compact subsets of \( D \), \( \| (C_\varphi V_g - V_g C_\varphi) f_k \|_* \to 0 \) as \( k \to \infty \).

Recall that the notation \( \mathbb{C}^N \) represents the \( N \) dimensional complex Euclidean space. Denote the unit ball of \( \mathbb{C}^N \) by \( B_N \). If \( z, w \in B_N \), we define Möbius transform by

\[
\Phi_w(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle},
\]

where \( P_a(z) = \frac{\langle z, a \rangle}{\langle a, a \rangle} a \), \( Q_a(z) = z - P_a(z) \) and \( s_a = \sqrt{1 - |a|^2} \). The following Lemma will be used in Section 4, which was first presented by Berndtsson in [Ber85].

**Lemma 2.3.** Let \( \{x_i\} \) be a sequence in the ball \( B_N \) satisfying

\[
\prod_{j \neq k \neq i} |\Phi_{x_j}(x_k)| \geq d > 0 \quad \text{for any} \quad k.
\]

Then there exists a number \( M = M(d) < \infty \) and a sequence of functions \( h_k \in H^\infty(B_N) \) such that

\[
(a) \quad h_k(x_j) = \delta_{kj}; \quad (b) \quad \sum_k |h_k(z)| \leq M \quad \text{for} \quad |z| < 1.
\]

(The symbol \( \delta_{kj} \) is equal to 1 if \( k = j \) and 0 otherwise.)
The next lemma was proved by Carl Toews in [Toe04].

**Lemma 2.4.** Let \(\{z_n\} \subset B_N\) be a sequence with \(|z_n| \to 1\) as \(n \to \infty\). Then for any given \(d \in (0, 1)\) there is a subsequence such that \(\{x_i\} := \{z_{n_i}\}\) satisfies (2.1).

From this lemma, there is always a subsequence which satisfies (2.1) for every sequence converging to the boundary of \(B_N\), and Lemma 2.2 holds for this subsequence. We just need the result in one dimension.

To get some simple consequences of our main problems, we will consider the situation in the little Bloch setting. The next lemma is well known, see [OSZ03].

**Lemma 2.5.** A closed set \(K\) in \(B_0\) is compact if and only if it is bounded and satisfies

\[
\lim_{|z| \to 1} \sup_{f \in K} (1 - |z|^2)|f'(z)| = 0.
\]

In our discussion, we will use the boundedness of operators \(J_g\) and \(I_g\), from \(H^\infty\) or \(B\) to \(B\). Several characterizations are listed below.

**Lemma 2.6.** Suppose that \(g \in H(D)\). Then

(a) \(J_g : H^\infty \to B\) is bounded if and only if \(g \in B\);
(b) \(J_g : B \to B\) is bounded if and only if \(g \in B_{\log}\);
(c) \(I_g : H^\infty \) or \(B \to B\) is bounded if and only if \(g \in H^\infty\).

Part (a) above was Corollary 4 in [LS08c], part (b) is from Theorem 3 in [LS09], and part (c) follows from Corollary 1 in [LS08c] and Theorem 14 in [LS09].

Some definitions and results in Geometric Function Theory are needed, and interested readers can refer to [Gar81] and [Kra06]. For \(\zeta \in \partial \mathbb{D}\) and \(M > 1\) the nontangential approaching region at \(\zeta\) is defined by

\[
\Gamma(\zeta, M) = \{z \in \mathbb{D} : |z - \zeta| < M(1 - |z|^2)\}.
\]

A function \(f\) is said to have a nontangential limit at \(\zeta\) if \(\lim_{z \to \zeta} f(z)\) exists in each nontangential region \(\Gamma(\zeta, M)\), and we denote it by \(\angle - \lim_{z \to \zeta} f(z)\). If \(\varphi \in S(\mathbb{D})\) and \(\zeta \in \partial \mathbb{D}\), we will call \(\zeta\) a boundary fixed point of \(\varphi\) if

\[
\angle - \lim_{z \to \zeta} \varphi(z) = \zeta.
\]

We say \(\varphi\) has a finite angular derivative at \(\zeta \in \partial \mathbb{D}\) if there is \(\eta \in \partial \mathbb{D}\) so that \((\varphi(z) - \eta)/(z - \zeta)\) has finite nontangential limit as \(z \to \zeta\). When it exists as a finite complex number, this limit is denoted \(\varphi'(\zeta)\). A \(\varphi \in S(\mathbb{D})\) is said to be parabolic type if \(\varphi\) has a boundary fixed point \(\zeta\) with \(\varphi'(\zeta) = 1\). If \(\varphi\) is parabolic type, \(\varphi(z) \to \zeta\) and \(\varphi'(z) \to 1\) as \(z \to \zeta\) unrestricted in the unit disk, we say \(\zeta\) is a \(C^1\) parabolic boundary fixed point of \(\varphi\).
3. The case of intertwining operator $I_g$

First we consider $C_\varphi I_g - I_g C_\varphi$ as an operator from the Bloch space to itself.

