Equivariant bundles and adapted connections

Indranil Biswas, Arjun Paul and Arideep Saha

Abstract. Given a complex manifold $M$ equipped with a holomorphic action of a connected complex Lie group $G$, and a holomorphic principal $H$–bundle $E_H$ over $X$ equipped with a $G$–connection $h$, we investigate the connections on the principal $H$–bundle $E_H$ that are (strongly) adapted to $h$. Examples are provided by holomorphic principal $H$–bundles equipped with a flat partial connection over a foliated manifold.

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1. Introduction

Let $X$ be a complex manifold, $G$ a connected complex Lie group and $\rho : G \times X \rightarrow X$ a holomorphic action of $G$ on $X$. The Lie algebra of $G$ is denoted by $\mathfrak{g}$. Let $p : E_H \rightarrow X$ be a holomorphic principal $H$–bundle, where $H$ is a complex Lie group. A $G$–connection on $E_H$ is a $\mathbb{C}$–linear map $h : \mathfrak{g} \rightarrow H^0(E_H, TE_H)^H$ such that for every $v \in \mathfrak{g}$, the vector field $dp \circ h(v)$ on $X$ coincides with the one defined by $v$ using the above action $\rho$. 

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I. BISWAS, A. PAUL AND A. SAHA (see Section 2.2). In [BP17], $G$–connections were investigated, in particular, a criterion was given for the existence of a $G$–connection.

Here we continue the investigations of $G$–connections. More precisely, we study the interactions of $G$–connections on $E_H$ with the holomorphic connections on the principal $H$–bundle $E_H$. There are two possible compatibility conditions between them which are called “adapted” and “strongly adapted” (see Section 3.1). To explain these conditions, if $h$ is given by a holomorphic action $\rho_E$ of $G$ on $E_H$, then a holomorphic connection $\eta$ on the principal $H$–bundle $E_H$ is adapted to $h$ if and only if $\eta$ is preserved by $\rho_E$; such an adapted connection $\eta$ is called strongly adapted if the image of the homomorphism $h$ is contained in the horizontal subbundle of $TE_H$ for the connection $\eta$.

The property of a holomorphic connection $\eta$ on a holomorphic principal $H$–bundle $E_H$ that it is strongly adapted to a $G$–connection $h$ on $E_H$ can also be formulated in the context of foliated manifolds and principal $H$–bundles on them equipped with a flat partial connection; the details are in Section 5.

2. Preliminaries

2.1. Atiyah bundle. Let $H$ be a complex Lie group. Its Lie algebra will be denoted by $\mathfrak{h}$. Let $X$ be a connected complex manifold and

$$p : E_H \rightarrow X$$

a holomorphic principal $H$–bundle over $X$. This means that $E_H$ is a complex manifold equipped with a holomorphic right action of $H$

$$a : E_H \times H \rightarrow E_H$$

such that

- $p \circ a = p \circ p_{E_H}$, where $p_{E_H}$ is the projection of $E_H \times H$ to $E_H$, and
- the map $(p_{E_H}, a) : E_H \times H \rightarrow E_H \times_X E_H$ is an isomorphism.

Note that the first condition means that the action of $H$ takes a fiber of $p$ to itself, so the image of the map $(p_{E_H}, a)$ is contained in the fiber product $E_H \times_X E_H$. The second condition above means that the action of $H$ on a fiber of $p$ is free and transitive.

The adjoint bundle for $E_H$

$$\text{ad}(E_H) := E_H \times^H \mathfrak{h} \rightarrow X$$

is the holomorphic vector bundle over $X$ associated to $E_H$ for the adjoint action of $H$ on the Lie algebra $\mathfrak{h}$.

The holomorphic tangent (respectively, cotangent) bundle of a complex manifold $Y$ will be denoted by $TY$ (respectively, $T^*Y$). The tangent bundle of a real manifold $Y$ will be denoted by $T^R Y$.

The Atiyah bundle for $E_H$

$$\text{At}(E_H) := (TE_H)/H \rightarrow E_H/H = X$$
is a holomorphic vector bundle over $X$ whose rank is $\dim X + \dim \mathfrak{h}$; see [At57]. Let

$$T_{E_H/X} \subset TE_H$$

be the relative tangent bundle for the projection $p$ in (2.1). The subbundle

$$(T_{E_H/X})/H \subset (TE_H)/H = At(E_H)$$

is identified with the adjoint vector bundle $ad(E_H)$. This identification is a consequence of the isomorphism of $T_{E_H/X}$ with the trivial vector bundle $E_H \times \mathfrak{h} \to E_H$ given by the action of $H$ on $E_H$. Therefore, the short exact sequence

$$0 \to T_{E_H/X} \to TE_H \xrightarrow{dp} p^*TX \to 0,$$

where $dp$ is the differential of $p$, produces a short exact sequence on $X$

(2.2) $$0 \to \text{ad}(E_H) \to \text{At}(E_H) \xrightarrow{dp} TX \to 0,$$

which is known as the Atiyah exact sequence for $E_H$. For simplicity, we have used the same notation $dp$ for the differential $TE_H \to p^*TX$ over $E_H$ as well as its descent $At(E_H) \to TX$ to $X$. A holomorphic connection on $E_H$ is a holomorphic homomorphism

