NODES ON SEXTIC HYPERSURFACES IN $\mathbb{P}^3$

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In this note we present a coding theory result which, together with Theorem 3.6.1 of [3], gives a short proof of a theorem of D. Jaffe and D. Ruberman:

**Theorem [5].** A sextic hypersurface in $\mathbb{P}^3$ has at most 65 nodes.

W. Barth [1] has constructed an example with 65 nodes. Following V. Nikulin [7] and A. Beauville [2], one must limit the size of an even set of nodes, and then prove a result about binary linear codes (i.e., linear subspaces of $\mathbb{F}^n$, where $\mathbb{F}$ is the field of two elements). The first step is the aforementioned result of Casnati–Catanese:

**Theorem [3].** On a sextic hypersurface, an even set of nodes has cardinality 24, 32 or 40.

The desired theorem will follow from:

**Theorem A.** Let $V \subset \mathbb{F}^{66}$ be a code, with weights from among 24, 32 and 40. Then $\dim(V) \leq 12$.

1. Codes from nodal hypersurfaces

(1.1) Let $\Sigma \subset \mathbb{P}^3$ be a hypersurface of degree $d$ having only $\mu$ ordinary double points as singularities. Let $\pi : S \to \Sigma$ be the minimal resolution of the singularities, with exceptional $(-2)$-curves $E_i$. Thus

$$E_i \cdot E_j = -2\delta_{ij},$$

$S$ is diffeomorphic to a smooth hypersurface of degree $d$.

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(1.2) The classes $[E_i]$ in $H^2(S; \mathbb{Z})$ span a not necessarily primitive sublattice of rank $\mu$. A subset $I \subset \{1, 2, \ldots, \mu\}$ for which $\Sigma[E_i]$ ($i \in I$) is divisible by 2 in $H^2(S; \mathbb{Z})$ (and therefore in $\text{Pic}(S)$) is called even (or strictly even in [4]). More generally, consider for any subset $I$ the homomorphism

$$\varphi : \mathbb{F}^I \to H^2(S, \mathbb{F}),$$

associating to each standard basis vector $e_i$ the mod 2 class of $[E_i]$. We define the code

$$\text{Code}(I) \equiv \text{Ker}(\varphi).$$

A non-0 element corresponds exactly to an even subset $J$ of $I$; the weight of such a “word” is its number of non-zero entries, i.e., $|J|$. $\text{Im}(\varphi)$ is totally isotropic by (1.1.1); thus, $\dim(\text{Im}(\varphi)) \leq \frac{1}{2}b_2(S)$, whence

$$(1.4.1) \quad \dim \text{Code}(I) \geq \text{Card}(I) - \frac{1}{2}b_2(S).$$

In particular, when $\mu > \frac{1}{2}b_2(S)$ one has a non-trivial code.

(1.5) It is an interesting question to determine for each $d$ the possible cardinality $t$ of an even set of nodes. By studying the corresponding double cover, one finds: For $d = 4$, one has $t = 8$ or 16 [7]; for $d = 5$, $t = 16$ or 20 [2]. The recent Theorem 3.6.1 of [3] proves that for $d = 6$, one has $t = 24, 32$ or 40. Since $b_2$ of a smooth sextic is 106, the result of [3] becomes

**Theorem 1.6.** Let $\Sigma \subset \mathbb{P}^3$ be a nodal sextic hypersurface with at least $\mu$ nodes. Then there is a code $V \subset \mathbb{F}^\mu$ of dimension $\geq \mu - 53$, all of whose weights are among $\{24, 32, 40\}$.

Let $I$ be any set of $\mu$ nodes. This result plus our Theorem A will imply the 63-node bound for sextics.

2. Proof of Theorem A

(2.1) The $\mathbb{F}$-inner product on $\mathbb{F}^\mu$ (counting mod 2 the number of overlaps of two words) makes $V^* \subset \mathbb{F}^\mu$. $V$ is called even if all words have even weight, double even if the weights are divisible by 4. Every doubly even code is automatically isotropic, i.e., $V \subset V^*$ (use (2.8.1) below). Since $\dim(V) = \dim(\mathbb{F}^\mu / V^*)$, a doubly even code satisfies $2d \leq n$ with equality iff the code is self-dual ($V = V^*$). The element $1 \in \mathbb{F}^\mu$ has a 1 in every position.
(2.2) Let \( V \subset \mathbb{F}^n \) be a \( d \)-dimensional code with \( a_i = a_i(V) \) words of weight \( i \). We have the simple equations

\[
\begin{align*}
(2.2.1) \quad \Sigma a_i &= 2^d - 1, \\
(2.2.2) \quad \Sigma i a_i &= n' \cdot 2^{d-1},
\end{align*}
\]

where \( n' \leq n \) is the number of entries containing 1's from words of \( V \). (2.2.1) is just an enumeration of \( V - \{0\} \). For (2.2.2) list all \( 2^d \) elements of \( V \) as rows of a \( 2^d \times n \) matrix of 0's and 1's. \( n' \) columns contains at least one 1; since \( V \) is a subspace, exactly half the entries are 1's. Now count the total number of 1's via rows or columns. If \( n' = n \), we say \( V \subset \mathbb{F}^n \) is a spanning code.