**Theorem 3.1.** Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi I_g - I_g C_\varphi$ is bounded on the Bloch space if and only if

(3.1) \[ \sup_{z \in \mathbb{D}} |\varphi^\#(z)| \cdot |g(\varphi(z)) - g(z)| < \infty. \]

**Proof.** Suppose (3.1) holds, we will show that $C_\varphi I_g - I_g C_\varphi$ is bounded on $\mathcal{B}$ by a direct computation. For any $f \in \mathcal{B}$,

\[
((C_\varphi I_g - I_g C_\varphi)f)(z) &= C_\varphi \int_0^z f'(\zeta)g(\zeta)d\zeta - I_g f(\varphi(z)) \\
&= \int_0^{\varphi(z)} f'(\zeta)g(\zeta)d\zeta - \int_0^z (f \circ \varphi)'(\zeta)g(\zeta)d\zeta.
\]

It follows that

\[
\|((C_\varphi I_g - I_g C_\varphi)f)\|_* = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \varphi(z)f'(\varphi(z))g(\varphi(z)) - \varphi'(z)f'(\varphi(z))g(z) \right| \\
&= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \varphi'(z)(g(\varphi(z)) - g(z)) \right| \\
&= \sup_{z \in \mathbb{D}} |\varphi^\#(z)| |g(\varphi(z)) - g(z)| (1 - |\varphi(z)|^2) |f'(\varphi(z))| \\
&\leq \sup_{z \in \mathbb{D}} |\varphi^\#(z)| |g(\varphi(z)) - g(z)| \cdot \|f\|_*
\]

From which we obtain that $C_\varphi I_g - I_g C_\varphi$ is bounded by (3.1).

Conversely, suppose that $C_\varphi I_g - I_g C_\varphi$ is bounded on $\mathcal{B}$, then there is a constant $C > 0$ such that

\[
\|((C_\varphi I_g - I_g C_\varphi)f)\|_* \leq \|(C_\varphi I_g - I_g C_\varphi)f\|_\mathcal{B} < C
\]

for $\|f\|_\mathcal{B} \leq 1$. We will prove condition (3.1). Suppose not, there exists a sequence $\{w_n\}$ in $\mathbb{D}$ such that

\[
\lim_{n \to \infty} |\varphi^\#(w_n)| |g(\varphi(w_n)) - g(w_n)| = \infty.
\]

Let $\alpha_n(z) = \frac{\varphi(w_n) - z}{1 - \varphi(w_n)z}$, and $\alpha'_n(z) = -\frac{1 - |\varphi(w_n)|^2}{(1 - \varphi(w_n)z)^2}$ for $n = 1, 2, \ldots$. It is easy to check that $\|\alpha_n\|_* = 1$.

\[
\|(C_\varphi I_g - I_g C_\varphi)\alpha_n\|_* = \sup_{z \in \mathbb{D}} |\varphi^\#(z)| |g(\varphi(z)) - g(z)| \cdot (1 - |\varphi(z)|^2) |\alpha'_n(\varphi(z))| \\
\geq (1 - |\varphi(w_n)|^2) \frac{1 - |\varphi(w_n)|^2}{(1 - \varphi(w_n)\varphi(w_n))^2} \cdot |\varphi^\#(w_n)| |g(\varphi(w_n)) - g(w_n)| \\
= |\varphi^\#(w_n)| |g(\varphi(w_n)) - g(w_n)| \to \infty.
\]
That contradicts to the boundedness of $C\varphi I_g - I_g C\varphi$. Hence (3.1) holds. $\Box$

When we investigate essential commutativity of $C\varphi$ and $I_g$, we need to add the condition $g \in H^\infty$ to ensure the boundedness of $I_g$ on the Bloch space, see Lemma 2.6.

**Theorem 3.2.** Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H^\infty(\mathbb{D})$. Then $C\varphi$ and $I_g$ are essentially commutative on $\mathcal{B}$ if and only if

$$
\lim_{|\varphi(z)| \to 1} |\varphi^\#(z)||g(\varphi(z)) - g(z)| = 0. 
$$

**Proof.** Sufficiency. Note that $g \in H^\infty$ and $||\varphi^\#||_\infty \leq 1$, we have

$$
\sup_{z \in \mathbb{D}} |\varphi^\#(z)| \cdot |g(\varphi(z)) - g(z)| < \infty. 
$$