(2.3) $$\eta : TX \to At(E_H)$$

such that $(dp) \circ \eta = \text{Id}_{TX}$, where $dp$ is the homomorphism in (2.2). For a holomorphic connection $\eta$ on $E_H$, the homomorphism

$$\bigwedge^2 TX \to \text{ad}(E_H), \ v \otimes w - w \otimes v \mapsto 2(\eta(v), \eta(w)) - \eta([v, w]),$$

where $v$ and $w$ are locally defined holomorphic sections of $TX$, produces a holomorphic section of $(\bigwedge^2 T^*X) \otimes \text{ad}(E_H)$. This holomorphic section of $(\bigwedge^2 T^*X) \otimes \text{ad}(E_H)$ is called the curvature of the connection $\eta$.

The vector bundle $TE_H \otimes p^*(TX)^*$ on $E_H$ has a natural action of $H$ given by the action of $H$ on $TE_H$ and the tautological action of $H$ on $p^*(TX)^*$. We note that a holomorphic connection on $E_H$ is an $H$–invariant holomorphic section of $TE_H \otimes p^*(TX)^*$.

2.2. $G$–connections on $E_H$. Let $G$ be a connected complex Lie group; its Lie algebra will be denoted by $\mathfrak{g}$. The identity element of $G$ will be denoted by $e$. Let

(2.4) $$\rho : G \times X \to X$$

be a holomorphic action of $G$ on $X$. Consider the holomorphic homomorphism

$$\rho' : At(E_H) \oplus (X \times \mathfrak{g}) \to TX, \ (v, w) \mapsto dp(v) - d'\rho(w),$$

where $dp$ is the homomorphism in (2.2), and

(2.5) $$d'\rho : X \times \mathfrak{g} \to TX, \ (x, v) \mapsto (dp)(e, x)(v, 0),$$
with \((dp)(e, x) : g \oplus T_x X \rightarrow T_x X\) being the differential of \(\rho\) at \((e, x) \in G \times X\). Define the subsheaf

\[
\text{At}_\rho(E_H) := (\rho')^{-1}(0) \subset \text{At}(E_H) \oplus (X \times g).
\]

Since the differential \(dp\) is surjective, it follows that \(\rho'\) is surjective. This implies that \(\text{At}_\rho(E_H)\) is a holomorphic subbundle of \(\text{At}(E_H) \oplus (X \times g)\). The vector bundle \(\text{At}_\rho(E_H)\) fits in a commutative diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \rightarrow & \text{ad}(E_H) & \rightarrow & \text{At}_\rho(E_H) & \rightarrow & X \times g & \rightarrow & 0 \\
\| & \downarrow J & & \downarrow d\rho' & & & & & \\
0 & \rightarrow & \text{ad}(E_H) & \rightarrow & \text{At}(E_H) & \rightarrow & TX & \rightarrow & 0
\end{array}
\]

where \(J\) (respectively, \(q\)) is given by the projection of \(\text{At}(E_H) \oplus (X \times g)\) to \(\text{At}(E_H)\) (respectively, \(X \times g\)). (See [BP17].)

A holomorphic \(G\)–connection on \(E_H\) is a holomorphic homomorphism of vector bundles

\[
h : X \times g \rightarrow \text{At}_\rho(E_H)
\]

such that \(q \circ h = \text{Id}_{X \times g}\), where \(q\) is the homomorphism in (2.7). The curvature of a \(G\)–connection \(h\)

\[
(s, t) \mapsto [h(s), h(t)] - h([s, t])
\]

is a holomorphic section

\[
\mathcal{K}(h) \in H^0(X, \text{ad}(E_H) \otimes \bigwedge^2(X \times g)^*) = H^0(X, \text{ad}(E_H)) \otimes \bigwedge^2 g^*.
\]

We will give examples of \(G\)–connection.

Let \(a : E_H \times H \rightarrow E_H\) be the action of \(H\) on the principal \(H\)–bundle \(E_H\).

A \(G\)–action on the principal bundle \(E_H\) is a holomorphic action of \(G\) on the total space of \(E_H\)

\[
\rho_E : G \times E_H \rightarrow E_H
\]

such that

1. \(p \circ \rho_E = \rho \circ (\text{Id}_G \times p)\), where \(p\) and \(\rho\) are the maps in (2.1) and (2.4) respectively, and
2. \(\rho_E \circ (\text{Id}_G \times a) = a \circ (\rho_E \times \text{Id}_H)\) as maps from \(G \times E_H \times H\) to \(E_H\) (this condition means that the actions of \(G\) and \(H\) on \(E_H\) commute).

An equivariant principal \(H\)–bundle is a holomorphic principal \(H\)–bundle with a \(G\)–action.

