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NODES ON SEXTIC HYPERSURFACES IN $\mathbb{P}^3$

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In this note we present a coding theory result which, together with Theorem 3.6.1 of [3], gives a short proof of a theorem of D. Jaffe and D. Ruberman:

Theorem [5]. *A sextic hypersurface in* $\mathbb{P}^3$ *has at most 65 nodes.*

W. Barth [1] has constructed an example with 65 nodes. Following V. Nikulin [7] and A. Beauville [2], one must limit the size of an even set of nodes, and then prove a result about binary linear **codes** (i.e., linear subspaces of $\mathbb{F}^n$, where $\mathbb{F}$ is the field of two elements). The first step is the aforementioned result of Casnati–Catanese:

Theorem [3]. *On a sextic hypersurface, an even set of nodes has cardinality 24, 32 or 40.*

The desired theorem will follow from:

Theorem A. *Let* $V \subset \mathbb{F}^{66}$ *be a code, with weights from among 24, 32 and 40. Then* $\dim(V) \leq 12$.

1. Codes from nodal hypersurfaces

(1.1) Let $\Sigma \subset \mathbb{P}^3$ be a hypersurface of degree d having only μ ordinary double points as singularities. Let $\pi : S \to \Sigma$ be the minimal resolution of the singularities, with exceptional (-2)-curves E_i. Thus

$$E_i \cdot E_j = -2\delta_{ij}. \tag{1.1.1}$$

S is diffeomorphic to a smooth hypersurface of degree d.

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(1.2) The classes $[E_i]$ in $H^2(S;\mathbb{Z})$ span a not necessarily primitive sublattice of rank μ. A subset $I \subset \{1, 2, \ldots, \mu\}$ for which $\Sigma[E_i]$ $(i \in I)$ is divisible by 2 in $H^2(S;\mathbb{Z})$ (and therefore in $\mathrm{Pic}(S)$) is called **even** (or strictly even in [4]). More generally, consider for any subset I the homomorphism

$$\varphi : \mathbb{F}^I \to H^2(S, \mathbb{F}),$$

associating to each standard basis vector e_i the mod 2 class of $[E_i]$. We define the code

$$\mathrm{Code}(I) \equiv \mathrm{Ker}(\varphi).$$

A non-0 element corresponds exactly to an even subset J of I; the **weight** of such a "word" is its number of non-zero entries, i.e., $|J|$. $\mathrm{Im}(\varphi)$ is totally isotropic by (1.1.1); thus, $\dim(\mathrm{Im}(\varphi)) \leq \frac{1}{2}b_2(S)$, whence

$$\dim \mathrm{Code}(I) \geq \mathrm{Card}(I) - \frac{1}{2}b_2(S). \tag{1.4.1}$$

In particular, when $\mu > \frac{1}{2}b_2(S)$ one has a non-trivial code.

(1.5) It is an interesting question to determine for each d the possible cardinality t of an even set of nodes. By studying the corresponding double cover, one finds: For $d = 4$, one has $t = 8$ or 16 [7]; for $d = 5$, $t = 16$ or 20 [2]. The recent Theorem 3.6.1 of [3] proves that for $d = 6$, one has $t = 24, 32$ or 40. Since b_2 of a smooth sextic is 106, the result of [3] becomes

Theorem 1.6. *Let $\Sigma \subset \mathbb{P}^3$ be a nodal sextic hypersurface with at least μ nodes. Then there is a code $V \subset \mathbb{F}^\mu$ of dimension $\geq \mu - 53$, all of whose weights are among* $\{24, 32, 40\}$.

Let I be any set of μ nodes. This result plus our Theorem A will imply the 65-node bound for sextics.

2. Proof of Theorem A

(2.1) The $\mathbb{F}$-inner product on $\mathbb{F}^n$ (counting mod 2 the number of overlaps of two words) makes $V^* \subset \mathbb{F}^n$. V is called **even** if all words have even weight, **double even** if the weights are divisible by 4. Every doubly even code is automatically isotropic, i.e., $V \subset V^*$ (use (2.8.1) below). Since $\dim(V) = \dim(\mathbb{F}^n/V^*)$, a doubly even code satisfies $2d \leq n$ with equality iff the code is self-dual $(V = V^*)$. The element $\mathbb{1} \in \mathbb{F}^n$ has a 1 in every position.