Hence $C\varphi I_g - I_g C\varphi$ is a bounded operator by Theorem 3.1. For any bounded sequence $\{f_k\}$ in $\mathcal{B}$ converging to zero uniformly on compact subsets of $\mathbb{D}$, we firstly find a positive $M > 0$ so that $\|f_k\|_\mathcal{B} \leq M$. It follows from (3.2) that for any small $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$
|\varphi^\#(z)||g(\varphi(z)) - g(z)| < \frac{\varepsilon}{M} 
$$

whenever $\varphi(z) \in \mathbb{D} \setminus (1 - \delta)\mathbb{D}$. Then we can compute as follows

$$
\|(C\varphi I_g - I_g C\varphi)f_k\|_* = \sup_{z \in \mathbb{D}} |\varphi^\#(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))| 
$$

$$
\leq (A) + (B), 
$$

where

$$
(A) = \sup_{\varphi(z) \in (1 - \delta)\mathbb{D}} |\varphi^\#(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))|, 
$$

and

$$
(B) = \sup_{\varphi(z) \in \mathbb{D} \setminus (1 - \delta)\mathbb{D}} |\varphi^\#(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))|. 
$$

Since $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, it is obvious that $(A) < \varepsilon$ for sufficiently large $k$. And by (3.3) we have

$$
(B) < \frac{\varepsilon}{M} \cdot \sup_{z \in \mathbb{D}}(1 - |\varphi(z)|^2)|f_k'(\varphi(z))| = \frac{\varepsilon}{M} \|f_k\|_* < \varepsilon. 
$$

Thus $C\varphi I_g - I_g C\varphi$ is compact on $\mathcal{B}$.

Necessity. If $C\varphi I_g - I_g C\varphi$ is compact on $\mathcal{B}$, it is certainly a bounded operator. Therefore

$$
\sup_{z \in \mathbb{D}} |\varphi^\#(z)| \cdot |g(\varphi(z)) - g(z)| < \infty 
$$

by Theorem 3.1. Suppose that

$$
\sup_{|\varphi(z)| \to 1} |\varphi^\#(z)||g(\varphi(z)) - g(z)| \neq 0, 
$$

Thus $C\varphi I_g - I_g C\varphi$ is not compact on $\mathcal{B}$. 

□
then there is a sequence \( \{w_n\} \) in \( \mathbb{D} \) with \(|\varphi(w_n)| \to 1\) and an \( \epsilon > 0 \) such that
\[
|\varphi^\#(w_n)||g(\varphi(w_n)) - g(w_n)| > \epsilon \quad \forall n.
\]
Let
\[
h_n(z) = \frac{1 - |\varphi(w_n)|^2}{1 - \varphi(w_n)z}.
\]
A simple computation shows that
\[
h_n'(z) = (1 - |\varphi(w_n)|^2) \frac{\varphi(w_n)}{(1 - \varphi(w_n)z)^2}.
\]
So \( \|h_n\|_* \leq 1 \) and \( \{h_n\} \) converges to zero uniformly on compact subsets of \( \mathbb{D} \). It follows from Lemma 2.2 that
\[
\|(C_\varphi I_g - I_g C_\varphi)h_n\|_* \to 0.
\]
On the other hand, by (3.4), we have
\[
\|(C_\varphi I_g - I_g C_\varphi)h_n\|_* = \sup_{z \in \mathbb{D}} |\varphi^\#(z)||g(\varphi(z)) - g(z)| \cdot (1 - |\varphi(z)|^2) |h_n'(\varphi(z))|
\]
\[
= \sup_{z \in \mathbb{D}} |\varphi^\#(z)||g(\varphi(z)) - g(z)| \cdot (1 - |\varphi(z)|^2) \frac{|\varphi(w_n)(1 - |\varphi(w_n)|^2)}{|1 - \varphi(w_n)\varphi(z)|^2}
\]
\[
\geq |\varphi^\#(w_n)||g(\varphi(w_n)) - g(w_n)||\varphi(w_n)| > \epsilon
\]
for any \( n \). It contradicts to (3.6). Hence (3.2) holds when \( C_\varphi \) and \( I_g \) are essentially commutative. \( \square \)

**Corollary 3.3.** Let \( \varphi \in S(\mathbb{D}) \) and \( g \in H(\mathbb{D}) \), then \( C_\varphi I_g - I_g C_\varphi : H^\infty \to \mathcal{B} \) is bounded if and only if (3.1) holds; \( C_\varphi I_g - I_g C_\varphi : H^\infty \to \mathcal{B} \) is compact if and only if (3.2) holds.

**Proof.** The proof of this corollary is analogous to those of Theorem 3.1 and 3.2. We only need to change the test function (3.5) to
\[
f_n(z) = \frac{1 - |\varphi(w_n)|^2}{1 - \varphi(w_n)z} \cdot \frac{\varphi(w_n) - z}{1 - \varphi(w_n)z}.
\]
The rest of the arguments are highly similar to those of Theorems 3.1 and 3.2. Hence we omit the details. \( \square \)

Using Lemma 2.4, we can easily get the following corollary on the little Bloch space. The method is the same as before and we omit its proof.

