Compactness of Hankel operators with conjugate holomorphic symbols on complete Reinhardt domains in $\mathbb{C}^2$

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Abstract. In this paper we characterize compact Hankel operators with conjugate holomorphic symbols on the Bergman space of bounded convex Reinhardt domains in $\mathbb{C}^2$. We also characterize compactness of Hankel operators with conjugate holomorphic symbols on smooth bounded pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$.

Contents

1. Introduction 1265
2. Preliminary lemmas 1267
3. Proof of Theorem 1 1270
4. Proof of Theorem 2 1272
References 1272

1. Introduction

We assume $\Omega \subset \mathbb{C}^2$ is a bounded convex Reinhardt domain. We denote the Bergman space with the standard Lebesgue measure on $\Omega$ as $A^2(\Omega)$. Recall that the Bergman space $A^2(\Omega)$ is the space of holomorphic functions on $\Omega$ that are square integrable on $\Omega$ under the standard Lebesgue measure. The Bergman space is a closed subspace of $L^2(\Omega)$. Therefore there exists an orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$ called the Bergman projection. The Hankel operator with symbol $\phi$ is defined as $H_\phi g = (I - P)(\phi g)$ for all $g \in A^2(\Omega)$. If $\phi \in L^\infty(\Omega)$, then $H_\phi$ is a bounded operator, however, the converse is not necessarily true. In one complex variable on the unit disk, Axler in [1] showed that the Hankel operator with conjugate holomorphic symbol $\phi$ is bounded if and only if $\overline{\phi}$ is in the Bloch space. There are unbounded, holomorphic functions in the Bloch space, as it only specifies a
growth rate of the derivative of the function near the boundary of the disk.
Namely, an analytic function \( \phi \) is in the Bloch space if
\[
\sup \{ (1 - |z|^2)|\phi'(z)| : z \in \mathbb{D} \} < \infty.
\]

Let \( h \in A^2(\Omega) \) so that the Hankel operator \( H_h^\tau \) is compact on \( A^2(\Omega) \). The Hankel operator with an \( L^2(\Omega) \) symbol may only be densely defined, since the product of \( L^2 \) functions may not be in \( L^2 \). However, if compactness of the Hankel operator is also assumed, then the Hankel operator with an \( L^2 \) symbol is defined on all of \( A^2(\Omega) \).

We wish to use the geometry of the boundary of \( \Omega \) to give conditions on \( h \). For example, if \( \Omega \) is the bidisk, Le in [5, Corollary 1] shows that if \( h \in A^2(\mathbb{D}^2) \) such that \( H_h^\tau \) is compact on \( A^2(\mathbb{D}^2) \) then \( h \equiv c \) for some \( c \in \mathbb{C} \).

In one variable, Axler in [1] showed that \( H_g^\tau \) is compact on \( A^2(\mathbb{D}) \) if and only if \( g \) is in the little Bloch space. That is, \( \lim_{|z| \to 1} (1 - |z|^2)|g'(z)| = 0 \).

If the symbol \( h \) is smooth up to the boundary of a smooth bounded convex domain in \( \mathbb{C}^2 \), Ćučković and Şahutoğlu in [3] showed that Hankel operator \( H_h^\tau \) is compact if and only if \( h \) is holomorphic along analytic disks in the boundary of the domain.

In this paper we will use the following notation.
\[
S_t = \{ z \in \mathbb{C} : |z| = t \},
\]
\[
T^2 = S_1 \times S_1 = \{ z \in \mathbb{C} : |z| = 1 \} \times \{ w \in \mathbb{C} : |w| = 1 \},
\]
\[
D_r = \{ z \in \mathbb{C} : |z| < r \}
\]
for any \( r, t > 0 \). If \( r = 1 \) we write
\[
\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}.
\]

We say \( \Delta \subset b\Omega \) is an analytic disk if there exists a function
\[
h = (h_1, h_2) : \mathbb{D} \to b\Omega
\]
so that each component function is holomorphic on \( \mathbb{D} \) and the image
\[
h(\mathbb{D}) = \Delta.
\]
An analytic disk is said to be trivial if it is degenerate (that is, \( \Delta = (c_1, c_2) \) for some constants \( c_1 \) and \( c_2 \)).

