SCREEN PSEUDO-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of screen pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions $D_1$, $D_2$ and $\text{RadTM}$ on screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold have been obtained. Further we obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic.

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1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu [3]. In [2], B.Y. Chen defined slant immersions in complex geometry as a natural generalization of both holomorphic immersions and totally real immersions. In [5], A. Lotta introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold. A. Carriazo defined and studied bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and further gave the notion of pseudo-slant submanifolds [1]. On other hand, the theory of invariant, screen slant, screen real, screen Cauchy-Riemann lightlike submanifolds have been studied in [3]. Thus motivated sufficiently, we introduce the notion of screen pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds. This new class of lightlike submanifolds of an indefinite Kaehler manifold includes invariant, screen slant, screen real, screen Cauchy-Riemann lightlike submanifolds as its subcases. The paper is arranged as follows. There are some basic results in section 2. In section 3, we study screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on screen pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds.

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2. Preliminaries

A submanifold \((M^m, g)\) immersed in a semi-Riemannian manifold \((\bar{M}^{m+n}, \bar{g})\) is called a lightlike submanifold \([3]\) if the metric \(g\) induced from \(\bar{g}\) is degenerate and the radical distribution \(\text{Rad}TM\) is of rank \(r\), where \(1 \leq r \leq m\). Let \(S(TM)\) be a screen distribution which is a semi-Riemannian complementary distribution of \(\text{Rad}TM\) in \(TM\), that is

\[
T M = \text{Rad}TM \oplus_{\text{orth}} S(TM).
\]

Now consider a screen transversal vector bundle \(S(TM^\perp)\), which is a semi-Riemannian complementary vector bundle of \(\text{Rad}TM\) in \(TM\). For any local basis \(\{\xi_i\}\) of \(\text{Rad}TM\), there exists a local null frame \(\{N_i\}\) of sections with values in the orthogonal complement of \(S(TM^\perp)\) in \([S(TM)]^\perp\) such that \(\bar{g}(\xi_i, N_j) = \delta_{ij}\) and \(\bar{g}(N_i, N_j) = 0\), it follows that there exists a lightlike transversal vector bundle \(ltr(TM)\) locally spanned by \(\{N_i\}\). Let \(tr(TM)\) be complementary (but not orthogonal) vector bundle to \(TM\) in \(T\bar{M}|_M\). Then

\[
(2.2) \quad tr(TM) = ltr(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

\[
(2.3) \quad T\bar{M}|_M = TM \oplus tr(TM),
\]

\[
(2.4) \quad T\bar{M}|_M = S(TM) \oplus_{\text{orth}} [\text{Rad}TM \oplus ltr(TM)] \oplus_{\text{orth}} S(TM^\perp).
\]

Following are four cases of a lightlike submanifold \((M, g, S(TM), S(TM^\perp))\):

Case.1 r-lightlike if \(r < \min(m, n)\),
Case.2 co-isotropic if \(r = n < m\), \(S(TM^\perp) = \{0\}\),
Case.3 isotropic if \(r = m < n\), \(S(TM) = \{0\}\),
Case.4 totally lightlike if \(r = m = n\), \(S(TM) = S(TM^\perp) = \{0\}\).

The Gauss and Weingarten formulae are given as

\[
(2.5) \quad \nabla_X Y = \nabla_X Y + h(X, Y),
\]

\[
(2.6) \quad \nabla_X V = -A_V X + \nabla^t_X V,
\]

for all \(X, Y \in \Gamma(TM)\) and \(V \in \Gamma(tr(TM))\), where \(\nabla_X Y, A_V X\) belong to \(\Gamma(TM)\) and \(h(X, Y), \nabla^t_X V\) belong to \(\Gamma(tr(TM))\). \(\nabla\) and \(\nabla^t\) are linear connections on \(M\) and on the vector bundle \(tr(TM)\) respectively. The second fundamental form \(h\) is a symmetric \(F(M)\)-bilinear form on \(\Gamma(TM)\) with values in \(\Gamma(tr(TM))\) and the shape operator \(A_V\) is a linear endomorphism of \(\Gamma(TM)\). From (2.5) and (2.6), for any \(X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))\) and \(W \in \Gamma(S(TM^\perp))\), we have

\[
(2.7) \quad \nabla_X Y = \nabla_X Y + h^I(X, Y) + h^s(X, Y),
\]