**Corollary 3.4.** Suppose \( \varphi \in S(\mathbb{D}) \) and \( g \in H(\mathbb{D}) \). Let \( X \) represent the space \( H^\infty \), \( \mathcal{B} \) or \( \mathcal{B}_0 \). The following three statements are equivalent:

(a) \( C_\varphi I_g - I_g C_\varphi : X \to \mathcal{B}_0 \) is bounded;

(b) \( C_\varphi I_g - I_g C_\varphi : X \to \mathcal{B}_0 \) is compact;

(c) \( \lim_{|z| \to 1} |\varphi^\#(z)||g(\varphi(z)) - g(z)| = 0. \)
Now we consider (Q1) raised in the first section:

- When does $C |_{X \propto K} C |_B (V_g)$ hold?

**Theorem 3.5.** The compact intertwining relation $C |_{X \propto K} C |_B (V_g)$ holds if and only if $g$ is a constant.

**Proof.** Sufficiency is obvious from a direct computation. To verify the necessity, we just need to consider $\varphi$ as an automorphism of $D$ in condition (3.2). Then maximum modulus theorem implies that $g$ must be a constant. □

The following proposition gives a sufficient condition which answers (Q2) in part.

**Proposition 3.6.** Suppose $\varphi \in S(D)$ and $g \in H^\infty$. Let $X$ denote either $B$ or $H^\infty$. If $\varphi$ satisfies

$$\lim_{|\varphi(z)| \to 1} \frac{|\varphi(z) - z|}{(1 - \max\{|z|, |\varphi(z)|\})^2} |\varphi^#(z)| = 0,$$

then $C_{\varphi |_{X \propto K} C_{\varphi}} |_B (I |_{X \to B})$.

**Proof.** By the Cauchy integral formula, one finds that

$$|g'(z)| \leq \frac{\|g\|_\infty}{(1 - |z|)^2} \quad (\forall z \in D)$$

for $g \in H^\infty$. The proposition will be proved immediately by noting that

$$|\varphi^#(z)||g(\varphi(z)) - g(z)| = |\varphi^#(z)| \left| \int_z^{\varphi(z)} g'(\zeta)d\zeta \right|$$

$$\leq |\varphi^#(z)| \frac{\|g\|_\infty}{(1 - \max\{|\varphi(z)|, |z|\})^2} \int_z^{\varphi(z)} |d\zeta|$$

$$\leq |\varphi^#(z)||g|_\infty \frac{|\varphi(z) - z|}{(1 - \max\{|\varphi(z)|, |z|\})^2}.$$

□

According to Lemma 2.6, the operator $I_g : X \to B$ is bounded if $g$ is analytic in $D$ and continuous to the boundary. The next theorem will answer (Q2) in part. We will get a necessary and sufficient condition of the compact intertwining relation, when $g$ is the function in the disk algebra.

**Theorem 3.7.** Let $A$ be the disk algebra, that is the subspace of $H^\infty$ whose elements are analytic in $D$ and continuous to the unit circle. Then

$$C_{\varphi |_{X \propto K} C_{\varphi}} |_B (I_g |_{X \to B})$$

holds for all $g \in A$ if and only if

$$\lim_{|\varphi(z)| \to 1} |\varphi^#(z)||\varphi(z) - z| = 0.$$ (3.8)
4. The case of intertwining operator $J_g$

First, we consider the case when $X = H^\infty$ in the main question.

**Theorem 4.1.** Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then

1. $C_\varphi J_g - J_g C_\varphi$ is bounded from $H^\infty$ to $\mathcal{B}$ if and only if

\begin{equation}
\label{eq:4.1}
\sup_{z \in \mathbb{D}} (1 - |z|^2) |(g \circ \varphi)'(z) - g'(z)| < \infty;
\end{equation}

2. $C_\varphi J_g - J_g C_\varphi$ is compact from $H^\infty$ to $\mathcal{B}$ if and only if \eqref{eq:4.1} holds and

\begin{equation}
\label{eq:4.2}
\lim_{|\varphi(z)| \to 1} (1 - |z|^2) |(g \circ \varphi)'(z) - g'(z)| = 0.
\end{equation}

**Proof.** Being similar to the proofs of Theorems 3.1 and 3.2, we just need some modifications. We firstly see that

\begin{equation}
\label{eq:4.3}
\begin{align*}
&\| (C_\varphi J_g - J_g C_\varphi) f \|_* \\
= &\| f \varphi(z) g'(\zeta) \, d\zeta - f \varphi(z) g'(\zeta) \, d\zeta \|_* \\
= &\sup_{z \in \mathbb{D}} (1 - |z|^2) |f(\varphi(z)) \varphi'(z) - f(\varphi(z)) g'(z)| \\
= &\sup_{z \in \mathbb{D}} (1 - |z|^2) |(g \circ \varphi)'(z) - g'(z)| |f(\varphi(z))|.
\end{align*}
\end{equation}

Sufficiency of the two items in the theorem is obvious from the last formula in \eqref{eq:4.3}.
Necessity of boundedness can be proved by computing test functions

\[ f_n(z) = 1 - \frac{\varphi(w_n) - z}{1 - \varphi(w_n)z}, \]

where sequence \( \{w_n\} \) violates equation (4.1).