(2.3) For a striking generalization of (2.2.1) and (2.2.2), define the weight enumerator of the code \( V \) as

\[
W_V(x, y) = \Sigma a_i x^{n-i} y^i
\]

with \( a_0 = 1 \). \( W \) is homogeneous of degree \( d \). The MacWilliams identity (e.g., [6]) states that the enumerator of the dual code \( V^* \) is

\[
W_{V^*}(x, y) = \left( \frac{1}{2^d} \right) W_V(x + y, x - y).
\]

Writing the coefficients of \( W_{V^*} \) as \( a_i^* = a_i^*(V) \), (2.3.1) takes the form

\[
\Sigma a_i^* x^{n-i} y^i = \left( \frac{1}{2^d} \right) \cdot \{(x + y)^d + \Sigma a_i (x + y)^{n-i} (x - y)^i\}.
\]

Equations (2.2.1) and (2.2.2) are respectively the statements \( a_0^* = 1 \) and \( a_1^* \) (=number of entries not appearing in \( V \)) \( = n - n' \). More generally, we deduce the

**Lemma 2.4.** Let \( V \subset \mathbb{F}^n \) be a \( d \)-dimensional code. Then

\[
\begin{align*}
(2.4.1) \quad \Sigma a_i &= 2^d - 1, \\
(2.4.2) \quad \Sigma i a_i &= 2^{d-1} (n - a_i^*), \\
(2.4.3) \quad \text{If } a_1^* = 0, \text{ then} \\
\Sigma i^2 a_i &= 2^{d-1} \{a_2^* + n(n + 1)/2\}. \\
(2.4.4) \quad \text{If } a_1^* = 0, \text{ then} \\
\Sigma i^2 a_i &= 2^{d-2} \{3(a_2^* n - a_3^*) + n^2(n + 3)/2\}.
\end{align*}
\]
Proof. Expand the right-hand side of (2.3.2), carefully.

Lemma 2.5. If \( V \subseteq \mathbb{F}^n \) is a \( d \)-dimensional spanning code with only one weight \( w \), then there is an integer \( s > 0 \), so that \( w = s \cdot 2^{d-1} \) and \( n = s(2^d - 1) \).

Proof. Use (2.2.1) and (2.2.2) and fact that \( 2^{d-1} \) and \( 2^d - 1 \) are relatively prime.

Lemma 2.6. If \( V \subseteq \mathbb{F}^n \) is a spanning code with weights 24 and 32, then \( n \leq 63 \) and \( d \leq 9 \).

Proof. Solving (2.2.1) and (2.2.2), one finds

\[
\begin{align*}
    a_{24} &= 2^{d-1}(64 - n) - 4, \\
    a_{32} &= 2^{d-1}(n - 48) + 3.
\end{align*}
\]

Since \( a_{24} \geq 0 \), one has \( n \leq 63 \). Next, by (2.4.3), \( 2^{d-1} \) divides

\[
24^2 a_{24} + 32^2 a_{32} = 2^8 \{ 2^{d-6} \cdot 9 \cdot (2^6 - n) + 2^{d-2} \cdot (n - 48) + 3 \}.
\]

So, if \( d \geq 8 \), then \( d \leq 9 \). (Of course, there are many more restrictions.)

(2.7) Suppose \( V \subseteq \mathbb{F}^n \) is a \( d \)-dimensional spanning code with weights among \{24, 32, 40\}. We solve equations (2.4.1)–(2.4.3) for the \( a_i \)'s; writing \( z = n(n+1)/2 + a_2^2 \), we find

\[
\begin{align*}
    a_{24} &= 2^{d-8} \{ z - 9 \cdot 2^3 n + 5 \cdot 2^7 \} - 10, \\
    a_{32} &= 2^{d-7} \{ -z + 2^6 n - 15 \cdot 2^7 \} + 15, \\
    a_{40} &= 2^{d-8} \{ z - 7 \cdot 2^3 n + 3 \cdot 2^9 \} - 6.
\end{align*}
\]

One can thus compute that

\[
\Sigma x^i a_i = 2^{d+4} \{ 3z - 2 \cdot 47 n + 3 \cdot 5 \cdot 2^7 \} - 2^{11} \cdot 3 \cdot 5.
\]