\[
(2.8) \quad \nabla_X N = -A_N X + \nabla^t_X N + D^s(X, N),
\]
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\[ \nabla_X W = -A_W X + \nabla^*_X W + D^l(X, W), \]

where \( h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l(X, W) = L(\nabla^l_X W), D^s(X, N) = S(\nabla^s_X N). \ L \) and \( S \) are the projection morphisms of \( tr(TM) \) on \( ltr(TM) \) and \( S(TM^\perp) \) respectively. \( \nabla^l \) and \( \nabla^s \) are linear connections on \( ltr(TM) \) and \( S(TM^\perp) \) called the lightlike connection and screen transversal connection on \( M \) respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection \( \nabla \), we obtain

\[ (2.10) \quad g(h^s(X, Y), W) + g(Y, D^l(X, W)) = g(A_W X, Y), \]

\[ (2.11) \quad g(D^s(X, N), W) = g(N, A_W X). \]

Denote the projection of \( TM \) on \( S(TM) \) by \( P \). Then from the decomposition of the tangent bundle of a lightlike submanifold, for any \( X, Y \in \Gamma(TM) \) and \( \xi \in \Gamma(RadTM) \), we have

\[ (2.12) \quad \nabla_X P Y = \nabla^*_X P Y + h^s(X, P Y), \]

\[ (2.13) \quad \nabla_X \xi = -A^s_\xi X + \nabla^*_X \xi, \]

By using above equations, we obtain

\[ (2.14) \quad g(h^l(X, P Y), \xi) = g(A^s_\xi X, P Y), \]

\[ (2.15) \quad g(h^s(X, P Y), N) = g(A_N X, P Y), \]

\[ (2.16) \quad g(h^l(X, \xi), \xi) = 0, \quad A^s_\xi \xi = 0. \]

It is important to note that in general \( \nabla \) is not a metric connection. Since \( \nabla \) is metric connection, by using (2.2), we get

\[ (2.17) \quad (\nabla_X g)(Y, Z) = g(h^l(X, Y), Z) + g(h^l(X, Z), Y). \]

An indefinite almost Hermitian manifold \((\bar{M}, \bar{g}, \bar{J})\) is a 2m-dimensional semi-Riemannian manifold \( \bar{M} \) with semi-Riemannian metric \( \bar{g} \) of constant index \( q \), \( 0 < q < 2m \) and a \((1, 1)\) tensor field \( \bar{J} \) on \( \bar{M} \) such that following conditions are satisfied:

\[ (2.18) \quad \bar{J}^2 X = -X, \]

\[ (2.19) \quad \bar{g}(\bar{J} X, \bar{J} Y) = \bar{g}(X, Y), \]

for all \( X, Y \in \Gamma(T\bar{M}) \).

An indefinite almost Hermitian manifold \((\bar{M}, \bar{g}, \bar{J})\) is called an indefinite Kaehler manifold if \( \bar{J} \) is parallel with respect to \( \nabla \), i.e.,

\[ (2.20) \quad (\nabla_X \bar{J}) Y = 0, \]

for all \( X, Y \in \Gamma(T\bar{M}) \), where \( \nabla \) is Levi-Civita connection with respect to \( \bar{g} \).
3. Screen Pseudo-Slant Lightlike Submanifolds

In this section, we introduce the notion of screen pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following Lemma for later use:

Lemma 3.1. Let $M$ be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$. Then the screen distribution $S(TM)$ of lightlike submanifold $M$ is Riemannian.

The proof of above Lemma follows as in Lemma 3.1 of [2], so we omit it.

Definition 3.1. Let $M$ be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$. Then we say that $M$ is a screen pseudo-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\text{Rad}(TM)$ is invariant with respect to $\overline{J}$, i.e. $\overline{J}(\text{Rad}(TM)) = \text{Rad}(TM)$,

(ii) there exists non-degenerate orthogonal distributions $D_1$ and $D_2$ on $M$ such that $S(TM) = D_1 \oplus_{\text{orth}} D_2$,

(iii) the distribution $D_1$ is anti-invariant, i.e. $\overline{J}D_1 \subset S(TM^\perp)$,

(iv) the distribution $D_2$ is slant with angle $\theta(\neq \pi/2)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle $\theta$ between $\overline{J}X$ and the vector subspace $(D_2)_x$ is a constant($\neq \pi/2$), which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle $\theta$ is called the slant angle of distribution $D_2$. A screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq 0$.