Let \(\rho_E : G \times E_H \rightarrow E_H\) be a \(G\)–action on \(E_H\). Consider the homomorphism

\[
\tilde{h} : E_H \times g \rightarrow TE_H
\]

given by the differential \(d\rho_E\) of the action \(\rho_E\); more precisely,

\[
\tilde{h}(z, v) = d\rho_E(e, z)(v, 0),
\]
so \(\tilde{h}\) is the homomorphism in (2.5) when \(X\) is substituted by \(E_H\). Since the actions of \(G\) and \(H\) on \(E_H\) commute, this homomorphism \(\tilde{h}\) produces a \(G\)–connection

\[
h_0 : X \times g \rightarrow \text{At}_p(E_H)
\]

on \(E_H\); the curvature of this \(G\)–connection \(h_0\) vanishes identically [BP17, p. 355, Lemma 4.1].

Let \(Y\) be a connected compact complex manifold such that \(TY\) is holomorphically trivial. Then \(Y\) is holomorphically isomorphic to \(G/\Gamma\), where \(G\) is a connected complex Lie group and \(\Gamma \subset G\) is a cocompact lattice [Wa54]; in fact, \(G\) is the connected component, containing the identity element, of the group of all holomorphic automorphisms of \(Y\). Consider the left–translation action of \(G\) on \(G/\Gamma = Y\). A \(G\)–connection on a holomorphic principal \(H\)–bundle \(E_H\) on \(Y\) is an usual holomorphic connection on the principal \(H\)–bundle.

### 2.3. Distributions under a flow.

Let \(Y\) be a connected \(C^\infty\) manifold and

\[
\mathcal{D} \subset T^\mathbb{R}Y
\]

a \(C^\infty\) subbundle. In other words, \(\mathcal{D}\) is a distribution on \(Y\). The fiber of \(\mathcal{D}\) over any point \(z \in Y\) will be denoted by \(\mathcal{D}_z\).

Let \(\xi\) be a \(C^\infty\) vector field on \(Y\). Given any point \(x \in Y\), there is an open neighborhood \(x \in U_x \subset Y\) and an open interval \(0 \in I_x \subset \mathbb{R}\), such that \(\xi\) integrates to a flow

\[
\Phi_x : U_x \times I_x \rightarrow Y.
\]

For any \(t \in I_x\), define

\[
\Phi_{x,t} : U_x \rightarrow Y, \quad z \mapsto \Phi_x(z,t).
\]

**Lemma 2.1.** The following two are equivalent:

1. For every \(x \in Y\) and \(z \in U_x\) as above,

\[
(d\Phi_{x,t})(z)(\mathcal{D}_z) = \mathcal{D}_{\Phi_x(z,t)}(z),
\]

where \(d\Phi_{x,t}(z) : T^\mathbb{R}Y \rightarrow T_{\Phi_x(z,t)}^\mathbb{R}Y\) is the differential of the map \(\Phi_{x,t}\) at \(z\).

2. \([\xi, \mathcal{D}] \subset \mathcal{D}\).

**Proof.** Let \(\mathcal{W}\) denote the space of all \(C^\infty\) 1–forms on \(Y\) that vanish on \(\mathcal{D}\). The first statement is equivalent to the statement that

\[
L_\xi(w) \in \mathcal{W} \quad \forall \ w \in \mathcal{W},
\]

where \(L_\xi\) denotes the Lie derivative with respect to the vector field \(\xi\).

First assume that

\[
[\xi, \mathcal{D}] \subset \mathcal{D}.
\]
To prove that (2.12) holds, take any \( w \in W \) and any \( C^\infty \) section \( \theta \) of \( \mathcal{D} \). We have
\[
(L_\xi (w))(\theta) = \xi (w(\theta)) - w(L_\xi \theta) = \xi (w(\theta)) - w([\xi, \theta]).
\]
Now, \( w(\theta) = 0 \), and \([\xi, \theta]\) is section of \( \mathcal{D} \) by (2.13). Hence \( (L_\xi (w))(\theta) = 0 \), which implies that (2.12) holds.

Now assume that (2.12) holds. To prove (2.13), let \( \theta \) be any \( C^\infty \) section of \( \mathcal{D} \). Take any \( w \in W \). We have
\[
w([\xi, \theta]) = w(L_\xi \theta) = \xi (w(\theta)) - (L_\xi w)(\theta).
\]
Now, \( w(\theta) = 0 \), and also \((L_\xi w)(\theta) = 0\) because \( L_\xi w \in W \) by (2.12). Hence (2.13) holds. \( \square \)

3. Connections and (strongly) adapted connections

3.1. Definitions. Let \( E_H \) be a holomorphic principal bundle over \( X \) such that \( E_H \) is equipped with a holomorphic connection \( \eta : TX \to At(E_H) \) (see (2.3)). Since \( At(E_H) = (TE_H)/H \), the image of \( \eta \) is a holomorphic distribution on \( E_H \); it is known as the horizontal distribution for the connection \( \eta \).