(2.2) Let $V \subset \mathbb{F}^n$ be a d-dimensional code with $a_i = a_i(V)$ words of weight i. We have the simple equations

$$\Sigma a_i = 2^d - 1, \tag{2.2.1}$$

$$\Sigma i a_i = n' \cdot 2^{d-1}, \tag{2.2.2}$$

where $n' \leq n$ is the number of entries containing 1's from words of V. (2.2.1) is just an enumeration of $V - \{0\}$. For (2.2.2) list all 2^d elements of V as rows of a $2^d \times n$ matrix of 0's and 1's. n' columns contains at least one 1; since V is a subspace, exactly half the entries are 1's. Now count the total number of 1's via rows or columns. If $n' = n$, we say $V \subset \mathbb{F}^n$ is a **spanning code**.

(2.3) For a striking generalization of (2.2.1) and (2.2.2), define the **weight enumerator** of the code V as

$$W_V(x, y) = \Sigma a_i x^{n-i} y^i$$

with $a_0 = 1$. W is homogeneous of degree d. The **MacWilliams identity** (e.g., [6]) states that the enumerator of the dual code V^* is

$$W_{V^*}(x, y) = \left(\frac{1}{2^d}\right) W_V(x + y, x - y). \tag{2.3.1}$$

Writing the coefficents of W_{V^*} as $a_i^* = a_i^*(V)$, (2.3.1) takes the form

$$\Sigma a_i^* x^{n-i} y^i = \left(\frac{1}{2^d}\right) \cdot \{(x + y)^d + \Sigma a_i (x + y)^{n-i} (x - y)^i\}. \tag{2.3.2}$$

Equations (2.2.1) and (2.2.2) are respectively the statements $a_0^* = 1$ and a_1^* (=number of entries not appearing in V)$= n - n'$. More generally, we deduce the

Lemma 2.4. *Let $V \subset \mathbb{F}^n$ be a d-dimensional code. Then*

$$\Sigma a_i = 2^d - 1, \tag{2.4.1}$$

$$\Sigma i a_i = 2^{d-1}(n - a_1^*). \tag{2.4.2}$$

(2.4.3) *If $a_1^* = 0$, then*

$$\Sigma i^2 a_i = 2^{d-1}\{a_2^* + n(n + 1)/2\}.$$

(2.4.4) *If $a_1^* = 0$, then*

$$\Sigma i^3 a_i = 2^{d-2}\{3(a_2^* n - a_3^*) + n^2(n + 3)/2\}.$$

Proof. Expand the right-hand side of (2.3.2), carefully.

Lemma 2.5. *If $V \subset \mathbb{F}^n$ is a d-dimensional spanning code with only one weight w, then there is an integer $s > 0$, so that $w = s \cdot 2^{d-1}$ and $n = s(2^d - 1)$.*

Proof. Use (2.2.1) and (2.2.2) and fact that 2^{d-1} and $2^d - 1$ are relatively prime.

Lemma 2.6. *If $V \subset \mathbb{F}^n$ is a spanning code with weights 24 and 32, then $n \leq 63$ and $d \leq 9$.*

Proof. Solving (2.2.1) and (2.2.2), one finds

$$\begin{aligned} a_{24} =& 2^{d-4}(64 - n) - 4, \\ a_{32} =& 2^{d-4}(n - 48) + 3. \end{aligned}$$

Since $a_{24} \geq 0$, one has $n \leq 63$. Next, by (2.4.3), 2^{d-1} divides

$$24^2 a_{24} + 32^2 a_{32} = 2^8\{2^{d-6} \cdot 9 \cdot (2^6 - n) + 2^{d-2} \cdot (n - 48) + 3\}.$$

So, if $d \geq 8$, then $d \leq 9$. (Of course, there are many more restrictions.)

(2.7) Suppose $V \subset \mathbb{F}^n$ is a d-dimensional spanning code with weights among $\{24, 32, 40\}$. We solve equations (2.4.1)–(2.4.3) for the a_i's; writing $z = n(n+1)/2 + a_2^*$, we find

$$\begin{aligned} a_{24} =& 2^{d-8}\{z - 9 \cdot 2^3 n + 5 \cdot 2^9\} - 10, \\ a_{32} =& 2^{d-7}\{-z + 2^6 n - 15 \cdot 2^7\} + 15, \\ a_{40} =& 2^{d-8}\{z - 7 \cdot 2^3 n + 3 \cdot 2^9\} - 6. \end{aligned}$$