From the above definition, we have the following decomposition

\begin{equation}
TM = \text{Rad}(TM) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2.
\end{equation}

In particular, we have

(i) if $D_1 = 0$, then $M$ is a screen slant lightlike submanifold,

(ii) if $D_2 = 0$, then $M$ is a screen real lightlike submanifold,

(iii) if $D_1 = 0$ and $\theta = 0$, then $M$ is an invariant lightlike submanifold,

(iv) if $D_1 \neq 0$ and $\theta = 0$, then $M$ is a screen CR-lightlike submanifold.

Thus the above new class of lightlike submanifolds of an indefinite Kaehler manifold includes invariant, screen slant, screen real, screen Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in [3].

Let $(\mathbb{R}_2^{2m}, \bar{g}, \bar{J})$ denote the manifold $\mathbb{R}_2^{2m}$ with its usual Kaehler structure given by

\begin{align*}
\bar{g} &= \frac{1}{4}(\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i), \\
\bar{J}(\sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i)) &= \sum_{i=1}^{m} (Y_i \partial x_i - X_i \partial y_i),
\end{align*}

where $(x^i, y^i)$ are the cartesian coordinates on $\mathbb{R}_2^{2m}$. Now, we construct some examples of screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold.

Example 1. Let $(\mathbb{R}_2^{12}, \bar{g}, \bar{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, - , + , + , + , + , + , +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$. 
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Suppose $M$ is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = y^2 = u_1$, $x^2 = -y^1 = u_2$, $x^3 = u_3 \cos \beta$, $y^3 = u_3 \sin \beta$, $x^4 = u_4 \sin \beta$, $y^4 = u_4 \cos \beta$, $x^5 = u_5 \cos \theta$, $y^5 = u_6 \cos \theta$, $x^6 = u_6 \sin \theta$, $y^6 = u_5 \sin \theta$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

\[
Z_1 = 2(\partial x_1 + \partial y_2), \quad Z_2 = 2(\partial x_2 - \partial y_1), \quad Z_3 = 2(\cos \beta \partial x_3 + \sin \beta \partial y_3), \quad Z_4 = 2(\sin \beta \partial x_4 + \cos \beta \partial y_4), \quad Z_5 = 2(\cos \theta \partial x_5 + \sin \theta \partial y_5), \quad Z_6 = 2(\sin \theta \partial x_6 + \cos \theta \partial y_5).
\]

Hence $RadTM = span \{Z_1, Z_2\}$ and $S(TM) = \{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2$, $N_2 = -\partial x_2 - \partial y_1$. $S(TM^\perp)$ is spanned by $\{W_1, W_2, W_3, W_4\}$,

\[
W_1 = 2(\sin \beta \partial x_3 - \cos \beta \partial y_3), \quad W_2 = 2(\cos \beta \partial x_4 - \sin \beta \partial y_4), \quad W_3 = 2(\sin \theta \partial x_5 - \cos \theta \partial y_5), \quad W_4 = 2(\cos \theta \partial x_6 - \sin \theta \partial y_5).
\]

It follows that $JZ_1 = Z_2$ and $JZ_2 = -Z_1$, which implies that $RadTM$ is invariant, i.e. $JRadTM = RadTM$. On the other hand, we can see that $D_1 = span \{Z_3, Z_4\}$ such that $JZ_3 = W_1$ and $JZ_4 = W_2$, which implies that $D_1$ is anti-invariant with respect to $J$ and $D_2 = span \{Z_5, Z_6\}$ is a slant distribution with slant angle $2\theta$. Hence $M$ is a screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

**Example 2.** Let $(\mathbb{R}^{12}_2, \bar{g}, \bar{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose $M$ is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = u_1$, $y^1 = -u_2$, $x^2 = -u_1 \cos \alpha - u_2 \sin \alpha$, $y^2 = -u_1 \sin \alpha + u_2 \cos \alpha$, $x^3 = y^4 = u_3$, $x^4 = y^3 = u_4$, $x^5 = u_5 \cos u_6$, $y^5 = u_5 \sin u_6$, $x^6 = \cos u_5$, $y^6 = \sin u_5$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

\[
Z_1 = 2(\partial x_1 - \cos \alpha \partial x_2 - \sin \alpha \partial y_2), \quad Z_2 = 2(\partial x_2 + \sin \alpha \partial x_3 + \cos \alpha \partial y_3), \quad Z_3 = 2(\partial x_3 + \partial y_1), \quad Z_4 = 2(\partial x_4 + \partial y_4), \quad Z_5 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 - \cos u_5 \partial x_6 + \sin u_5 \partial y_6), \quad Z_6 = 2(\sin u_6 \partial x_5 + u_5 \cos u_6 \partial y_5 - u_5 \sin u_6 \partial x_6 + u_5 \cos u_5 \partial y_6).
\]