As before, a connected complex Lie group \( G \) acts holomorphically on \( X \). Given a holomorphic \( G \)-connection \( h : X \times g \to At(\rho(E_H)) \) on \( E_H \) (see (2.8)), the connection \( \eta \) is said to be adapted to \( h \) if

\[
(J \circ h)(X \times \{v\}), \eta(TX) \subset \eta(TX) \quad \forall \ v \in g,
\]

where \( J \) is the homomorphism in (2.7). Note that a \( C^\infty \) section of \( At(E_H) \) defines a \( H \)-invariant vector field on \( E_H \) of type \((1,0)\).

The connection \( \eta \) is said to be strongly adapted to \( h \) if it is adapted to \( h \), and furthermore

\[
\text{image}(J \circ h) \subset \text{image}(\eta). \tag{3.2}
\]

We will now give examples to show that the conditions in (3.1) and (3.2) are independent.

Consider the trivial action of the multiplicative group \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) on \( X \). Let \( E \) be a holomorphic principal \( GL(\mathbb{C}) \)-bundle on \( X \) admitting a holomorphic connection, for example \( E \) can be the trivial holomorphic principal \( GL(\mathbb{C}) \)-bundle \( X \times GL(\mathbb{C}) \) on \( X \). The center of \( GL(\mathbb{C}) \) is identified with \( \mathbb{C}^* \) by sending any \( c \in \mathbb{C}^* \) to \( c \cdot \text{Id}_{\mathbb{C}^r} \in GL(\mathbb{C}) \). Using this identification, the action of the center of \( GL(\mathbb{C}) \) on \( E \) produces an action of \( \mathbb{C}^* \) on \( E \). Since \( \mathbb{C}^* \) is in the center of \( GL(\mathbb{C}) \), the actions of \( \mathbb{C}^* \) and \( GL(\mathbb{C}) \) on \( E \) commute. If \( E' \) is the vector bundle of rank \( r \) associated to \( E \) by the standard representation of \( GL(\mathbb{C}) \), then this action of \( \mathbb{C}^* \) on \( E \) corresponds to the action of \( \mathbb{C}^* \) on \( E' \) as scalar multiplications. Let \( h \) be the holomorphic \( \mathbb{C}^* \)-connection on \( E \) given by this action of \( \mathbb{C}^* \) on \( E \) (see
Any holomorphic connection on the principal $GL(r, \mathbb{C})$–bundle $E$ is adapted to $h$. But (3.2) fails for every holomorphic connection on $E$.

Now take $X = \mathbb{C}^2$ and $G = \mathbb{C} = H$. Let $E_H$ be the trivial principal $\mathbb{C}$–bundle $\mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. Take $\rho$ to be the action of $\mathbb{C}$ on $\mathbb{C}^2$ defined by $(z, (x, y)) \mapsto (x + z, y)$, $z \in \mathbb{C}$, $(x, y) \in \mathbb{C}^2$.

This action of $\mathbb{C}$ on $X$ and the trivial action of $\mathbb{C}$ on $\mathbb{C}$ together define an action of $\mathbb{C}$ on $E_H = X \times \mathbb{C}$. Let $h$ be the holomorphic $\mathbb{C}$–connection on $E_H$ associated to this action of $\mathbb{C}$ on $E_H$ (see (2.11)). Let $D$ be the holomorphic connection on the principal $H$–bundle $E_H$ defined by $f \mapsto df + xf \cdot dy$, where $f$ is any holomorphic function on $\mathbb{C}^2$ (holomorphic sections of $E_H$ are holomorphic functions) and $d$ denotes the standard de Rham differential. Then (3.2) holds while (3.1) fails.

### 3.2. Equivariant bundles and adaptable connections

As in (2.10), take a $G$–action $\rho_E$ on $E_H$. As mentioned earlier, there is a natural $G$–connection on $E_H$ (3.3) $h_0 : X \times g \to At_\rho(E_H)$ corresponding to $\rho_E$.

Let $p_X : G \times X \to X$ be the natural projection. The action $\rho_E$ produces a holomorphic isomorphism of principal $H$–bundles

(3.4) \[ \beta : p_X^*E_H \to \rho^*E_H, \quad \beta(g, x)(z) = \rho_E(g, z) \]

for all $g \in G, x \in X$ and $z \in (E_H)_x$, where $\rho$ is the map in (2.4).

For any $g \in G$, let

(3.5) \[ \beta^g : E_H \to (\rho \circ j_g)^*E_H, \quad z \mapsto \beta(g, x)(z) = \rho_E(g, z) \]

for all $x \in X$ and $z \in (E_H)_x$. The map from the holomorphic connections on $E_H$ to the holomorphic connections on $(\rho \circ j_g)^*E_H$ induced by the above isomorphism $\beta^g$ will be denoted by $\beta^g_\ast$; note that $\beta^g_\ast$ is a bijection.

**Proposition 3.1.** A holomorphic connection $\eta$ on $E_H$ is adapted to the $G$–connection $h_0$ in (3.3) associated to $\rho_E$ if and only if for all $g \in G$,

(3.6) \[ (\rho \circ j_g)^*\eta = \beta^g_\ast(\eta) \]

(both are connections on the principal $H$–bundle $(\rho \circ j_g)^*E_H$).

