Power bounded composition operators on weighted Dirichlet spaces

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Abstract. In this paper, we study power bounded composition operators on weighted Dirichlet spaces $D_{\alpha}$. As applications, we give the necessary and sufficient conditions for the composition operators to be Riesz operator on $D_{\alpha}$, when $C_{\varphi}$ is power bounded on $D_{\beta}$, for some $0 < \beta < \alpha$. For $\alpha > 1$, we completely characterize the Riesz composition operators on $D_{\alpha}$. Moreover, we investigate the functions $f \in D_{\alpha}$, when $f \circ \varphi_n$ is convergent or $\lim_{n \to \infty} f \circ \varphi_n = 0$, in $D_{\alpha}$. Some of the techniques developed in the paper are not new but lead to new results.

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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Let $\varphi$ be a function analytic on the unit disk such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. A composition operator on $H(\mathbb{D})$ is defined by $C_{\varphi} f = f \circ \varphi$ for every $f \in H(\mathbb{D})$.

An operator $T$, on a Hilbert space $H$, is called power bounded if $\{T^n\}$ is a bounded sequence in $B(H)$, the space of all bounded operators on $H$. Many authors studied the power bounded composition operators on different spaces, see [1, 2, 3, 4, 8, 15, 16]. In this paper, we study these operators on weighted Dirichlet spaces $D_{\alpha}$, when $-1 < \alpha < 1$.  

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The operator $T : H \rightarrow H$ is said to be a Riesz operator if
\[
\lim_{n \to \infty} \|T^n\|_e^{1/n} = 0.
\]
Where $\| \cdot \|_e$ denotes the essential norm on $H$. J. H. Shapiro and P. D. Taylor in [14] have shown that if $C_\varphi$ is compact on $H^2$, then $\varphi$ cannot have an angular derivative at any point of the boundary of the unit disk. Using Carleson measure techniques, MacCluer and Shapiro [11] proved the Shapiro-Taylor result in the more general setting of the weighted Dirichlet spaces, $D_\alpha$, and showed that, for composition operators $C_\varphi$ acting on $A^p_\alpha (\alpha > -1)$, the non-existence of the angular derivative for $\varphi$ is also sufficient condition for compactness of the composition operator $C_\varphi$. In this paper, we show that the Riesz composition operators, also, have a straight relationship with the angular derivative. Indeed, we prove that if $0 < \beta < \alpha$ and $C_\varphi$ is power bounded on $D_\beta$, then
\[
C_\varphi \text{ is a Riesz operator on } D_\alpha \iff \lim_{n \to \infty} \left( \min_{\zeta \in \partial D} d(\zeta, \varphi_n) \right)^{1/n} = \infty,
\]
where $\varphi_n$ denotes the n-th iterate of $\varphi$ and $d(\zeta, \varphi_n)$ is the angular derivative of $\varphi_n$ at $\zeta$. Moreover, we show that when $\alpha > 1$, the above statement holds without assuming the power boundedness of $C_\varphi$. In [5] and [13], some results about Riesz composition operators have been given.

Our manuscript is organized as follows: In section 3, we give the necessary and sufficient conditions for the power boundedness of composition operators on $D_\alpha$. In Theorem 3.2, the characterization is done by using Carleson measure. In Theorem 3.4 we give another characterization for the power boundedness of composition operators on $D_\alpha$ when $0 < \alpha < 1$. In section 4, we investigate the Riesz composition operators on $D_\alpha$. As an another application, for a power bounded composition operator $C_\varphi$ on $D_\alpha$, we characterize the following sets
\[
U_{c,\alpha}(\varphi) = \{ f \in D_\alpha : C_{\varphi_n}f \text{ is convergent} \}
\]
and
\[
U_{0,\alpha}(\varphi) = \{ f \in D_\alpha : \lim_{n \to \infty} \|C_{\varphi_n}f\| = 0 \}.
\]
Finally in section 5, we present several examples related to our results.

Throughout this paper, $A(z) \lesssim B(z)$ on a set $S$ means that there exists some positive constant $C$ such that for each $z \in S$, we have $A(z) \leq CB(z)$. Also we use the notation $A(z) \asymp B(z)$ on $S$, to say that there are some positive constants $C$ and $D$ such that $CB(z) \leq A(z) \leq DB(z)$ for each $z \in S$.

2. Preliminaries

Let $\alpha > -1$, the weighted Bergman space $A_\alpha$ is the space of all $f \in H(\mathbb{D})$ for which
\[
\|f\|_{A_\alpha}^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty,
\]
where $A$ is the normalized area measure on $\mathbb{D}$. Also, the space of all analytic functions on the unit disk $\mathbb{D}$, whose derivatives are in $A_\alpha$ with the norm given by

$$||f||^2_{A_\alpha} = ||f(0)||^2 + ||f'||^2_{A_\alpha},$$

is called the weighted Dirichlet space and is denoted by $D_{\alpha}$. These spaces with the above norms are Hilbert spaces. The space $D_{\alpha}$ is a reproducing kernel Hilbert space with kernel functions $K_w(z) = \sum_{k=0}^{\infty} \frac{w^k z^k}{(k+1)^{1-\alpha}}$ and $||K_w||^2_{A_\alpha} = \sum_{k=0}^{\infty} \frac{|w|^{2k}}{(k+1)^{1-\alpha}}$.