Hence $RadTM = span \{Z_1, Z_2\}$ and $S(TM) = span \{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 - \cos \alpha \partial x_2 - \sin \alpha \partial y_2$, $N_2 = \partial y_1 - \sin \alpha \partial x_2 + \cos \alpha \partial y_2$. $S(TM^\perp)$ is spanned by $\{W_1, W_2, W_3, W_4\}$,

\[
W_1 = 2(\partial x_3 - \partial y_4), \quad W_2 = 2(\partial x_4 - \partial y_3), \quad W_3 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 + u_5 \partial x_6 - \cos u_5 \partial y_6), \quad W_4 = 2(u_5 \cos u_6 \partial x_5 + u_5 \sin u_6 \partial y_5).
\]

It follows that $JZ_1 = Z_2$ and $JZ_2 = -Z_1$, which implies that $RadTM$ is invariant, i.e. $JRadTM = RadTM$. On the other hand, we can see that $D_1 = span \{Z_3, Z_4\}$ such that $JZ_3 = W_1$ and $JZ_4 = W_2$, which implies that $D_1$ is anti-invariant with respect to $J$ and $D_2 = span \{Z_5, Z_6\}$ is a slant distribution with slant angle $\pi/4$. Hence $M$ is a screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Now, for any vector field $X$ tangent to $M$, we put $JX = PX + FX$, where $PX$ and $FX$ are tangential and transversal parts of $JX$ respectively. We denote the projections on $RadTM$, $D_1$ and $D_2$ in $TM$ by $P_1$, $P_2$ and $P_3$ respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$, $J(D_1)$ and $D'$ by $Q_1$, $Q_2$ and $Q_3$, respectively.
$Q_2$ and $Q_3$ respectively, where $D'$ is non-degenerate orthogonal complementary subbundle of $\mathcal{J}(D_1)$ in $S(TM^\perp)$. Then, for any $X \in \Gamma(TM)$, we get

\[(3.2)\quad X = P_1X + P_2X + P_3X.\]

Now applying $\mathcal{J}$ to (3.2), we have

\[(3.3)\quad \mathcal{J}X = \mathcal{J}P_1X + \mathcal{J}P_2X + \mathcal{J}P_3X,\]

which gives

\[(3.4)\quad \mathcal{J}X = \mathcal{J}P_1X + \mathcal{J}P_2X + fP_3X + FP_3X,\]

where $fP_3X$ (resp. $FP_3X$) denotes the tangential (resp. transversal) component of $\mathcal{J}P_3X$. Thus we get $\mathcal{J}P_1X \in \Gamma(Rad TM)$, $\mathcal{J}P_2X \in \Gamma(\mathcal{J}(D_1)) \subset \Gamma(S(TM^\perp))$, $fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(D')$. Also, for any $W \in \Gamma(tr(TM))$, we have

\[(3.5)\quad W = Q_1W + Q_2W + Q_3W.\]

Applying $\mathcal{J}$ to (3.3), we obtain

\[(3.6)\quad \mathcal{J}W = \mathcal{J}Q_1W + \mathcal{J}Q_2W + \mathcal{J}Q_3W,\]

which gives

\[(3.7)\quad \mathcal{J}W = \mathcal{J}Q_1W + \mathcal{J}Q_2W + BQ_3W + CQ_3W,\]

where $BQ_3W$ (resp. $CQ_3W$) denotes the tangential (resp. transversal) component of $\mathcal{J}Q_3W$. Thus we get $\mathcal{J}Q_1W \in \Gamma(ltr(TM))$, $\mathcal{J}Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$ and $CQ_3W \in \Gamma(D')$.