**Proof.** First assume that $\eta$ is adapted to $h_0$. Take any $v \in \mathfrak{g}$. The flow on $E_H$ generated by $v$ sends any $t \in \mathbb{R}$ to the biholomorphism

(3.7) \[ F_t : E_H \to E_H, \quad z \mapsto \rho_E(\exp(tv), z). \]
Note that $F_t$ coincides with $\beta^{\exp(tv)}$ constructed in (3.5). Consider the $H$-invariant distribution

$$D^n := \image(\eta) \subset TE_H.$$  

Its fiber over any point $z \in E_H$ will be denoted by $D^n_z$. Since $\eta$ is adapted to $h_0$, from Lemma 2.1 it follows that

(3.7) \hspace{1cm} (dF_t)(z)(D^n_z) = D^n_{F_t(z)}  

for all $z \in E_H$ and $t \in \mathbb{R}$, where $(dF_t)(z) : T_z E_H \rightarrow T_{F_t(z)} E_H$ is the differential of the map $F_t$. Since the subset $\{\exp(tv)\}_{v \in \mathbb{R}, t \in \mathbb{R}} \subset G$ is dense in the analytic topology (recall that $G$ is connected), and also $F_t = \beta^{\exp(tv)}$, from (3.7) we conclude that (3.6) holds for all $g \in G$.

Now assume that (3.6) holds for all $g \in G$. This implies that (3.7) holds for all $z \in E_H$ and $t \in \mathbb{R}$. Consequently, from Lemma 2.1 we conclude that $\eta$ is adapted to $h_0$. \hfill $\Box$

Take any point $x \in X$. Define

$$\rho_x : G \rightarrow X, \hspace{1cm} g \mapsto \rho \circ j_g(x) = \rho(g, x).$$

Consider the map

$$\rho_{E,x} : G \times (E_H)_x \rightarrow \rho_x^* E_H, \hspace{1cm} (g, z) \mapsto \rho_E(g, z).$$

Since this $\rho_{E,x}$ is $H$–equivariant (recall that the actions of $G$ and $H$ on $E_H$ commute), it identifies the pulled back principal $H$–bundle $\rho_x^* E_H$ with the trivial principal $H$–bundle $G \times (E_H)_x \rightarrow G$. Let $D^0_x$ be the holomorphic connection on the principal $H$–bundle $\rho_x^* E_H$ induced by the trivial connection on $G \times (E_H)_x$ using the above isomorphism $\rho_{E,x}$. Note that $\rho_x^* E_H$ is identified with the restriction of $\rho^* E_H$ to $G \times \{x\}$, because $\rho_x$ is the restriction of $\rho$ to $G \times \{x\}$. Therefore, $\rho^* \eta|_{G \times \{x\}}$ is also a connection on $\rho_x^* E_H$.

**Proposition 3.2.** A holomorphic connection $\eta$ on $E_H$ is strongly adapted to the $G$–connection $h_0$ in (3.3) if and only if the following two hold:

1. For all $g \in G$,

   $$(\rho \circ j_g)^* \eta = \beta^0_g(\eta).$$

2. For every $x \in X$, the connection $D^0_x$ on $\rho_x^* E_H$ coincides with the connection $\rho^* \eta|_{G \times \{x\}}$.

**Proof.** First assume that $\eta$ is strongly adapted to $h_0$. Since $\eta$ is adapted to $h_0$, Proposition 3.1 says that $(\rho \circ j_g)^* \eta = \beta^0_g(\eta)$ for all $g \in G$. The given condition (3.2) implies that the connection $\rho^* \eta|_{G \times \{x\}}$.

The converse is similarly proved. Assume that the two statements in the proposition hold. From Proposition 3.1 we know that $\eta$ is adapted to $h_0$. The second condition in the proposition implies that (3.2) holds. \hfill $\Box$
4. Criterion for adapted connection

Let $\eta : TX \rightarrow \text{At}(E_H)$ be a holomorphic connection on $E_H$. Let

\begin{equation}
\tilde{\eta} : X \times g \rightarrow \text{At}(E_H) \oplus (X \times g)
\end{equation}

be the $O_X$–linear homomorphism defined by

$$(x, v) \mapsto (\eta(d'\rho(x, v)), (x, v)),$$

where $d'\rho$ is the homomorphism in (2.5). Since we have $(dp) \circ \eta = \text{Id}_{TX}$, where $dp$ is the homomorphism in (2.2), it follows immediately that the image of $\tilde{\eta}$ is contained in $\text{At}_\rho(E_H) := (\rho')^{-1}(0)$ (see (2.6)). The homomorphism $\tilde{\eta}$ evidently is a $G$–connection on $E_H$.

Let $K(\eta) \in H^0(X, \Omega_X^2 \otimes \text{ad}(E_H))$ be the curvature of the connection $\eta$, where $\Omega_X^2 = \bigwedge^2 T^*X$. For any $w \in T_x X$, let

\begin{equation}
i_w(K(\eta)(x)) \in (T^*X)_x \otimes \text{ad}(E_H)_x = (T^*X \otimes \text{ad}(E_H))_x
\end{equation}

be the contraction of $K(\eta)(x) \in (\Omega_X^2 \otimes \text{ad}(E_H))_x$ by the tangent vector $w \in T_x X$.