Necessity of compactness can also be proved by contradiction. Suppose that we can find a sequence \( \{\varphi(w_n)\} \) converging to the boundary of \( D \) and \( \epsilon_0 > 0 \) such that

\[
\lim_{n \to \infty} (1 - |w_n|^2)|(g \circ \varphi)'(w_n) - g'(w_n)| > \epsilon_0.
\]

Further we may assume that \( \{\varphi(w_n)\} \) is interpolating. Then there exist functions \( \{h_n\} \) in \( H^\infty \) for \( \{\varphi(w_n)\} \) such that

\[
h_n(\varphi(w_k)) = \begin{cases} 1 & n = k, \\ 0 & n \neq k. \end{cases}
\]

and

\[
\sum_n |h_n(z)| \leq M < \infty,
\]

by Lemma 2.4 and 2.3, or see [Gar81]. Equation (4.6) guarantees that \( \{h_n\} \) is bounded in \( H^\infty \) and converges to zero on compact subsets of \( D \). Then we have

\[
\|(C_{\varphi}J_g - J_gC_{\varphi})h_n\|_* \\
= \sup_{z \in \mathbb{B}} (1 - |z|^2)|(g \circ \varphi)'(z) - g'(z)||h_n(\varphi(z))| \\
\geq (1 - |w_n|^2)|(g \circ \varphi)'(w_n) - g'(w_n)||h_n(\varphi(w_n))| \\
= (1 - |w_n|^2)|(g \circ \varphi)'(w_n) - g'(w_n)|,
\]

where the last equation follows by (4.5). Letting \( n \to \infty \), we find a contradiction by (4.4).

Corollary 4.2. Let \( \varphi \in S(\mathbb{D}) \) and \( g \in H(\mathbb{D}) \). The following three conditions are equivalent:

(a) \( C_{\varphi}J_g - J_gC_{\varphi} : H^\infty \to \mathcal{B}_0 \) is bounded;
(b) \( C_{\varphi}J_g - J_gC_{\varphi} : H^\infty \to \mathcal{B}_0 \) is compact;
(c) \( \lim_{|z| \to 1} (1 - |z|^2)|(g \circ \varphi)'(z) - g'(z)| = 0. \)

When we consider (Q1) for composition operator and \( J_g \), the result turns out to be interesting. Note that \( g \in \mathcal{B} \) implies that \( \sup_{z \in \mathbb{D}} (1 - |z|^2)|(g \circ \varphi)'(z) - g'(z)| < \infty \), thus \( C_{\varphi}J_g - J_gC_{\varphi} \) is a bounded operator from \( H^\infty \) to \( \mathcal{B} \). And

\[
C_{\varphi}|_{H^\infty \to \mathcal{B}} \quad (J_g|_{H^\infty \to \mathcal{B}})
\]

if and only if (4.2) holds. Now, we can answer (Q1).
Corollary 4.3. If \( g \in B_0 \), then

\[
C |_{H^\infty \times K} C |_{B} \quad (J_g |_{H^\infty \to B}).
\]

Proof. Since \( g \) is in the little Bloch space, \((1 - |z|^2)|g'(z)|\) tends to 0 whenever \( z \) tends to the boundary of the disk. Then we have

\[
\lim_{|\varphi(z)| \to 1} (1 - |z|^2)|g'(\varphi(z))\varphi'(z) - g'(z)| \leq \lim_{|\varphi(z)| \to 1} |\varphi^#(z)|(1 - |\varphi(z)|^2)|g'(\varphi(z))| + \lim_{|\varphi(z)| \to 1} (1 - |z|^2)|g'(z)|.
\]

Conditions \( g \in B \) and \( |\varphi^#| \leq 1 \) imply that \( C_{\varphi}J_g - J_gC_{\varphi} \) is compact from \( H^\infty \) to \( B \) for every self-mapping \( \varphi \). \( \square \)

In contrast to the case of intertwining operator \( I_g \), there are many non-constant functions \( g \), which are actually the little Bloch functions, so that \( J_g \) can commute with any composition operator essentially. Naturally, we are going to ask the inverse question: if \( J_g \) commute with every composition operator essentially, does that imply \( g \in B_0 \)? The answer is positive. The next lemma is the key lemma to investigate the necessary condition on \( g \) in (Q1). The method of proof is the same as in our recent papers [TZ14, TZ13].

Lemma 4.4. If \( g \) is a Bloch function on the unit disk with the property that, for any rotation \( \tau(z) = e^{it}z \), \( g \circ \tau - g \) is in the little Bloch space, then \( g \) itself must be in the little Bloch space.