In [2] we considered bounded convex Reinhardt domains in \( \mathbb{C}^2 \). We characterized nontrivial analytic disks in the boundary of such domains.

We defined
\[
\Gamma_\Omega = \bigcup \{ \phi(\mathbb{D}) : \phi : \mathbb{D} \to b\Omega \text{ are holomorphic, nontrivial} \}
\]
and showed that
\[
\Gamma_\Omega = \Gamma_1 \cup \Gamma_2
\]
where either \( \Gamma_1 = \emptyset \) or
\[
\Gamma_1 = \overline{D}_{r_1} \times S_{s_1}
\]
and likewise either $\Gamma_2 = \emptyset$ or

$$\Gamma_2 = S_{s_2} \times \mathbb{D}_{r_2}$$

for some $r_1, r_2, s_1, s_2 > 0$.

**Remark 1.** We only consider domains in $\mathbb{C}^2$ as opposed to domains in $\mathbb{C}^n$ for $n \geq 3$ because a full geometric characterization of analytic structure in higher dimensions is unknown.

The main results are the following theorems.

**Theorem 1.** Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Let $f \in A^2(\Omega)$ so that $H_f$ is compact on $A^2(\Omega)$. If $\Gamma_1 \neq \emptyset$, then $f$ is a function of $z_2$ alone. If $\Gamma_2 \neq \emptyset$, then $f$ is a function of $z_1$ alone.

**Corollary 1.** Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Suppose $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. Let $f \in A^2(\Omega)$ so that $H_f$ is compact on $A^2(\Omega)$. Then there exists $c \in \mathbb{C}$ so that $f \equiv c$.

**Theorem 2.** Let $\Omega \subset \mathbb{C}^2$ be a $C^\infty$-smooth bounded pseudoconvex complete Reinhardt domain. Let $f \in A^2(\Omega)$ such that $H_f$ is compact on $A^2(\Omega)$. Suppose either of the following conditions hold:

1. There exists a holomorphic function $F = (F_1, F_2) : \mathbb{D} \to b\Omega$ so that both $F_1$ and $F_2$ are not identically constant.
2. $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.

Then $f \equiv c$ for some $c \in \mathbb{C}$.

2. **Preliminary lemmas**

As a bit of notation to simplify the reading, we will use the multi-index notation. That is, we will write

$$z = (z_1, z_2)$$

and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$$

and $|\alpha| = \alpha_1 + \alpha_2$. We say $\alpha = \beta$ if $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. If either $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$ we say $\alpha \neq \beta$.

It is well known that for bounded complete Reinhardt domains in $\mathbb{C}^2$, the monomials

$$\left\{ \frac{z^\alpha}{\|z^\alpha\|_{L^2(\Omega)}} : \alpha \in \mathbb{Z}^2_+ \right\}$$

form an orthonormal basis for $A^2(\Omega)$.

We denote

$$\frac{z^\alpha}{\|z^\alpha\|_{L^2(\Omega)}} = e_\alpha(z)$$
Definition 1. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$, we define

$$G_\beta := \left\{ \psi \in L^2(\Omega) : \psi(\zeta z) = \zeta^\beta \psi(z) \text{ a.e. } z \in \Omega \text{ a.e. } \zeta \in \mathbb{T}^2 \right\}.$$ 

Note this definition makes sense in the case $\Omega$ is a Reinhardt domain, and is the same as the definition of quasi-homogeneous functions in [5].