**Lemma 4.1.** The connection $\eta$ on $E_H$ is strongly adapted to the above constructed $G$–connection $\tilde{\eta}$ if and only if for all $v \in g$ and $x \in X$,

\begin{equation}
i_d'\rho(x,v)(K(\eta)(x)) = 0,
\end{equation}

where $d'\rho$ is defined in (2.5) (see (4.2) for the contraction).

**Proof.** From the construction of $\tilde{\eta}$ in (4.1) it follows immediately that the condition in (3.2) holds. We need to show that (3.1) holds if and only if (4.3) holds.

To prove this, we recall a construction of the curvature $K(\eta)$. Given a point $x \in X$ and holomorphic tangent vectors $v, w \in T_x X$, extend $v, w$ to vector fields $\tilde{v}, \tilde{w}$ of type $(1, 0)$ on some open neighborhood of the point $x$. Let $\tilde{v} = \eta(\tilde{v})$ and $\tilde{w} = \eta(\tilde{w})$ be the horizontal lifts of $\tilde{v}$ and $\tilde{w}$ respectively, for the connection $\eta$. Then

$$K(\eta)(x)(v, w) = ([\tilde{v}, \tilde{w}]_{\text{Vert}})|_{p^{-1}(x)},$$

where $[\tilde{v}, \tilde{w}]_{\text{Vert}}$ is the component of the Lie bracket $[\tilde{v}, \tilde{w}]$ in the vertical direction (for the projection $p$). We note that the section $([\tilde{v}, \tilde{w}]_{\text{Vert}})|_{p^{-1}(x)}$ of $TE_H/X$ over $p^{-1}(x)$ is $H$–invariant and hence it defines an element of the fiber $\text{ad}(E_H)_x$ over $x$; recall that $\text{ad}(E_H)$ is identified with $(TE_H/X)/H$. The element $([\tilde{v}, \tilde{w}]_{\text{Vert}})|_{p^{-1}(x)} \in \text{ad}(E_H)_x$ does not depend on the choice of the extensions $\tilde{v}$ and $\tilde{w}$ of $v$ and $w$ respectively. From this description of $K(\eta)$ it follows immediately that (3.1) holds if and only if (4.3) holds.

From the proof of Lemma 4.1 we have the following:

**Corollary 4.2.** The connection $\eta$ on $E_H$ is adapted to the above constructed $G$–connection $\tilde{\eta}$ if and only if the condition in (4.3) holds. In other words, the connection $\eta$ on $E_H$ is strongly adapted to $\tilde{\eta}$ if $\eta$ is adapted to $\tilde{\eta}$.
Take a \(\mathbb{C}\)-linear map
\[
\varphi_0 : \mathfrak{g} \longrightarrow H^0(X, \text{ad}(E_H)).
\] (4.4)
For any \(v \in \mathfrak{g}\), the section \(\varphi_0(v) \in H^0(X, \text{ad}(E_H))\) defines a holomorphic vertical tangent vector field on \(E_H\) for the projection \(p\). This vertical tangent vector field on \(E_H\) will be denoted by \(\varphi(v)\). Let \(U \subset X\) be an open subset and \(V\) a \(C^\infty\) vector field on \(U\) of type \((1, 0)\). Let \(V' = \eta(V)\) be the horizontal lift of \(V\) on \(p^{-1}(U)\) for the holomorphic connection \(\eta\) on \(E_H\). Let \(f_0\) be any \(C^\infty\) function on \(U\). Then \(V'(f_0 \circ p)\) is a \(H\)-invariant function on \(p^{-1}(U)\), and hence
\[
\varphi(v)(V'(f_0 \circ p)) = 0.
\] (5.5)
On the other hand,
\[
\varphi(v)(f_0 \circ p) = 0
\] (5.6)
because \(\varphi(v)\) is a vertical vector field. From (5.5) and (5.6) we conclude that
\[
[\varphi(v), V'](f_0 \circ p) = 0.
\]
In other words,
\[
[\varphi(v), V'] = [\varphi(v), V']_{\text{Vert}},
\] (5.7)
where \([\varphi(v), V']_{\text{Vert}}\) is the vertical component of \([\varphi(v), V']\). The vector field \([\varphi(v), V']\) is \(H\)-invariant because both \(\varphi(v)\) and \(V'\) are \(H\)-invariant. If \(f_1\) is a \(C^\infty\) function on \(U\), then note that
\[
[\varphi(v), (f_1 \circ p) \cdot V'] = (f_1 \circ p) \cdot [\varphi(v), V']
\]
because \(\varphi(v)(f_1 \circ p) = 0\). Clearly, the vector field \((f_1 \circ p) \cdot V'\) is the horizontal lift of the vector field \(f_1 \cdot V\) on \(U\) for the connection \(\eta\). From these observations we conclude that there is a homomorphism
\[
\tilde{\varphi} : \mathfrak{g} \otimes \mathbb{C} T_x X \longrightarrow \text{ad}(E_H)
\] (5.8)
that sends \(v \otimes w \in \mathfrak{g} \otimes T_x X\) to \([\varphi(v), V'](x)\), where \(V' = \eta(V)\) is the horizontal lift, with respect to the connection \(\eta\), of a vector field \(V\) defined on a neighborhood of the point \(x \in X\) with \(V(x) = w\). Note that \([\varphi(v), V'](x)\) does not depend on the choice of the extension \(V\) of \(w\).