Proof. Since \( g \circ \tau - g \) is in the little Bloch space, we have

\[
\lim_{|z| \to 1} (1 - |z|^2) \left| g'(e^{it}z)e^{it} - g'(z) \right| = 0.
\]

We estimate the upper bound of left hand side in equation (4.8) as follows,

\[
(1 - |z|^2) \left| g'(e^{it}z)e^{it} - g'(z) \right| \\
\leq (1 - |z|^2) \left| g'(e^{it}z)e^{it} \right| + (1 - |z|^2) \left| g'(z) \right| \\
= (1 - |e^{it}z|^2) \left| g'(e^{it}z) \right| + (1 - |z|^2) \left| g'(z) \right| \\
\leq 2\|g\|_*.
\]

Thus \( g \in B \) implies that the left hand side in equation (4.8) is finite uniformly in \( t \).
Suppose that \( g(z) = \sum_{n=0}^{\infty} a_{n} z^{n} \), then \( g'(z) = \sum_{n=1}^{\infty} n a_{n} z^{n-1} \). Integrating with respect to \( t \) from 0 to \( 2\pi \), we have that

\[
\int_{0}^{2\pi} \lim_{|z| \to 1} (1 - |z|^{2}) \left| g'(e^{it} z) e^{i t} - g'(z) \right| dt
\]

\[
= \lim_{|z| \to 1} (1 - |z|^{2}) \left| \int_{0}^{2\pi} g'(e^{it} z) e^{i t} - g'(z) dt \right|
\]

\[
= \lim_{|z| \to 1} (1 - |z|^{2}) \left| \int_{0}^{2\pi} \sum_{n=1}^{\infty} (na_{n}(e^{i t} z)^{n-1} e^{i t} - na_{n} z^{n-1}) dt \right|
\]

\[
= \lim_{|z| \to 1} (1 - |z|^{2}) \sum_{n=1}^{\infty} na_{n} z^{n-1} \int_{0}^{2\pi} (e^{i n t} - 1) dt
\]

\[
= 2\pi \cdot \lim_{|z| \to 1} (1 - |z|^{2}) |g'(z)|,
\]

where the equation in the second line is true by the dominated convergence theorem. Thus we get \( g \in \mathcal{B}_{0} \) from (4.8). \( \square \)

**Theorem 4.5.** \( C |_{H^\infty K} \mathcal{B} \) \( (J_{g} |_{H^\infty \to \mathcal{B}}) \) if and only if \( g \in \mathcal{B}_{0} \).

**Proof.** Sufficiency is stated in Corollary 4.3. To prove necessity, just consider the rotation of the disk. That is to say, by putting \( \varphi(z) = \tau(z) = e^{i t} z \) in the condition (4.2), and we have \( g \in \mathcal{B}_{0} \) by Lemma 4.4. \( \square \)

Considering the composition operator \( C_{\varphi} \) and Volterra operator \( J_{g} \), we can obtain a necessary condition (Proposition 4.6) and a sufficient condition (Proposition 4.7) to answer \( \text{(Q2)} \) partially.

**Proposition 4.6.** Let \( \zeta \in \partial \mathbb{D} \), \( \varphi \in S(\mathbb{D}) \) and \( \varphi(z) \) tend to the unit circle when \( z \) converges to \( \zeta \) nontangentially. If

\[
C_{\varphi} |_{H^\infty K} \mathcal{B} \quad (J_{ \varphi - g} |_{H^\infty \to \mathcal{B}}),
\]

then \( \zeta = \lim_{z \to \zeta} \varphi(z) = \zeta \).

**Proof.** By Theorem 4.5, \( \lim_{|\varphi(z)| \to 1} (1 - |z|^{2}) |(g \circ \varphi - g)'(z)| = 0 \) holds for every \( g \in \mathcal{B} \). Suppose that \( \varphi(z) \to \partial \mathbb{D} \) as \( z \to \zeta \) in some nontangential approaching region, such that \( \eta \neq \zeta \) and \( \varphi(z_{n}) \to \eta \). Let \( g(z) = -\log(1 - \bar{\zeta} z) \) in equation (4.7), then

\[
\lim_{z \to \zeta} (1 - |z|^{2}) \left| \frac{\bar{\zeta} \varphi'(z)}{1 - \bar{\zeta} \varphi(z)} - \frac{\bar{\zeta}}{1 - \bar{\zeta}} \right|
\]

\[
= \lim_{z \to \zeta} \frac{1 - |z|^{2}}{|1 - \bar{\zeta} z|} \left| \frac{\zeta - z}{\bar{\zeta} - \varphi(z)} \right| \varphi'(z) - 1 = 0.
\]
Note that

\[ \angle \lim_{z \to \zeta} \left| \frac{\zeta - z}{\zeta - \varphi(z)} \varphi'(z) - 1 \right| = 0. \]

By noting that \(|\zeta - z||\varphi'(z)| = \frac{|\zeta - z|}{1 - |z|^2} \cdot (1 - |\varphi(z)|^2)|\varphi'(z)| \rightarrow 0\), one has

\[ (4.9) \quad \angle \lim_{z \to \zeta} \varphi(z) = \zeta. \square \]

**Proposition 4.7.** Let \( \varphi \in S(\mathbb{D}) \) be of parabolic type with \( C^1 \) boundary fixed point \( \zeta \in \partial \mathbb{D} \). If

\[ (4.10) \quad \sup_{z \in \mathbb{D}} \frac{|\varphi(z) - z|}{(1 - \max \{|z|, |\varphi(z)|\})^2} \leq \infty, \]

then \( C_\varphi |_{H^\infty K} C_\varphi |_\mathcal{B} (J |_{H^\infty \rightarrow \mathcal{B}}) \).