Lemma 1. Let $\Omega \subset \mathbb{C}^2$ be a bounded complete Reinhardt domain. $G_\alpha$ as defined above are closed subspaces of $L^2(\Omega)$ and for $\alpha \neq \beta$,

$$G_\alpha \perp G_\beta.$$ 

Proof. The proof that $G_\beta$ is a closed subspace of $L^2(\Omega)$ is similar to [5]. Without loss of generality, suppose $\alpha_1 \neq \beta_1$. Since $\Omega$ is a complete Reinhardt domain, one can 'slice' the domain similarly to [4]. That is,

$$\Omega = \bigcup_{z_2 \in H_\Omega} (\Delta_{|z_2|} \times \{z_2\})$$

where $H_\Omega \subset \mathbb{C}$ is a disk centered at 0 and

$$\Delta_{|z_2|} = \{ z \in \mathbb{C} : |z| < r_{|z_2|} \}$$

is a disk with radius depending on $|z_2|$. As we shall see, the proof relies on the radial symmetry of both $H_\Omega$ and $\Delta_{|z_2|}$.

Let $f \in G_\alpha$, $g \in G_\beta$, $z_1 = r_1 \zeta_1$, $z_2 = r_2 \zeta_2$ for $(\zeta_1, \zeta_2) \in \mathbb{T}^2$, and $r_1, r_2 \geq 0$. Then we have

$$\langle f, g \rangle$$

$$= \int_{\Omega} f(z) \overline{g(z)} dV(z)$$

$$= \int_{H_\Omega} \int_{0 \leq r_1 \leq r_{|z_2|}} \int_{\mathbb{T}} \zeta_1^{\alpha_1} \overline{\zeta_1^{\beta_1}} f(r_1, z_2) \overline{g(r_1, z_2)} r_1 d\sigma(\zeta_1) d\sigma(\zeta_1) dV(z_2).$$

Since $\alpha_1 \neq \beta_1$,

$$\int_{\mathbb{T}} \zeta_1^{\alpha_1} \overline{\zeta_1^{\beta_1}} d\sigma(\zeta_1) = 0.$$

This completes the proof.

In the case of a bounded convex Reinhardt domain in $\mathbb{C}^2$, one can use the 'slicing' approach in [4] to explicitly compute $P(\overline{z}^j e_n)$.

Lemma 2. Let $\Omega \subset \mathbb{C}^2$ be a bounded complete Reinhardt domain. Then the Hankel operator with symbol $\overline{z}^j \overline{w}^k$ applied to the orthonormal basis vector $e_n$ has the following form:

$$H_{\overline{z}^j e_n}(z) = \frac{\overline{z}^j z^n}{\|z^n\|}.$$
if either \( n_1 - j_1 < 0 \) or \( n_2 - j_2 < 0 \). If \( n_1 - j_1 \geq 0 \) and \( n_2 - j_2 \geq 0 \) then we can express the Hankel operator applied to the standard orthonormal basis as

\[
H_{\bar{z}^j} e_n(z) = \frac{\bar{z}^j z^n}{\| z^n \|} - \frac{z^{n-j} \| z^n \|}{\| z^{n-j} \|^2}.
\]

Furthermore, for any monomial

\[
\bar{w}^j w^n \in G_{n-j}
\]

the projection

\[
(I - P)(\bar{w}^j w^n) \in G_{n-j}.
\]

**Proof.** We have

\[
P(\bar{z}^j e_n)(z)
\]

\[
= \int \frac{\bar{w}^j w^n}{\| w^n \|} \sum_{l \in \mathbb{Z}^2} e_l(w) e_l(z) dV(z, w)
\]

\[
= \int_{H_\Omega} \int_{w_1 \in \Delta_{|w_2|}} \frac{\bar{w}_1^{j_1} w_2^{j_2} w_1^{n_1} w_2^{n_2}}{\| z^n \|} \\
\cdot \sum_{l_1, l_2 = 0}^\infty e_{l_1, l_2}(w_1, w_2) e_{l_1, l_2}(z_1, z_2) dA_1(w_1) dA_2(w_2)
\]

\[
= \sum_{l_1, l_2 = 0}^\infty \frac{z_1^{l_1} z_2^{l_2}}{\| z^n \| \| z^n \|^2} \int_{H_\Omega} \int_{w_1 \in \Delta_{|w_2|}} \frac{\bar{w}_2^{j_2 + l_2} w_1^{n_1}}{\| z_1^{n_1} \|} dA_1(w_1) dA_2(w_2).
\]