The contraction in (4.2) produces a homomorphism
\[
\Pi : \mathfrak{g} \otimes \mathbb{C} T_x X \longrightarrow \text{ad}(E_H)
\] (5.9)
that sends \(v \otimes w \in \mathfrak{g} \otimes T_x X\) to
\[
i_w i_{d'(p(x,v))}(\mathcal{K}(\eta)(x)) \in \text{ad}(E_H)_x,
\]
which is the contraction of \(i_{d'(p(x,v))}(\mathcal{K}(\eta)(x)) \in (T^*X)_x \otimes \text{ad}(E_H)_x\) (see (2.5), (4.2)) by the tangent vector \(w \in T_x X\).
Theorem 4.3. Let $X$ be a complex manifold equipped with a holomorphic action of $G$ and $E_H$ a holomorphic principal $H$–bundle on $X$ equipped with a holomorphic connection $\eta$. Then there is a $G$–connection $h$ on $E_H$ such that $\eta$ is adapted to $h$ if and only if there is a homomorphism $\varphi_0$ as in (4.4) such that the homomorphism $\tilde{\varphi}$ in (4.8) coincides with the homomorphism $-\Pi$, where $\Pi$ is constructed in (4.9).

Proof. Let $h : g \rightarrow H^0(X, \text{At}_\rho(E_H))$ be a $G$–connection on $E_H$ such that $\eta$ is adapted to $h$. For any $v \in g$, consider $J \circ h(v) - \eta(v') \in H^0(X, \text{At}(E_H))$, where $J$ is the homomorphism in (2.7) and $v'$ is the holomorphic vector field on $X$ defined by $x \mapsto d'\rho(x, v)$ (see (2.5)). Note that $dp \circ J \circ h(v) = v'$, where $dp$ is the homomorphism in (2.2). Therefore, we have $J \circ h(v) - \eta(v') \in H^0(X, \text{ad}(E_H)) \subset H^0(X, \text{At}(E_H))$ (see (2.7)). Now define $\varphi_0 : g \rightarrow H^0(X, \text{ad}(E_H)), \ v \mapsto J \circ h(v) - \eta(v')$.

We will show that the homomorphism $\tilde{\varphi}$ in (4.8) for this $\varphi_0$ coincides with the homomorphism $-\Pi$.

Take any $v \in g$. Given any $x \in X$ and any $w \in T_x X$, let $V$ be any $C^\infty$ vector field of type $(1, 0)$, defined on an open neighborhood of $x \in X$, such that $[v', V] = 0$.

Since $\eta$ is adapted to $h$, the Lie bracket $[J \circ h(v), \eta(V)]$ lies in the horizontal subbundle $\eta(TX) \subset TE_H$. In other words, the vertical component of $[J \circ h(v), \eta(V)]$ vanishes identically.

The Lie bracket $[\eta(v'), \eta(V)]$ is vertical because $dp([\eta(v'), \eta(V)]) = [v', V] = 0$.

From (4.7) we know that the Lie bracket $[\varphi(v), \eta(V)]$ is vertical, where $\varphi(v)$ is the vertical vector field corresponding to $\varphi_0(v) \in H^0(X, \text{ad}(E_H))$.

This and the fact that $[\eta(v'), \eta(V)]$ is vertical together imply that

$$[\varphi(v) + \eta(v'), \eta(V)] = [J \circ h(v), \eta(V)]$$

is vertical. But it was shown above that the vertical component of $[J \circ h(v), \eta(V)]$ vanishes identically. Hence we conclude that $[J \circ h(v), \eta(V)] = 0$.

Consequently, we have

$$[\varphi(v), \eta(V)] = -[\eta(v'), \eta(V)]$$
for all \( v \in g \). Since \([\varphi(v), \eta(V)] = \tilde{\varphi}(v \otimes V)\) and \([\eta(v'), \eta(V)] = \Pi(v \otimes V)\), from (4.11) it follows that

\[
\tilde{\varphi} = -\Pi.
\]

To prove the converse, take any homomorphism \( \varphi_0 \) as in (4.4) such that

\[
(4.12) \quad \tilde{\varphi} = -\Pi.
\]

Now define a \( G \)-connection

\[
h : g \rightarrow H^0(X, At_\rho(E_H)), v \mapsto (\varphi_0(v) + \eta(v'), X \times \{v\}).
\]

We will show that \( \eta \) is adapted to \( h \).