Which means that the functions $K_w$ are in $D_{\alpha}$ for all $w \in \mathbb{D}$ and $\langle f, K_w \rangle = f(w)$. Also evaluation of the derivative of functions in $D_{\alpha}$ at $w$ is a bounded linear functional and $\langle f, K'_w \rangle = f'(w)$, where by [6, Theorem 2.16]

$$K'_w(z) = \sum_{k=1}^{\infty} k \frac{w^k z^k}{(k+1)^{1-\alpha}}$$

and $||K'_w||^2_{A_\alpha} = \sum_{k=1}^{\infty} k^2 \frac{|w|^{2(k-1)}}{(k+1)^{1-\alpha}}$.

For $\alpha > 0$ we can see that

$$||K_w||^2_{A_\alpha} \asymp \frac{1}{(1 - |w|^2)^{\alpha}}$$

and $||K'_w||^2_{A_\alpha} \asymp \frac{1}{(1 - |w|^2)^{\alpha+2}}$.

The pseudohyperbolic distance between the points $z$ and $a$ in $\mathbb{D}$ is defined as $\rho(z, a) = |\varphi_a(z)|$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. The pseudohyperbolic disk with center $a$ and radius $r \in (0, 1)$ is

$$\Delta(a, r) = \{ z : \rho(z, a) < r \} = \varphi_a(\Delta(0, r)) = \varphi_a(\{ z : |z| < r \}).$$

For $\varphi$ an analytic self-map of the unit disk and $w \neq \varphi(0)$, a point of the plane, let $z_j(w)$ be the points of the disk for which $\varphi(z_j(w)) = w$, with their multiplicities. Let $\alpha > -1$, the generalized Nevanlinna counting function is

$$N_{\varphi, \alpha}(w) = \sum_j (1 - |z_j(w)|^2)^\alpha,$$

where we understand $N_{\varphi, \alpha}(w) = 0$ for $w$ which is not in $\varphi(\mathbb{D})$. For convenience, we introduce two notations:

- $\overset{u,c}{\longrightarrow}$, that is, the sequence $\{f_n\}$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$.
- $\overset{D_{\alpha}}{\longrightarrow}$, that is, the sequence $\{f_n\}$ converges to $f$ in the norm of $D_{\alpha}$.

The following theorems are key theorems of this paper, for the proofs see [6, Theorem 2.35, Theorem 2.44 and Theorem 2.51].

**Theorem 2.1.** (*Julia-Carathéodory Theorem*) For $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ analytic and $\zeta$ in $\partial \mathbb{D}$, the following are equivalent:

1. $d(\zeta, \varphi) = \lim_{z \rightarrow \zeta} (1 - |\varphi(z)|)/(1 - |z|) < \infty$,
2. $\varphi$ has finite angular derivative $\varphi'(\zeta)$ at $\zeta$.
3. Both $\varphi$ and $\varphi'$ have finite nontangential limits at $\zeta$, with $|\eta| = 1$ for $\eta = \lim_{r \rightarrow 1} \varphi(r\zeta)$. 

Moreover, when these conditions hold, we have \( \lim_{r \to 1} \varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta, \varphi) \bar{\zeta} \eta \) and \( d(\zeta, \varphi) \) is the nontangential limit \( \lim_{z \to \zeta}(1 - |\varphi(z)|)/(1 - |z|) \).

**Theorem 2.2** (Denjoy-Wolff Theorem). If \( \varphi \), not the identity and not an elliptic automorphism of \( \mathbb{D} \), is an analytic map of unit disk into itself, then there is a point \( w \) in \( \overline{\mathbb{D}} \) so that \( \frac{u \cdot \varphi}{\varphi_n} \to w \).

The point in the above theorem is called the Denjoy-Wolff point of \( \varphi \). Indeed, the Denjoy-Wolff point of \( \varphi \) can be described as the unique fixed point of \( \varphi \) in \( \overline{\mathbb{D}} \) with \( |\varphi'(a)| \leq 1 \), see [6, page 59].

**Theorem 2.3** (Change of Variable Theorem). If \( g \) and \( W \) are non-negative measurable functions on \( \mathbb{D} \) and \( \varphi \) is a holomorphic self-map of \( \mathbb{D} \), then

\[
\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2W(z)dA(z) = \int_{\varphi(\mathbb{D})} g(w)N_{\varphi,W}(w)dA(w).
\]

### 3. Conditions for Power Boundedness

In this section, we characterize the power bounded composition operators on weighted Dirichlet spaces \( \mathcal{D}_\alpha \), when \(-1 < \alpha < 1\). When \( \alpha \geq 1 \), the result is obvious. Indeed, if \( \alpha \geq 1 \), then \( \mathcal{D}_\alpha = A_{\alpha-2} \) and their norms are equivalent and

\[
(\frac{1}{1 - |\varphi_n(0)|^2})^\alpha \leq \|C_{\varphi_n}\|^2_{A_{\alpha-2}} \leq (\frac{1 + |\varphi_n(0)|^2}{1 - |\varphi_n(0)|^2})^\alpha.
\]