Now, by using (2.20), (3.3), (3.4) and (2.4)-(2.8) and identifying the components on $Rad TM$, $D_1$, $D_2$, $ltr(TM)$, $\mathcal{J}(D_1)$ and $D'$, we obtain

\[(3.8)\quad \nabla_X^*\mathcal{J}P_1Y + P_1(\nabla_X fP_3Y) = P_1(A_{\mathcal{J}P_3Y}X) + P_1(A_{\mathcal{J}P_2Y}X)\]

\[+ \mathcal{J}P_1\nabla_X Y,\]

\[(3.9)\quad P_2(A_{\mathcal{J}P_1Y}^*X) + P_2(A_{\mathcal{J}P_2Y}^*X) + P_2(A_{FP_3Y}X) = P_2(\nabla_X fP_3Y)\]

\[\quad - \mathcal{J}Q_2h^s(X,Y),\]

\[(3.10)\quad P_3(\mathcal{J}P_1Y^*X) + P_3(A_{\mathcal{J}P_2Y}^*X) + P_3(A_{FP_3Y}X) = P_3(\nabla_X fP_3Y)\]

\[\quad - fP_3(\nabla_X Y) - BQ_3h^s(X,Y),\]

\[(3.11)\quad h^l(X,\mathcal{J}P_1Y) + D^l(X,\mathcal{J}P_2Y) + h^l(X,fP_3Y) + D^l(X,FP_3Y)\]

\[= \mathcal{J}h^l(X,Y),\]

\[(3.12)\quad Q_2\nabla_X^*\mathcal{J}P_2Y + Q_2\nabla_X^*FP_3Y = \mathcal{J}P_2\nabla_X Y - Q_2h^s(X,\mathcal{J}P_1Y)\]

\[\quad - Q_2h^s(X,fP_3Y),\]

\[(3.13)\quad Q_3\nabla_X^*\mathcal{J}P_2Y + Q_3\nabla_X^*FP_3Y - FP_3\nabla_X Y = CQ_3h^s(X,Y)\]

\[\quad - Q_3h^s(X,fP_3Y) - Q_3h^s(X,\mathcal{J}P_1Y).\]
Theorem 3.2. Let $M$ be a 2$q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $M$ is a screen pseudo-slant lightlike submanifold of $\overline{M}$ if and only if

(i) $ltr(TM)$ is invariant and $D_1$ is anti-invariant with respect to $\overline{J}$,

(ii) there exists a constant $\lambda \in (0,1]$ such that $P^2X = -\lambda X$.

Moreover, there also exists a constant $\mu \in [0,1)$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where $D_1$ and $D_2$ are non-degenerate orthogonal distributions on $M$ such that $S(TM) = D_1 \oplus_{orth} D_2$ and $\lambda = \cos^2 \theta$, $\theta$ is slant angle of $D_2$.

Proof. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ is anti-invariant and $RadTM$ is invariant with respect to $\overline{J}$. For any $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(S(TM))$, using (3.14) and (3.15), we obtain $\overline{g}(\overline{J}N, X) = -\overline{g}(N, \overline{J}X) = -\overline{g}(N, \overline{J}P_2X + fP_3X + FP_3X) = 0$. Thus $\overline{J}N$ does not belong to $\Gamma(S(TM))$. For any $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, from (3.16) and (3.17), we have $\overline{g}(\overline{J}N, W) = -\overline{g}(N, \overline{J}W) = -\overline{g}(N, \overline{J}Q_2W + BQ_3W + CQ_3W) = 0$. Hence, we conclude that $\overline{J}N$ does not belong to $\Gamma(S(TM^\perp))$.

Now suppose that $\overline{J}N \in \Gamma(RadTM)$. Then $\overline{J}(\overline{J}N) = \overline{J}^2N = -N \in \Gamma(ltr(TM))$, which contradicts that $RadTM$ is invariant. Thus $ltr(TM)$ is invariant with respect to $\overline{J}$. Now for any $X \in \Gamma(D_2)$ we have $|PX| = |\overline{J}X| \cos \theta$, which implies

\begin{equation}
\cos \theta = \frac{|PX|}{|\overline{J}X|}.
\end{equation}

In view of (3.14), we get $\cos^2 \theta = \frac{|PX|^2}{|\overline{J}X|^2} = \frac{g(PX, PX)}{g(\overline{J}X, \overline{J}X)} = \frac{g(X, P^2X)}{g(X, \overline{J}^2X)}$, which gives

\begin{equation}
g(X, P^2X) = \cos^2 \theta g(X, \overline{J}^2X).
\end{equation}