Let \( h : g \rightarrow H^0(X, At_\rho(E_H)) \) be a \( G \)-connection on \( E_H \). Take any \( \theta \in C^\infty(X, At(E_H)) \), where \( a \) and \( b \) are nonnegative integers. Note that \( \theta \) defines a \( H \)-invariant section of the vector bundle \((TE_H)^{\otimes a} \otimes (T^*E_H)^{\otimes b}\) on \( E_H \); this section of \((TE_H)^{\otimes a} \otimes (T^*E_H)^{\otimes b}\) will be denoted by \( \Theta \). We say that \( \theta \) is preserved by \( h \) if

\[
L_{J\circ h(v)} \Theta = 0 \quad \forall \quad v \in g,
\]

where \( L_\mathcal{J}(v) \) is the Lie derivative with respect to the vector field \( J \circ h(v) \) on \( E_H \) (the homomorphism \( J \) is constructed in (2.7)).

If \( h \) is the \( G \)-connection associated to a \( G \)-action \( \rho_E \) on \( E_H \), then it is straight-forward to check that \( \theta \) is preserved by \( h \) if and only if the section \( \Theta \) is preserved by the action \( \rho_E \) on \( E_H \).

5. Holomorphic foliations and strongly adapted connections

As before, \( X \) is a complex manifold. Let

\[
\mathcal{F} \subset TX
\]

be a holomorphic foliation on \( X \), which means that \( \mathcal{F} \) is a holomorphic subbundle of \( TX \) such that for any two sections \( s \) and \( t \) of \( \mathcal{F} \) defined over some open subset of \( X \), the Lie bracket \([s, t]\) is also a section of \( \mathcal{F} \) [La77].

Let \( E_H \) be a holomorphic principal \( H \)-bundle on \( X \).

Consider the Atiyah exact sequence for \( E_H \) in (2.2). Define

\[
At_\mathcal{F}(E_H) := (dp)^{-1}(\mathcal{F}) \subset At(E_H).
\]
So, from (2.2) we have the short exact sequence of holomorphic vector bundles

\[(5.1) \quad 0 \rightarrow \text{ad}(E_H) \rightarrow \text{At}_F(E_H) \xrightarrow{dp} \mathcal{F} \rightarrow 0, \]

where \( \widetilde{dp} \) is the restriction of \( dp \) to \( \text{At}_F(E_H) \). A holomorphic partial connection on \( E_H \) is a homomorphism

\[ D : \mathcal{F} \rightarrow \text{At}_F(E_H) \]

such that \( \widetilde{dp} \circ D = \text{Id}_F \) [La77].

Given such a holomorphic partial connection \( D \), the homomorphism

\[ \bigwedge^2 \mathcal{F} \rightarrow \text{ad}(E_H), \quad v \otimes w - w \otimes v \mapsto 2([D(v), D(w)] - D([v, w])), \]

where \( v \) and \( w \) are locally defined holomorphic sections of \( \mathcal{F} \), produces a holomorphic section of \( \bigwedge^2 \mathcal{F}^* \otimes \text{ad}(E_H) \). This holomorphic section of \( \bigwedge^2 \mathcal{F}^* \otimes \text{ad}(E_H) \) is called the curvature of the partial connection \( D \). A holomorphic partial connection is called flat if its curvature vanishes identically.

Let \( \eta : TX \rightarrow \text{At}(E_H) \) be a holomorphic connection on the principal \( H \)-bundle \( E_H \). As before, the curvature of \( \eta \) will be denoted by \( \mathcal{K}(\eta) \). Let

\[ D : \mathcal{F} \rightarrow \text{At}_F(E_H) \]

be a flat holomorphic partial connection on \( E_H \).

The connection \( \eta \) is said to be strongly adapted to \( D \) if

- the restriction \( \eta|_F : \mathcal{F} \rightarrow \text{At}(E_H) \) coincides with \( D \), and
- for any \( x \in X \) and \( w \in \mathcal{F}_x \), the contraction

\[ i_w \mathcal{K}(\eta)(x) \in T^*_x X \otimes \text{ad}(E_H)_x \]

vanishes.

**Corollary 5.1.** Suppose that \( \mathcal{F} \) is given by a holomorphic action \( \rho \) of a connected complex Lie group \( G \) on \( X \) (so the leaves of \( \mathcal{F} \) are the orbits of \( G \)), and also assume that \( D \) is given by a \( G \)-action \( \rho_E \) on \( E_H \) (so the tangent spaces to the leaves in \( E_H \) are the horizontal subspaces). Then \( \eta \) is strongly adapted to \( D \) if and only if \( \eta \) is strongly adapted to the \( G \)-connection on \( E_H \) given by \( \rho_E \).

**Proof.** The above condition that \( \eta|_F = D \) is equivalent to the condition that the \( G \)-connection \( \tilde{\eta} \) constructed in (4.1) from \( \eta \) coincides with the \( G \)-connection on \( E_H \) given by the above \( G \)-action \( \rho_E \). Therefore, the result follows from Lemma 4.1. \( \square \)

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