Therefore, if \( \varphi \) has a Denjoy-Wolff point, then \( C_{\varphi} \) is power bounded on \( \mathcal{D}_\alpha \), for \( \alpha \geq 1 \), if and only if the Denjoy-Wolff point of \( \varphi \) is in \( \mathbb{D} \). It is clear that if \( \varphi \) is the identity or an elliptic automorphism of \( \mathbb{D} \) and \( \alpha > -1 \), then \( C_{\varphi} \) is power bounded. Indeed there are some \( \lambda \in \partial \mathbb{D} \) and some disk automorphism \( \varphi_\alpha \) such that \( \psi = \varphi_\alpha \circ \varphi \circ \varphi_\alpha(z) = \lambda z \). So \( \psi_n = \varphi_\alpha \circ \varphi_n \circ \varphi_\alpha(z) = \lambda^n z \). Throughout this paper, \( \varphi \) is an analytic self-map of \( \mathbb{D} \) which is not the identity and not an elliptic automorphism, so \( \varphi \) has a Denjoy-Wolff point. Now we are going to prove our main results. First, we need the following lemma.

**Lemma 3.1.** (i) [7, Lemma 4, page 42] In each pseudohyperbolic disk \( \Delta(a, r) \), the function \( k_\alpha(z) = (1 - \bar{a}z)^{-2} \) satisfies the sharp inequalities

\[
\left(\frac{1 - |r|a}{1 - |a|^2}\right)^2 \leq |k_\alpha(z)| \leq \left(\frac{1 + |r|a}{1 - |a|^2}\right)^2, \quad \text{for all } z \text{ in } \Delta(a, r).
\]

(ii) [17, Proposition 4.5] If \( r \in (0, 1) \) is fixed and \( z \in \Delta(a, r) \), then

\[
A(\Delta(z, r)) \asymp (1 - |z|^2)^2 \asymp (1 - |a|^2)^2 \asymp A(\Delta(a, r)).
\]

(iii) [7, Lemma 12, page 62] For each pseudohyperbolic radius \( r \in (0, 1) \), there exists a sequence \( \{a_k\} \) of points in \( \mathbb{D} \) and an integer \( N \) such
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that

\[ \bigcup_{k=1}^{\infty} \Delta(a_k, r) = \mathbb{D} \]

and no point \( z \in \mathbb{D} \) belong to more than \( N \) of the dilated disk \( \Delta(a_k, R) \), where \( R = \frac{1}{2}(1 + r) \).

(iv) [7, Lemma 13, page 63] If \( 0 < r < 1 \), and \( f \) is analytic in \( \mathbb{D} \), then for arbitrary \( a \in \mathbb{D} \) and for all \( z \in \Delta(a, r) \),

\[ |f(z)|^2 \leq \frac{4(1 - R)^{-4}}{|\Delta(a, R)|} \int_{\Delta(a, R)} |f(\zeta)|^2 dA(\zeta), \quad \text{where } R = \frac{1}{2}(1 + r). \]

(v) [9, Theorem 1.7] Independently of \( a \) in \( \mathbb{D} \),

\[ \int_{\mathbb{D}} \frac{(1 - |z|^2)^c dA(z)}{|1 - az|^{2+c+d}} \asymp \frac{1}{(1 - |a|^2)^d}, \quad \text{if } d > 0, \ c > -1. \]

Theorem 3.2. Let \( \varphi \) be an analytic self-map of the unit disk which is not the identity or an elliptic automorphism. Then

(i) if \( 0 \leq \alpha < 1 \), then \( C_\varphi \) is power bounded on \( \mathcal{D}_\alpha \) if and only if \( \varphi \) has its Denjoy-Wolff point in \( \mathbb{D} \) and for every \( 0 < r < 1 \),

\[ \sup_{n \in \mathbb{N}, a \in \mathbb{D}} \frac{\int \Delta(a, r) N_{\varphi, a}(z) dA(z)}{(1 - |a|^2)^{\alpha+2}} < \infty; \tag{3.2} \]

(ii) if \( -1 < \alpha < 0 \), then \( C_\varphi \) is power bounded on \( \mathcal{D}_\alpha \) if and only if for all \( 0 < r < 1 \), Equation 3.2 holds.

Proof. (i): Let \( C_\varphi \) be power bounded on \( \mathcal{D}_\alpha \). Hence, there is some positive constant \( C \) such that for any \( f \) in the unit ball of \( \mathcal{D}_\alpha \) and \( n \in \mathbb{N} \), \( |f(\varphi_n(0))| < C \). Thus, if \( n \in \mathbb{N} \), then \( \| K_{\varphi_n} \| \leq C \). But we know that \( \lim_{|z| \to 1} \| K_z \| = \infty \), hence there exists some \( 0 < r < 1 \) such that \( \varphi_n(0) \in r\mathbb{D}, \ n \in \mathbb{N} \). If \( w \in \overline{\mathbb{D}} \) is the Denjoy-Wolff point of \( \varphi \), then \( \lim_{n \to \infty} \varphi_n(0) = w \). Therefore, \( w \) must be in \( \mathbb{D} \). Now we show that Equation 3.2 holds. Let