Since $M$ is a screen pseudo-slant lightlike submanifold, $\cos^2 \theta = \lambda(constant) \in (0,1]$ and therefore from (3.15), we get $g(X, P^2X) = \lambda g(X, \overline{J}^2X) = g(X, \lambda \overline{J}^2X)$, which implies

\begin{equation}
g(X, (P^2 - \lambda \overline{J}^2)X) = 0.
\end{equation}

Since $(P^2 - \lambda \overline{J}^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2 \times D_2}$ is non-degenerate(positive definite), from (3.16), we have $(P^2 - \lambda \overline{J}^2)X = 0$, which implies

\begin{equation}
P^2X = \lambda \overline{J}^2X = -\lambda X.
\end{equation}

Now, for any vector field $X \in \Gamma(D_2)$, we have

\begin{equation}
\overline{J}X = PX + FX,
\end{equation}

where $PX$ and $FX$ are tangential and transversal parts of $\overline{J}X$ respectively. Applying $\overline{J}$ to (3.18) and taking tangential component, we get

\begin{equation}
- X = P^2X + BFX.
\end{equation}
From (3.17) and (3.19), we get

\[(3.20) \quad BFX = -\mu X, \quad \forall X \in \Gamma(D_2),\]

where \(1 - \lambda = \mu(\text{constant}) \in [0, 1)\). This proves (ii).

Conversely suppose that conditions (i) and (ii) are satisfied. We can show that \(\text{Rad}TM\) is invariant in similar way that \(l\text{tr}(TM)\) is invariant. From (3.19), for any \(X \in \Gamma(D_2)\), we get

\[(3.21) \quad X = P^2 X - \mu X,\]

which implies

\[(3.22) \quad P^2 X = -\lambda X,\]

where \(1 - \mu = \lambda(\text{constant}) \in (0, 1]\).

Now \(\cos \theta = \frac{g(JX, PX)}{|JX||PX|} = -\frac{g(X, JPX)}{|JX||PX|} = -\frac{g(X, P^2 X)}{|JX||PX|} = -\lambda \frac{g(X, JX)}{|JX||PX|}.\)

From above equation, we get

\[(3.23) \quad \cos \theta = \lambda \frac{|JX|}{|PX|}.\]

Therefore (5.20) and (5.21) give \(\cos^2 \theta = \lambda(\text{constant})\).

Hence \(M\) is a screen pseudo-slant lightlike submanifold.

**Corollary 3.1.** Let \(M\) be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \(\overline{M}\) with slant angle \(\theta\), then for any \(X, Y \in \Gamma(D_2)\), we have

(i) \(g(PX, PY) = \cos^2 \theta g(X, Y)\),

(ii) \(g(FX, FY) = \sin^2 \theta g(X, Y)\).

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [B].

**Theorem 3.3.** Let \(M\) be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \(\overline{M}\). Then \(\text{Rad}TM\) is integrable if and only if

(i) \(Q_2 h^*(Y, JP_1 X) = Q_2 h^*(X, JP_1 Y)\),

(ii) \(Q_3 h^*(Y, JP_1 X) = Q_3 h^*(X, JP_1 Y)\),

(iii) \(P_3 A_{JP_1}^* X = P_3 A_{JP_1}^* Y\), for all \(X, Y \in \Gamma(\text{Rad}TM)\).

**Proof.** Let \(M\) be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \(\overline{M}\). Let \(X, Y \in \Gamma(\text{Rad}TM)\). From (5.12), we have \(Q_2 h^*(X, JP_1 Y) = JP_2 \nabla_X Y\), which gives \(Q_2 h^*(Y, JP_1 Y) - Q_2 h^*(Y, JP_1 X) = JP_2 [X, Y]\). In view of (4.23), we get \(Q_3 h^*(X, JP_1 Y) = C Q_3 h^*(X, Y) + FP_3 \nabla_X Y\), which implies \(Q_3 h^*(X, JP_1 Y) - Q_3 h^*(Y, JP_1 X) = FP_3 [X, Y]\). Also from (3.10), we have \(P_3 A_{JP_1}^* X = f P_3 \nabla_X Y + B Q_3 h^*(X, Y)\), which gives \(P_3 A_{JP_1}^* X - P_3 A_{JP_1}^* Y = f P_3 [X, Y]\). Thus, we obtain the required results.
Theorem 3.4. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$. Then $D_1$ is integrable if and only if
\begin{enumerate}[(i)]  
  \item $P_1 A_{\overline{JP}_2} Y = P_1 A_{\overline{JP}_2} Y$ and $P_3 A_{\overline{JP}_2} X = P_3 A_{\overline{JP}_2} X$,  
  \item $Q_3(\nabla^*_Y \overline{JP}_2 X) = Q_3(\nabla^*_X \overline{JP}_2 Y)$, for all $X,Y \in \Gamma(D_1)$.
\end{enumerate}