Converting to polar coordinates and using the orthogonality of \{e^{in\theta} : n \in \mathbb{Z}\} and the fact that

\[
\int_{w_1 \in \Delta_{|w_2|}} \frac{\bar{w}_1^{j_1 + l_1} w_1^{n_1}}{\| z_1^{n_1} \|} dA_1(w_1)
\]

is a radial function of \( w_2 \) and \( H_\Omega \) is radially symmetric, we have the only nonzero term in the previous sum is when \( n_2 - j_2 = l_2 \) and \( n_1 - j_1 = l_1 \). Therefore, we have \( P(\bar{w}^j e_n)(z) = 0 \) if \( n_2 - j_2 < 0 \) or \( n_1 - j_1 < 0 \). Otherwise, if \( n_2 - j_2 \geq 0 \) and \( n_1 - j_1 \geq 0 \), we have

\[
P(\bar{w}^j e_n)(z) = \frac{z^{n-j} \| z^n \|}{\| z^{n-j} \|^2}.
\]

Therefore, we have

\[
H_{\bar{w}^j} e_n(z) = \frac{\bar{z}^j z^n}{\| z^n \|} - \frac{z^{n-j} \| z^n \|}{\| z^{n-j} \|^2}
\]

if \( n_2 - k \geq 0 \) and \( n_1 - j \geq 0 \) otherwise

\[
H_{\bar{w}^j} e_n(z) = \frac{\bar{z}^j z^n}{\| z^n \|}
\]
if either $n_2 - k < 0$ or $n_1 - j < 0$. This also shows that the subspaces $G_\alpha$ remain invariant under the projection $(I - P)$, at least for monomial symbols. □

**Lemma 3.** For every $\alpha \geq 0$, the product Hankel operator

$$H_{z^\alpha}^* H_{z^\alpha} : A^2(\Omega) \to A^2(\Omega)$$

is a diagonal operator with respect to the standard orthonormal basis

$$\{e_j : j \in \mathbb{Z}_+^2\}.$$  

**Proof.** Assume without loss of generality, $j \neq l$. We have

$$\langle H_{z^\alpha}^* H_{z^\alpha} e_j, e_l \rangle = \langle H_{z^\alpha} e_j, H_{z^\alpha} e_l \rangle = \langle (I - P)(z^\alpha e_j), z^\alpha e_l \rangle.$$  

We have $z^\alpha e_j \in G_{j-\alpha}$, $z^\alpha e_l \in G_{l-\alpha}$. By Lemma 2,

$$(I - P)z^\alpha e_j \in G_{j-\alpha}.$$  

By Lemma 1, $G_\alpha$ are mutually orthogonal. Therefore,

$$\langle (I - P)(z^\alpha e_j), z^\alpha e_l \rangle = 0$$  

unless $j = l$. □

Using Lemma 2 and Lemma 3, let us compute the eigenvalues of $H_{z^\alpha}^* H_{z^\alpha}$.

Let us first assume $n - \alpha \geq 0$. We have

$$\langle H_{z^\alpha}^* H_{z^\alpha} e_n, e_n \rangle = \frac{\|z^\alpha z^n\|^2}{\|z^n\|^2} - \frac{\|z^{n-\alpha}\|^2}{\|z^n\|^2} \frac{\|z^\alpha z^n\|^2}{\|z^n\|^2}.$$  

If $n - \alpha < 0$, we have

$$\langle H_{z^\alpha}^* H_{z^\alpha} e_n, e_n \rangle = \frac{\|z^{n-\alpha}\|^2}{\|z^n\|^2}.$$  