\[ f_a(z) = (1 - |a|^2)^{1+\alpha} \int_0^{|z|} \frac{d\zeta}{(1 - \overline{a}\zeta)^{2+\alpha}}. \]
Therefore, there is some $C > 0$. By using power boundedness of $C_\varphi$ and Lemma 3.1, part (v),
\[
\frac{\int_{\Delta(a,r)} N_{\varphi_n,\alpha}(z)dA(z)}{(1 - |a|^2)^{\alpha+2}} \lesssim \int_{\Delta(a,r)} \frac{(1 - |a|^2)^{\alpha+2}}{|\alpha z|^{4+2\alpha}} N_{\varphi_n,\alpha}(z)dA(z) 
\]
\[
\leq \int_{\mathbb{D}} \frac{(1 - |a|^2)^{\alpha+2}}{|\alpha z|^{4+2\alpha}} N_{\varphi_n,\alpha}(z)dA(z) \leq \|f_n \circ \varphi_n\|_\alpha^2 
\]
\[
\lesssim \|f_n\|_\alpha^2 = \int_{\mathbb{D}} \frac{(1 - |a|^2)^{\alpha+2}}{|\alpha z|^{4+2\alpha}} (1 - |z|^2)^\alpha dA(z) \approx 1. 
\]

Conversely, let $w$ in $\mathbb{D}$ be the Denjoy-Wolff point of $\varphi$ and Equation (3.1) holds. So, $\lim_{n \to \infty} \varphi_n(0) = w$. Thus, there is some $0 < r < 1$ such that $\{\varphi_n(0)\}_{n \in \mathbb{N}} \subseteq r\mathbb{D}$. Therefore, for $f$ in the unit ball of $\mathcal{D}_\alpha$
\[
|f(\varphi_n(0))|^2 \leq \|K_{\varphi_n(0)}\|_\alpha^2 \leq \|K_r\|_\alpha^2. 
\]

Let $\{a_k\}$ be the sequence in Lemma 3.1, part (ii). By using Lemma 3.1, Fubini’s theorem and Equation 3.2,
\[
\int_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \leq \sum_{k=1}^{\infty} \frac{1}{\Delta(a_k,r)} \int_{\Delta(a_k,r)} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) 
\]
\[
\lesssim \sum_{k=1}^{\infty} \frac{1}{(1 - |a_k|^2)^2} \int_{\Delta(a_k,r)} |f'(\zeta)|^2 N_{\varphi_n,\alpha}(z)dA(\zeta)dA(z) 
\]
\[
\lesssim \sum_{k=1}^{\infty} \int_{\Delta(a_k,R)} |f'(\zeta)|^2 \left(\frac{\Delta(a_k,r)}{(1 - |a_k|^2)^{\alpha+2}}\right) (1 - |\zeta|^2)^\alpha dA(\zeta) 
\]
\[
\lesssim \sum_{k=1}^{\infty} \int_{\Delta(a_k,R)} |f'(\zeta)|^2 (1 - |\zeta|^2)^\alpha dA(\zeta) \leq N. 
\]

Therefore, there is some $C > 0$ such that
\[
\|f \circ \varphi_n\|_\alpha^2 = |f(\varphi_n(0))|^2 + \int_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \leq \|K_r\|_\alpha^2 + CN.
\]

(ii): Let $f$ be in the unit ball of $\mathcal{D}_\alpha$. Then
\[
|f \circ \varphi_n(0)|^2 \leq \|K_{\varphi_n(0)}\|_\alpha^2 = \sum_{j=0}^{\infty} \frac{\varphi_n(0)|^2}{(j+1)^{1-\alpha}} \leq \sum_{j=0}^{\infty} \frac{1}{(j+1)^{1-\alpha}}.
\]
Since $\alpha < 0$
\[\sum_{j=0}^{\infty} \frac{1}{(j+1)^{1-\alpha}} < \infty.\]

Therefore, $C_\varphi$ is power bounded if and only if
\[
\sup_{n \in \mathbb{N}, f \in \text{Ball}_D} \int_D |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) < \infty.
\]

Similar to the proof of part $(i)$ we can show that the above inequality is equivalent to Inequality 3.2. \hfill \Box

By using the following proposition, we give a better characterization, Theorem 3.4, for the power boundedness of composition operators on $D_\alpha$, when $0 < \alpha < 1$.

**Proposition 3.3.** [12, Proposition 2.1] Let $0 < \alpha < 1$ and $0 < p < \infty$. Suppose that $\varphi$ be an analytic self-map of the unit disk. Then there is a positive constant $C = C_p < \infty$ such that
\[
N_{\varphi,\alpha}(\zeta)^p \leq \frac{C}{|B|} \int_B N_{\varphi,\alpha}(w)^p dA(w),
\]
where $\zeta \in \mathbb{D} \setminus \{\varphi(0)\}$ and $B$ is any Euclidean disk centered at $\zeta$ contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Moreover, one can take $C = 1$ if $p \geq 1$.