One can thus compute that

$$\Sigma i^3 a_i = 2^{d+4}\{3z - 2 \cdot 47n + 3 \cdot 5 \cdot 2^7\} - 2^{11} \cdot 3 \cdot 5.$$

By (2.4.4), this expression is divisible by 2^{d-2}; we conclude that

$$d \leq 13 \tag{2.7.1}$$

Equating with (2.4.4) and simplifying yield

$$\begin{aligned} 3\{a_2^*(2^6 - n) + a_3^*\} =& n^3/2 - (189/2)n^2 + 2^5 \cdot 185n \\ & - 3 \cdot 5(2^{13} - 2^{13-d}). \end{aligned} \tag{2.7.2}$$

We record this equation for special pairs (n, d):

$$\begin{aligned} (n,d) =& (66,13) \qquad a_3^* - 2a_2^* = -13, \\ (n,d) =& (65,13) \qquad a_3^* - a_2^* = -5. \end{aligned} \tag{2.7.3}$$

Proposition 2.8. *Let $V \subset \mathbb{F}^n$ be a code with weights among $\{w_1, \dots, w_t\}$. Let $v \in V$ have weight w. Consider the projection $\pi : \mathbb{F}^n \to \mathbb{F}^{n-w}$ onto the places off the support of v. Then*

(a) $\pi(V) = V'$ is a code of dimension $= d - \dim(V \cap \mathbb{F}^w)$; in particular, if v is not a sum of two disjoint words in V, then $\dim(V') = d - 1$.

(b) The weights of V' are all of the form $(\frac{1}{2})(w_i + w_j - w)$.

Proof. For (a), the kernel of $\pi|V$ consists of words of V in the support of v. If it contained another word v', one could write a disjoint sum $v = v' + (v - v')$. For (b), the weight of $\pi(v') \in V'$ is the number of positions of v' not in the support of v; this equals $w' - r$, where r is the number of overlaps between v and v'. If $v + v' = v''$, then on the weight level

$$w + w' - 2r = w''. \tag{2.8.1}$$

Therefore, $w' - r = (w' + w'' - w)/2$, as claimed.

Proof of Theorem A. We may assume $V \subset \mathbb{F}^n$ is spanning code, where $n \leq 66$. By (2.7.1) it suffices to rule out the case of $d = 13$. By Lemma 2.6, V contains a word of length 40; we project off it, and apply Proposition 2.8. Since 40 is not the sum of two weights, the projected $V' \subset \mathbb{F}^{n-40}$ has dimension 12; the weights are among $\{4, 8, 12, 16, 20\}$. So, V' is a doubly even code, hence $V' \subset V'^*$; as

$$n - 40 = \dim V' + \dim V'^* \geq 2 \cdot \dim V' = 24,$$

one has $n \geq 64$. But V' could not be self-dual, as $\mathbb{1} \in V'^* - V'$ has weight $n - 40 > 20$. This leaves the cases $n = 65$ and 66.

Return to the projected 12-dimensional doubly even code V' in $\mathbb{F}^{25}$ or $\mathbb{F}^{26}$. We claim $a_2^*(V') = 0$. Otherwise, there is a weight 2 word f orthogonal to V'; the span V'' of f and V' is even (by definition), dimension 13, and orthogonal to itself. In $\mathbb{F}^{25}$ this is impossible for dimension reasons. In $\mathbb{F}^{26}$ the span could not contain $\mathbb{1}$ (which is clearly in V'^*), as its weight of 26 is not 2 plus a weight of V'. This proves the claim.

On the other hand, (2.7.3) implies that V satisfies $a_2^*(V) > 0$; thus, there exists a word of the form $e_\alpha + e_\beta$ in the dual of V. A word in V thus contains either both e_α and e_β or neither. On the other hand, projecting off a word of weight 40 gives a V' with no such word of length 2; thus, every word in V of weight 40 must contain both e_α and e_β.

Intersecting V with the codimension-2 subspace $\mathbb{F}^{n-2} \subset \mathbb{F}^n$ of words containing neither e_α nor e_β gives 12-dimensional space $\tilde{V}$, but now the only weights can be 24 and 32. By Lemma 2.6, this is a contradiction.

Remark 2.9. Note that the inequality $\mu > \frac{1}{2}b_2(S)$, needed to assure a non-trivial code, cannot be true for $d = \text{degree}(\Sigma) \geq 18$. For, Miyaoka's inequality implies $\mu \leq (\frac{4}{9})d(d-1)^2$, while

$$b_2(S) = d^3 - 4d^2 + 6d - 2.$$

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