**Proof.** It is well known that \( g \in \mathcal{B} \) if and only if

\[ \sup_{z \in \mathbb{D}} (1 - |z|^2)^n |g^{(n)}(z)| < \infty. \]

Note that

\[
(1 - |z|^2)((g \circ \varphi)'(z) - g'(z)) \\
\leq (1 - |z|^2)|\varphi'(z)||g'((\varphi(z)) - g'(z)) + (1 - |z|^2)|g'(z)||\varphi'(z) - 1| \\
= (1 - |z|^2)|\varphi'(z)| \int_{\varphi(z)}^{\varphi'(z)} g''(\zeta) d\zeta + (1 - |z|^2)|g'(z)||\varphi'(z) - 1| \\
\leq (1 - |z|^2)|\varphi'(z)| \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |g''(z)| \int_{\varphi(z)}^{\varphi'(z)} \frac{|d\zeta|}{(1 - |\zeta|)^2} \\
+ (1 - |z|^2)|g'(z)||\varphi'(z) - 1| \\
\leq (1 - |z|^2)|\varphi'(z)| \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |g''(z)| \frac{|\varphi(z) - z|}{(1 - \max \{|z|, |\varphi(z)|\})^2} \\
+ (1 - |z|^2)|g'(z)||\varphi'(z) - 1|
\]

where the integral path is chosen to be the segment from \( z \) to \( \varphi(z) \). Then \( C_\varphi |_{H^\infty K} C_\varphi |_\mathcal{B} (J |_{H^\infty \rightarrow \mathcal{B}}) \) follows immediately from those conditions in the proposition. \( \square \)

The next several propositions concern \( C_\varphi \) and \( J_\varphi \) as maps from \( \mathcal{B} \) to itself.

**Proposition 4.8.** Assume that \( \varphi \in S(\mathbb{D}) \) and \( g \in H(\mathbb{D}) \). Then \( C_\varphi J_\varphi - J_\varphi C_\varphi \) is bounded from \( \mathcal{B} \) to itself if and only if

\[ (4.11) \quad \sup_{z \in \mathbb{D}} (1 - |z|^2)^3 |(g \circ \varphi)'(z) - g'(z)| \log \frac{2}{1 - |\varphi(z)|^2} < \infty. \]
Proof. Sufficiency can be verified by some straightforward computations and inequalities. Necessity will be proved by choosing the test function

\[ f_w(z) = \log \frac{2}{1 - wz}. \]

\[ \square \]

Proposition 4.9. Assume that \( \varphi \in S(\mathbb{D}) \) and \( g \in H(\mathbb{D}) \). Then the following statements are equivalent:

(a) \( C\varphi J g - J g C\varphi : B \rightarrow B \) is compact and (4.11) holds;
(b) \( C\varphi J g - J g C\varphi : B_0 \rightarrow B_0 \) is compact;
(c) \( C\varphi J g - J g C\varphi : B_0 \rightarrow B_0 \) is weakly compact;
(d) Condition (4.11) holds and

\[ \lim_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)|(g \circ \varphi)'(z) - g'(z)| \log \frac{2}{1 - |\varphi(z)|^2} = 0; \]

(e) \( C\varphi J g - J g C\varphi : B \rightarrow B_0 \) is compact;
(f) \( C\varphi J g - J g C\varphi : B \rightarrow B_0 \) is bounded;

The compactness of operators \( C\varphi J g \) and \( J g C\varphi \) are characterized in Theorem 4 and Theorem 11 in [LS09]. The proofs are similar to those in Theorem 4 in [LS09], so we omit them.

To continue our discussion, we need an upper bound on the modulus of \( \varphi(z) \) (see Corollary 2.40 in [CM95]).

Lemma 4.10. If \( \varphi \in S(\mathbb{D}) \), then

\[ |\varphi(z)| \leq \frac{|z| + |\varphi(0)|}{1 + |z||\varphi(0)|}. \]

Now we are ready to consider (Q1) for \( C\varphi \) and \( J g \), where both of them are operators acting on \( B \).

Theorem 4.11. Assume \( g \in H(\mathbb{D}) \) is such that \( J g \) is bounded on the Bloch space. Then

\[ C |_{B} \propto_K C |_{B} \quad (J g |_{H^{\infty} \rightarrow B}) \]

if and only if \( g \in B_{\log,0} \), that is

\[ \lim_{|z| \rightarrow 1} (1 - |z|^2)|g'(z)| \log \frac{2}{1 - |z|^2} = 0. \]

Proof. Following the method in the proof of Lemma 4.10, the necessity can be proved similarly.