**Theorem 3.4.** Let $0 < \alpha < 1$ and $\varphi$ be an analytic self-map of the unit disk which is not the identity or an elliptic automorphism with $w$ as its Denjoy-Wolff point. Then $C_\varphi$ is power bounded on $D_\alpha$ if and only if

- $w$ is in $\mathbb{D}$,
- $\{\varphi_n\}$ is a bounded sequence in $D_\alpha$,
- there exists some $C > 0$ such that if $n \in \mathbb{N}$ and $|a| \geq \frac{1+|\varphi_n(0)|}{2}$, then
  \[
  N_{\varphi_n,\alpha}(a) \left(\frac{1}{1-|a|^2}\right)^\alpha < C.
  \]

**Proof.** Let $C_\varphi$ be power bounded. By using the preceding theorem, $w$ must be in $\mathbb{D}$. Since $\varphi_n = C_{\varphi_n} z$, the second condition also holds. For the third condition, suppose that $|a| > \frac{1+|\varphi_n(0)|}{2}$ and $D(a) = \{z : |z-a| < \frac{1}{2}(1-|a|)\}$. Easily we can see that every point in $D(a)$ has modulus greater than $|\varphi_n(0)|$. 


Therefore, by Proposition 3.3 and Lemma 3.1,
\[
N_{\varphi_n, \alpha}(a) \leq \frac{\int_{D(a)} N_{\varphi_n, \alpha}(z)dA(z)}{(1 - |a|^2)^{\alpha+2}} \leq \int_{D(a)} \frac{(1 - |a|^2)^{\alpha+2}}{|1 - \overline{a}z|^{4+2\alpha}}N_{\varphi_n, \alpha}(z)dA(z)
\leq \int_{D} \frac{(1 - |a|^2)^{\alpha+2}}{|1 - \overline{a}z|^{4+2\alpha}}N_{\varphi_n, \alpha}(z)dA(z)
\lesssim \int_{D} \frac{(1 - |a|^2)^{\alpha+2}}{|1 - \overline{a}z|^{4+2\alpha}}(1 - |z|^2)^{\alpha}dA(z) \asymp 1.
\]

Conversely, let the above conditions hold. Let \( f \) be in the unit ball of \( D_\alpha \). Then
\[
\int_{D} |f'(z)|^2 N_{\varphi_n, \alpha}(z)dA(z) = \int_{|z| \geq \frac{1 + |\varphi_n(0)|}{2}} |f'(z)|^2 N_{\varphi_n, \alpha}(z)dA(z)
+ \int_{|z| \leq \frac{1 + |\varphi_n(0)|}{2}} |f'(z)|^2 N_{\varphi_n, \alpha}(z)dA(z)
\leq C \int_{|z| \geq \frac{1 + |\varphi_n(0)|}{2}} |f'(z)|^2(1 - |z|^2)^{\alpha}dA(z)
+ \|K'_{1 + |\varphi_n(0)|}||^{2}_{\alpha} \int_{|z| \leq \frac{1 + |\varphi_n(0)|}{2}} N_{\varphi_n, \alpha}(z)dA(z)
\leq C + \|K'_{1 + |\varphi_n(0)|}||^{2}_{\alpha}||\varphi_n||^{2}_{\alpha}.
\]

Where the first two conditions of the theorem show that the last quantity is bounded above.

**Remark 3.5.** In Example 5.3, we present an analytic self-map of the unit disk which has its Denjoy-Wolff point in \( D \), but is not power bounded on \( D_\alpha \), when \( 0 < \alpha < 1 \). Also, we give another analytic self-map of the unit disk in Example 5.5 whose Denjoy-Wolff point is in the unit circle, however, it is power bounded on \( D_\alpha \), for \(-1 < \alpha < 0 \).

**Remark 3.6.** By Lemmas [10, Lemma 2.2 and Lemma 2.3], if \( \alpha > 0 \) and \( \varphi(0) = 0 \), then
\[
N_{\varphi, \alpha}(\zeta) \leq \frac{2\pi}{|B|} \int_{B} N_{\varphi, \alpha}(w)dA(w),
\]
where \( \zeta \in \mathbb{D} \setminus \{0\} \) and \( B \) is any Euclidean disk centered at \( \zeta \) contained in \( \mathbb{D} \setminus \{0\} \). Now if \( \varphi(0) \neq 0 \), then by [10, Lemma 2.1], there exists some positive
constant $C(\alpha)$ depending only on $\alpha$ such that
\[ N_{\varphi,\alpha}(\zeta) \leq \frac{C(\alpha)}{|B|(1-|\varphi(0)|^2)^{\alpha}} \int_B N_{\varphi,\alpha}(w) dA(w). \]

Therefore, by using an argument similar to the proof of Theorem 3.4, we can show that if $C_\varphi$ is power bounded on $D_\alpha$, $\alpha > 0$, then there exists some $C > 0$ such that if $n \in \mathbb{N}$ and $|a| \geq 1 + |\varphi_n(0)|^2$, then $N_{\varphi_n,\alpha}(a) < C$.

4. Applications

In this section, we give some applications of our results obtained from the preceding section.