By (2.4.4), this expression is divisible by \( 2^{d-2} \); we conclude that

(2.7.1) \quad d \leq 13

Equating with (2.4.4) and simplifying yield

\[
\begin{align*}
3 \{ a_2^* (2^6 - n) + a_3^* \} &= n^3/2 - (189/2)n^2 + 2^5 \cdot 185n \\
&
- 3 \cdot 5(2^{13} - 2^{13-d}).
\end{align*}
\]
We record this equation for special pairs \((n, d)\):

\[
(2.7.3) \quad (n, d) = (66, 13) \quad a_3^* - 2a_2^* = -13, \\
(n, d) = (65, 13) \quad a_3^* - a_2^* = -5.
\]

**Proposition 2.8.** Let \( V \subset \mathbb{P}^n \) be a code with weights among \( \{w_1, \ldots, w_t\} \). Let \( v \in V \) have weight \( w \). Consider the projection \( \pi : \mathbb{P}^n \to \mathbb{P}^{n\setminus w} \) onto the places off the support of \( v \). Then

(a) \( \pi(V) = V' \) is a code of dimension \( d - \dim(V \cap \mathbb{P}^w) \); in particular, if \( v \) is not a sum of two disjoint words in \( V \), then \( \dim(V') = d - 1 \).

(b) The weights of \( V' \) are all of the form \((\frac{1}{2})(w_i + w_j - w)\).

**Proof.** For (a), the kernel of \( \pi|V \) consists of words of \( V \) in the support of \( v \). If it contained another word \( v' \), one could write a disjoint sum \( v = v' + (v - v') \). For (b), the weight of \( \pi(v') \in V' \) is the number of positions of \( v' \) not in the support of \( v \); this equals \( w' - r \), where \( r \) is the number of overlaps between \( v \) and \( v' \). If \( v + v' = v'' \), then on the weight level

\[
(2.8.1) \quad w + w' - 2r = w''.
\]

Therefore, \( w' - r = (w' + w'' - w)/2 \), as claimed.

**Proof of Theorem A.** We may assume \( V \subset \mathbb{P}^n \) is spanning code, where \( n \leq 66 \). By (2.7.1) it suffices to rule out the case of \( d = 13 \). By Lemma 2.6, \( V \) contains a word of length \( 40 \); we project off it, and apply Proposition 2.8. Since \( 40 \) is not the sum of two weights, the projected \( V' \subset \mathbb{P}^{n-40} \) has dimension 12; the weights are among \( \{4, 8, 12, 16, 20\} \). So, \( V' \) is a doubly even code, hence \( V'' \subset \mathbb{P}^* \); as

\[
n - 40 = \dim(V') + \dim(V'' \geq 2 \cdot \dim V' = 24,
\]

one has \( n \geq 64 \). But \( V' \) could not be self-dual, as \( \| \in V^* \subset V' \) has weight \( n - 40 > 20 \). This leaves the cases \( n = 65 \) and 66.

Return to the projected 12-dimensional doubly even code \( V' \in \mathbb{P}^{25} \) or \( \mathbb{P}^{26} \). We claim \( a_2^*(V') = 0 \). Otherwise, there is a weight 2 word \( f \) orthogonal to \( V' \); the span \( V'' \) of \( f \) and \( V' \) is even (by definition), dimension 13, and orthogonal to itself. In \( \mathbb{P}^{25} \) this is impossible for dimension reasons. In \( \mathbb{P}^{26} \) the span could not contain \( \| \) (which is clearly in \( V^* \)), as its weight of 26 is not 2 plus a weight of \( V' \). This proves the claim.
On the other hand, (2.7.3) implies that $V$ satisfies $a_0^2(V) > 0$; thus, there exists a word of the form $e_\alpha + e_\beta$ in the dual of $V$. A word in $V$ thus contains either both $e_\alpha$ and $e_\beta$ or neither. On the other hand, projecting off a word of weight 40 gives a $V'$ with no such word of length 2; thus, every word in $V$ of weight 40 must contain both $e_\alpha$ and $e_\beta$.

Intersecting $V$ with the codimension-2 subspace $\mathbb{P}^{n-2} \subset \mathbb{P}^n$ of words containing neither $e_\alpha$ nor $e_\beta$ gives 12-dimensional space $\tilde{V}$, but now the only weights can be 24 and 32. By Lemma 2.6, this is a contradiction.

**Remark 2.9.** Note that the inequality $\mu > \frac{1}{2} b_2(S)$, needed to assure a non-trivial code, cannot be true for $d = \text{degree}(\Sigma) \geq 18$. For, Miyaoka's inequality implies $\mu \leq (\frac{1}{3})d(d-1)^2$, while

$$b_2(S) = d^3 - 4d^2 + 6d - 2.$$

**References**


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