**3. Proof of Theorem 1**

**Proof.** Assume $f \in A^2(\Omega)$ and $H_T$ is compact on $A^2(\Omega)$. Then, we can represent

$$f = \sum_{j, k=0}^{\infty} c_{j, k, f} z_1^j z_2^k$$  

almost everywhere (with respect to the Lebesgue volume measure on $\Omega$). Let

$$\{e_m : m \in \mathbb{Z}_+^2\}$$  

be the standard orthonormal basis for $A^2(\Omega)$. Then

$$\|H_T e_m\|^2 \to 0.$$  

as $|m| \to \infty$. Using the mutual orthogonality of the subspaces $G_\alpha$, we get
\[
\|H_f e_m\|^2 = \langle (I - P)(f e_m), (I - P)(f e_m) \rangle
\]
\[
= \left\langle \sum_{j,k=0}^\infty (I - P)(c_{j,k,f}z_1^j z_2^k e_m), \sum_{s,p=0}^\infty c_{s,p,f}z_1^s z_2^p e_m \right\rangle
\]
\[
= \sum_{j,k=0}^\infty \left\| H_{c_{j,k,f}z_1^j z_2^k} e_m \right\|^2
\]
\[
\geq \left\| H_{c_{j,k,f}z_1^j z_2^k} e_m \right\|^2
\]
for every $(j,k) \in \mathbb{Z}_+^2$. Taking limits as $|m| \to \infty$, we have
\[
\lim_{|m| \to \infty} \left\| H_{c_{j,k,f}z_1^j z_2^k} e_m \right\|^2 = 0
\]
for all $(j,k) \in \mathbb{Z}_+^2$. The Hankel operators
\[
H^*_{c_{j,k,f}z_1^j z_2^k} H_{c_{j,k,f}z_1^j z_2^k}
\]
are diagonal by Lemma 3, with eigenvalues
\[
\lambda_{j,k,m} = \left\| H_{c_{j,k,f}z_1^j z_2^k} e_m \right\|^2.
\]
This shows that
\[
H^*_{c_{j,k,f}z_1^j z_2^k} H_{c_{j,k,f}z_1^j z_2^k}
\]
are compact for every $(j,k) \in \mathbb{Z}_+^2$. Then
\[
H_{c_{j,k,f}z_1^j z_2^k}
\]
are compact on $A^2(\Omega)$.

Without loss of generality, assume $\Gamma_1 \neq \emptyset$. Then there exists a holomorphic function $F = (F_1, F_2) : \mathbb{D} \to b\Omega$ so that $F_2$ is identically constant and $F_1$ is nonconstant. Therefore, by [2], the composition
\[
c_{j,k,f}F_1(z)^j F_2(z)^k
\]
must be holomorphic in $z$. This cannot occur unless $\overline{c_{j,k,f}} = 0$ for $j > 0$. Therefore, using the representation
\[
f = \sum_{j,k=0}^\infty c_{j,k,f}z_1^j z_2^k
\]
we have $f = \sum_{k=0}^\infty c_{0,k,f}z_2^k$ almost everywhere. By holomorphicity of $f$ and the identity principle, this implies
\[
f \equiv \sum_{k=0}^\infty c_{0,k,f}z_2^k.
Hence $f$ is a function of only $z_2$. The proof is similar if $\Gamma_2 \neq \emptyset$. \hfill \Box

4. Proof of Theorem 2

Using the same argument in the proof of Theorem 1, one can show compactness of $H_T$ implies compactness of

$$H_{c_{j,k,f}z_1^jz_2^k}$$

for every $j, k \in \mathbb{Z}_+$. Hence by [3, Corollary 1], for any holomorphic function $\phi = (\phi_1, \phi_2) : \mathbb{D} \to \mathcal{B}_\Omega$, we have

$$c_{j,k,f}\phi_1^j\phi_2^k$$

must be holomorphic. If we assume condition two in Theorem 2, then it follows that $f \equiv c_{0,0,f}$. Assuming condition one in Theorem 2, we may assume $\phi_1$ and $\phi_2$ are not identically constant. Thus $c_{j,k,f} = 0$ for $j > 0$ or $k > 0$ and so $f \equiv c_{0,0,f}$.

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