4.1. Riesz composition operators. We denote by $\|\cdot\|_{e,\alpha}$ the essential norm of operators on $D_\alpha$. Pau and Perez in [12, Theorem 3.2], for $0 < \alpha < 1$, independently of $\varphi$, showed that
\[ \|C_\varphi\|_{e,\alpha}^2 \approx \limsup_{|z| \to 1} N_{\varphi,\alpha}(z) \frac{1}{(1-|z|)^\alpha}. \]

By using [6, page 136] and Remark 3.6 and with an argument similar to the proof of [12, Theorem 3.2] we can show that the above inequality is also true for $\alpha \geq 1$. Thus, if $\alpha > 0$, then $C_\varphi$ is a Riesz operator on $D_\alpha$ if and only if
\[ \lim_{n \to \infty} \left( \limsup_{|z| \to 1} N_{\varphi_n,\alpha}(z) \frac{1}{(1-|z|)^\alpha} \right)^{\frac{1}{n}} = 0. \]

**Theorem 4.1.** Let $0 < \beta < \alpha$ and $C_\varphi$ be power bounded on $D_\beta$. Then $C_\varphi$ is a Riesz operator on $D_\alpha$ if and only if
\[ \lim_{n \to \infty} \left( \min_{\zeta \in \partial \mathbb{D}} d(\zeta, \varphi_n) \right)^{\frac{1}{n}} = \infty. \]

**Proof.** ($\Leftarrow$): Let $r = \sup_{n \in \mathbb{N}} \frac{1 + |\varphi_n(0)|^2}{2}$, so $\frac{1}{2} < r < 1$. By using Remark 3.6, there is some $C > 0$ such that if $a \in \mathbb{D} \setminus r\mathbb{D}$, then
\[ N_{\varphi_n,\beta}(a) \leq C \frac{(1 - |a|^2)^\beta}{(1 - |a|^2)^\alpha}. \]

Also, let $z(a)$ be a point in $\mathbb{D}$ with minimum modulus where $\varphi(z(a)) = a$. Hence
\[ \limsup_{|a| \to 1} \frac{N_{\varphi_n,\alpha}(a)}{(1 - |a|)^\alpha} \leq \limsup_{|a| \to 1} \left( \frac{1 - |z(a)|^2}{1 - |a|^2} \right)^{\alpha-\beta} \frac{N_{\varphi_n,\beta}(a)}{(1 - |a|^2)^\beta} \leq C \frac{(1 - |z(a)|^2)^{\alpha-\beta}}{(1 - |\varphi_n(z)|^2)^{\alpha-\beta}} = C \min_{\zeta \in \partial \mathbb{D}} d(\zeta, \varphi_n)^{\alpha-\beta}. \]
Therefore,
\[
\lim_{n \to \infty} \left( \lim_{|a| \to 1} \frac{N_{\varphi_n,a}(a)}{(1-|a|)^{\alpha}} \right)^{\frac{1}{2n}} \leq \lim_{n \to \infty} \left( \min_{\zeta \in \partial \mathbb{D}} d(\zeta, \varphi_n) \right)^{-\frac{\alpha - \beta}{n}} = 0.
\]

(\Rightarrow): it is trivial by the known estimate
\[
\|C_{\varphi_n}\|_{e,\alpha} \geq \limsup_{|z| \to 1} \left( \frac{1 - |z|^2}{1 - |\varphi_n(z)|^2} \right)^{\frac{\alpha}{2}}.
\]

\[\square\]

**Corollary 4.2.** Let \( \alpha > 1 \) and \( \varphi \) be an analytic self-map of the unit disk which is not the identity or an elliptic automorphism. Then \( C_{\varphi} \) is a Riesz operator on \( D_{\alpha} \) if and only if Equation 4.2 holds.

**Proof.** We show that both of our conditions imply that \( \varphi \) has its Denjoy-Wolff point in \( \mathbb{D} \). So \( C_{\varphi} \) is power bounded on every \( D_{\beta} \), where \( \beta > 1 \). Then by using Theorem 4.1, the proof is complete. Let \( C_{\varphi} \) be a Riesz operator on \( D_{\alpha} \) and \( w \), the Denjoy-Wolff point of \( \varphi \), be in the unit circle. We can easily see that \( w \) is the Denjoy-Wolff point of any iterate function \( \varphi_n \) and \( d(\zeta, \varphi_n) \leq 1 \). Hence,
\[
\|C_{\varphi_n}\|_{e,\alpha} \geq \limsup_{|z| \to 1} \left( \frac{1 - |z|^2}{1 - |\varphi_n(z)|^2} \right)^{\frac{\alpha}{2}} = \left( \frac{1}{\min_{\zeta \in \partial \mathbb{D}} d(\zeta, \varphi_n)} \right)^{\frac{\alpha}{2}} \geq 1.
\]

This contradicts the assumption that \( C_{\varphi} \) is a Riesz operator. Thus, \( w \) is in \( \mathbb{D} \). Now let Equation 4.2 hold. Hence, the angular derivative of \( \varphi_n \) at any point of unit circle converges to infinity as \( n \to \infty \). Thus, again the Denjoy-Wolff point of \( \varphi \) cannot be in \( \partial \mathbb{D} \). \[\square\]

**4.2. Characterization of sets \( \mathcal{U}_{c,\alpha}(\varphi) \) and \( \mathcal{U}_{0,\alpha}(\varphi) \).** For a positive constant \( \delta \) and an analytic function \( f \) on \( \mathbb{D} \) we define
\[
\Omega_{\delta}(f) = \{ z \in \mathbb{D} : |f(z)|^2(1 - |z|^2)^{\alpha+2} \geq \delta \}.
\]