Now suppose that (4.13) holds. We have

\[ (1 - |z|^2)|(g \circ \varphi)'(z) - g'(z)| \log \frac{2}{1 - |\varphi(z)|^2} \leq |\varphi^\#(z)|(1 - |\varphi(z)|^2)|g'(\varphi(z))| \log \frac{2}{1 - |\varphi(z)|^2} \]

\[ +(1 - |z|^2)|g'(z)| \log \frac{2}{1 - |z|^2} \cdot \frac{\log 2 - \log(1 - |\varphi(z)|^2)}{\log 2 - \log(1 - |z|^2)}. \]
Just consider those \( z \)'s tending to \( \partial \mathbb{D} \) with \( |\varphi(z)| \to 1 \).

\[
\lim_{|z| \to 1^-} \frac{\log 2 - \log(1 - |\varphi(z)|^2)}{\log 2 - \log(1 - |z|^2)} = \lim_{|z| \to 1^-} \frac{-\log(1 - |\varphi(z)|)}{-\log(1 - |z|)} \leq \lim_{|z| \to 1^-} \frac{-\log(1 - |z| + |\varphi(0)|)}{-\log(1 - |z|)} = \lim_{|z| \to 1^-} \frac{\log(1 - |z|)(1 - |\varphi(0)|) - \log(1 + |z||\varphi(0)|)}{\log(1 - |z|)} - \frac{1 - |z| - \frac{|\varphi(0)|}{1 + |z||\varphi(0)|}}{1 - |z|} = 1
\]

where Lemma 4.8 and L'Hospital Law are applied. Thus (4.12) holds for every \( \varphi \in S(\mathbb{D}) \) by (4.13). The proof is completed by Proposition 4.9. \( \square \)

By the same method as in Proposition 4.6, we can obtain a necessary condition for \((Q2)\) when \( X = B \). Hence we omit the proof.

**Proposition 4.12.** Let \( \zeta \in \partial \mathbb{D} \), \( \varphi \in S(\mathbb{D}) \) and \( \varphi(z) \) tend to the unit circle when \( z \) converges to \( \zeta \) nontangentially. If

\[
C_{\varphi} |_{B} \propto \frac{C_{\varphi}}{|_{B}} (J |_{B}),
\]

then \( \angle - \lim_{z \to \zeta} \varphi(z) = \zeta \).

The following proposition gives a sufficient condition on \( g \) to answer \((Q2)\).

**Proposition 4.13.** Let \( \varphi \in S(\mathbb{D}) \) be parabolic type with \( C^1 \) boundary fixed point \( \zeta \in \partial \mathbb{D} \). If

\[
(4.14) \quad \sup_{z \in \mathbb{D}} \frac{|\varphi(z) - z|}{1 - \max \{|z|, |\varphi(z)|\}} \leq \infty,
\]

then \( C_{\varphi} |_{B} \propto K_{C_{\varphi}} |_{B} (J |_{B}) \).

**Proof.** The proposition follows by \( B_{\log} \subset B \). In fact,

\[
\log \frac{2}{1 - |\varphi(z)|^2} \approx \log \frac{2}{1 - |\varphi(z)|} \leq \log \frac{2(1 + |\varphi(z)||z|)}{(1 - |z|)(1 - |\varphi(z)|)} \leq \log \frac{2}{1 - |z|^2}
\]
for $z$ close enough to the unit circle. We can conduct the following estimate

\[
(1 - |z|^2)(g \circ \varphi)'(z) - g'(z) \log \frac{2}{1 - |\varphi(z)|^2}
\]

\[
\lesssim (1 - |z|^2)(g \circ \varphi)'(z) - g'(z) \log \frac{2}{1 - |z|^2}
\]

\[
\leq (1 - |z|^2) \log \frac{2}{1 - |z|^2} |\varphi'(z)||g'(\varphi(z)) - g'(z)|
\]

\[
+ (1 - |z|^2) \log \frac{2}{1 - |z|^2} |g'(z)||\varphi'(z) - 1|.
\]

The proof will be completed by the same argument as in the proof of Proposition 4.7. □

Remark. Question 1 concerns the subclass of the bounded Volterra operators, whose elements’ essential commutants contain all the composition operators. Question 2 concerns the subclass of the composition operators, whose elements’ essential commutants contain all the bounded Volterra operators. Answers to (Q1) are complete, but we can only find some sufficient or necessary conditions for (Q2). It seems very difficult to answer (Q2) completely, since the boundary behavior of a function either in $H^\infty$ or $B$ can be rather wild.

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C. Z. TONG, C. YUAN AND Z.H. ZHOU


(C. Z. Tong) Department of Mathematics, Hebei University Technology, Tianjin 300401, P.R. China.
ctong@hebut.edu.cn

(C. Yuan) School of Applied Mathematics, Guangdong University of Technology, Guangdong 510520, P.R. China
yuancheng1984@163.com

(Z.H. Zhou) School of Mathematics, Tianjin University, Tianjin 300354, P.R. China.
zhzhou@tju.edu.cn

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