**Theorem 4.3.** Let \( \alpha > 0 \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \) with Denjoy-Wolff point \( w \) and let \( C_{\varphi} \) be power bounded on \( D_{\alpha} \). Then \( f \) is in \( \mathcal{U}_{c,\alpha}(\varphi) \) if and only if for each \( \delta > 0 \),
\[
\lim_{n \to \infty} \int_{\Omega_{\delta}(f')} \frac{N_{\varphi_n,a}(z)dA(z)}{(1 - |z|^2)^{\alpha+2}} = 0.
\]

Moreover, \( f \) is in \( \mathcal{U}_{0,\alpha}(\varphi) \) if and only if \( f(w) = 0 \) and equation 4.3 holds.
Proof. Let $f$ be in $D_\alpha$. Since $w$ is the Denjoy-Wolff point of $\phi$, we have $\underset{\phi \circ \phi_n \rightarrow f(w)}{\overset{u.c}{\longrightarrow}}$. Thus, $f$ is in $\mathcal{U}_{c,\alpha}(\phi)$ if and only if

$$\lim_{n \to \infty} \int_{D} |f'(z)|^2 N_{\phi_n,\alpha}(z) dA(z) = 0.$$ 

If for some $\delta > 0$, Equation 4.3 does not hold, then there is a sequence $\{n_k\}$ in $\mathbb{N}$ and some positive constant $\varepsilon$ such that for any $k \in \mathbb{N}$ we have

$$\int_{\Omega_\delta(f')} \frac{N_{\phi_{n_k},\alpha}(z) dA(z)}{(1 - |z|^2)^{\alpha+2}} > \varepsilon.$$ 

Thus

$$\int_{D} |f'(z)|^2 N_{\phi_{n_k},\alpha}(z) dA(z) \geq \int_{\Omega_\delta(f')} |f'(z)|^2 N_{\phi_{n_k},\alpha}(z) dA(z) \geq \delta \int_{\Omega_\delta(f')} \frac{N_{\phi_{n_k},\alpha}(z) dA(z)}{(1 - |z|^2)^{\alpha+2}} > \delta \varepsilon.$$ 

Conversely, let $f$ be in $D_\alpha$ such that Equation 4.3 holds. Let $\varepsilon > 0$ be arbitrary. We choose $0 < \delta < \varepsilon$ sufficiently small such that

$$\int_{\Omega_\delta(f')} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \varepsilon.$$ 

Now for this $\delta$, there is some $N \in \mathbb{N}$ such that for each $n \geq N$

$$\int_{\Omega_\delta(f')} \frac{N_{\phi_n,\alpha}(z) dA(z)}{(1 - |z|^2)^{\alpha+2}} < \varepsilon.$$ 

Thus,

$$\int_{\Omega_\delta(f')} |f'(z)|^2 N_{\phi_n,\alpha}(z) dA(z) \leq \|f\|^2 \int_{\Omega_\delta(f')} \frac{N_{\phi_n,\alpha}(z) dA(z)}{(1 - |z|^2)^{\alpha+2}} < \varepsilon \|f\|^2.$$
Also,

\[
\int_{\Omega_{\delta}(f')^c} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) = \int_{\Omega_{\delta}(f') \cap \mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) \\
+ \int_{\Omega_{\delta}(f') \setminus \mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) \\
< \delta \int_{\Omega_{\delta}(f') \cap \mathbb{D}} \frac{N_{\varphi_n,\alpha}(z)}{(1-|z|^2)^{\alpha+2}} dA(z) \\
+ C \int_{\Omega_{\delta}(f') \setminus \mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha dA(z) \\
\leq \varepsilon \frac{\|\varphi_n\|^2}{(1-r^2)^{\alpha+2}} + C\varepsilon.
\]

Therefore,

\[
\int_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) = \int_{\Omega_{\delta}(f')} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) \\
+ \int_{\Omega_{\delta}(f')^c} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) \\
\leq (\|f\|^2_{\alpha} + \|\varphi_n\|^2_{\alpha} + \varepsilon + C\varepsilon).
\]

\[\Box\]

5. Examples

A well-known fact is that if \( C_{\varphi} \) is compact on \( D_\alpha \) then \( \varphi \) has its Denjoy-Wolff point \( w \) in \( \mathbb{D} \). So for \( \alpha \geq 1 \), if \( C_{\varphi} \) is compact on \( D_\alpha \) then it is power bounded.

**Example 5.1.** Let \(-1 < \alpha < 0\) and \( \varphi \) be an analytic self-map of the unit disk. If \( C_{\varphi} \) is compact on \( D_\alpha \), then it is power bounded.

**Proof.** Since \( C_{\varphi} \) is compact, we have \( \varphi(\mathbb{D}) \subset \mathbb{D} \). Thus, there is some positive constant \( C \) such that

\[
\|K'_{\varphi_n(z)}\|_{\alpha}^2 \leq C, \quad \forall z \in \mathbb{D}, \forall n \in \mathbb{N}.
\]

Also, \( \frac{u.c}{\varphi_n} \xrightarrow{\varphi_n} w \), so \( \frac{u.c}{\varphi_n} \xrightarrow{} \pi \). Hence, there exists a \( D > 0 \) such that

\[
|\varphi'_n(\varphi(z))|^2 \leq D, \quad \forall z \in \mathbb{D}, \forall n \in \mathbb{N}.
\]

Finally, if \( f \) is in the unit ball of \( D_\alpha \), then

\[
\int_{\mathbb{D}} |f'(\varphi_{n+1}(z))|^2 |\varphi'_{n+1}(z)|^2 (1-|z|^2)^\alpha dA(z)
\]
Example 5.2. Let $\alpha > -1$, $\varphi$ be an analytic self-map of the unit disk, and $w$ be its Denjoy-Wolff point. If $C_{\varphi}$ is compact and power bounded on $D_\alpha$, then for any $f$ in $D_\alpha$, we have \( \frac{f \circ \varphi_n}{f(w)} \). Moreover, \( \mathcal{U}_{\alpha}(\varphi) = D_\alpha \) and \( \mathcal{U}_{0,\alpha}(\varphi) = \left\{ f \in D_\alpha : f(w) = 0 \right\} \). Indeed, if $f$ is in $D_\alpha$, then the sequence \( \{ f \circ \varphi_n \} \) is bounded and \( \frac{f \circ \varphi_n}{f(w)} \). Therefore, by the compactness of $C_{\varphi}$, \( \frac{f \circ \varphi_n}{f(w)} \) is in unit circle, then \( d(\zeta, \varphi_n) \geq 2^n \). Therefore, Corollary 4.2 implies that $C_{\varphi}$ is a Riesz operator on every $D_\alpha$, when $\alpha > 1$.

Example 5.3. Let $\varphi(z) = z^2$, so for any $n$ in $\mathbb{N}$, $\varphi_n(z) = z^{2^n}$ and
\[
\|\varphi_n\|^2 = (1 + 2^n)^{1-\alpha}.
\]
So $C_{\varphi}$ is not power bounded on $D_\alpha$, for $-1 < \alpha < 1$, however, since zero is the Denjoy-Wolff point of $\varphi$, $C_{\varphi}$ is power bounded on each $D_\alpha$, for $\alpha \geq 1$. Also, by using Schwartz Lemma
\[
2^n|z|^{2^n-1} = |\varphi_n'(z)| \leq \frac{1 - |\varphi_n(z)|^2}{1 - |z|^2}.
\]
Thus, if $\zeta$ is in unit circle, then $d(\zeta, \varphi_n) \geq 2^n$. Therefore, Corollary 4.2 implies that $C_{\varphi}$ is a Riesz operator on every $D_\alpha$, when $\alpha > 1$.

Example 5.4. Let $\varphi$ be a univalent self-map of $\mathbb{D}$ with Denjoy-Wolff point in $\mathbb{D}$. Then for any $\alpha \geq 0$, $C_{\varphi}$ is power bounded on $D_\alpha$. We can easily see that there is a positive constant $C$, independently of $\varphi$, such that if $z \in \mathbb{D}$, then
\[
1 - |z|^2 \leq \frac{C}{1 - |\varphi(0)|^2}(1 - |\varphi(z)|^2).
\]
Thus, there is a $D > 0$ such that for any $z \in \mathbb{D}$ and any $n \in \mathbb{N}$
\[
1 - |z|^2 \leq D(1 - |\varphi_n(z)|^2).
\]
Thus,
\[
\int_{\Delta(a,r)} (1 - |\varphi_n^{-1}(z)|^2)^\alpha dA(z) \leq \int_{\Delta(a,r)} (1 - |z|^2)^\alpha dA(z)
\]
\[
\frac{(1 - |a|^2)^{2+\alpha}}{(1 - |a|^2)^{2+\alpha}} \leq D^\alpha C_r.
\]

Example 5.5. Consider $\varphi(z) = \frac{1}{2}(1 + z)$. Then for any $-1 < \alpha < 0$, $C_{\varphi}$ is power bounded on $D_\alpha$. However, the Denjoy-Wolff point of $\varphi$ is $1 \in \partial \mathbb{D}$.

Proof. We can see that
\[
\varphi_n(z) = \sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^n} z.
\]
Hence, $\varphi_n(0) = 1 - \frac{1}{2\pi}$ and $\varphi_n'(z) = \frac{1}{2\pi}$. Let $f$ be in the ball of $D_\alpha$ so $f'$ is in the ball of $A_\alpha$. Therefore,

$$
\int_D |f'(\varphi_n(z))|^2 |\varphi_n'(z)|^2 (1 - |z|)^\alpha dA(z)
= \frac{1}{2^{2n}} \int_D |f'(\varphi_n(z))|^2 (1 - |z|)^\alpha dA(z)
\leq \frac{2^{\alpha+2}}{2^{2n}} \left( \frac{1}{1 - |\varphi_n(0)|^2} \right)^2
= 2^{\alpha+2} \left( \frac{1}{1 - \left| 1 - \frac{1}{2\pi} \right|^2} \right)^2
\leq 2^{\alpha+2}.
$$

□

References


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