Proof. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$. Let $X,Y \in \Gamma(D_1)$. From (3.13), we have $P_1 A_{\overline{JP}_2} Y = -\overline{JP}_1 \nabla_X Y$, which gives $P_1 A_{\overline{JP}_2} Y - P_1 A_{\overline{JP}_2} X = \overline{JP}_1 [X,Y]$. In view of (3.14), we obtain $P_3 A_{\overline{JP}_2} X - P_3 A_{\overline{JP}_2} X = -P_3 \nabla_X Y$, which implies $P_3 A_{\overline{JP}_2} X = -P_3 A_{\overline{JP}_2} X$.

Theorem 3.5. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$. Then $D_2$ is integrable if and only if
\begin{enumerate}[(i)]  
  \item $P_1 (\nabla_X fP_2 Y - \nabla_Y fP_3 X) = P_1 (A_{FP_3} X - A_{FP_3} Y)$,  
  \item $Q_2(\nabla^*_X fP_3 Y - \nabla^*_Y fP_3 X) = Q_2(h^*(X,fP_3 X) - h^*(X,fP_3 Y))$, for all $X,Y \in \Gamma(D_2)$.
\end{enumerate}

Proof. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$. Let $X,Y \in \Gamma(D_2)$. In view of (3.13), we get $P_1 (\nabla_X fP_3 Y) = P_1 (A_{FP_3} X) + \overline{JP}_1 \nabla_X Y$, from (3.14), we get $Q_2(\nabla^*_X fP_3 Y - \nabla^*_Y fP_3 X) = Q_2(h^*(X,fP_3 X) - h^*(X,fP_3 Y))$.

Theorem 3.6. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$. Then the induced connection $\nabla$ is a metric connection if and only if
\begin{enumerate}[(i)]  
  \item $\overline{JP}_2 h^*(X,Y) = 0$ and $BQ_3 h^*(X,Y) = 0$,  
  \item $A^*_Y$ vanishes on $\Gamma(TM)$, for all $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$.
\end{enumerate}

Proof. Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$. Then the induced connection $\nabla$ on $\mathcal{M}$ is a metric connection if and only if $RadTM$ is parallel distribution with respect to $\nabla$. From (3.16), (3.13) and (3.14), for any $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$, we have $\nabla_X \nabla Y = \nabla^*_X \nabla Y - \nabla A^*_X X + \nabla h^*(X,Y) + \nabla Q_2 h^*(X,Y) + \nabla Q_3 h^*(X,Y)$. On comparing tangential components of both sides of above equation, we get $\nabla_X \nabla Y = \nabla^*_X \nabla Y - \nabla A^*_X X + \nabla Q_2 h^*(X,Y) + BQ_3 h^*(X,Y)$, which completes the proof.

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

Definition 4.1. A screen pseudo-slant lightlike submanifold $M$ of an indefinite Kaehler manifold $\mathcal{M}$ is said to be a mixed geodesic screen pseudo-slant lightlike
submanifold if its second fundamental form $h$ satisfies $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus $M$ is mixed geodesic screen pseudo-slant lightlike submanifold if $h^1(X, Y) = 0$ and $h^s(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

**Theorem 4.1.** Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $\text{RadTM}$ defines a totally geodesic foliation if and only if $\overline{g}(D^I(\mathcal{T}P_2Z) + D^I(\mathcal{F}P_3Z), \mathcal{J}Y) = -\overline{g}(h^I(X, fP_3Z), \mathcal{J}Y)$, for all $X, Y \in \Gamma(\text{RadTM})$ and $Z \in \Gamma(S(TM))$.

**Proof.** Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. To prove the distribution $\text{RadTM}$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(\text{RadTM})$, for all $X, Y \in \Gamma(\text{RadTM})$. Since $\nabla$ is a metric connection, using (2.7), (2.11), (2.20) and (3.4), for any $X, Y \in \Gamma(\text{RadTM})$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X (\mathcal{J}P_2Z + fP_3Z + fP_3Z), \mathcal{J}Y)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(D^I(X, \mathcal{J}P_2Z) + h^I(X, fP_3Z) + D^I(X, fP_3Z), \mathcal{J}Y)$. This completes the proof.

**Theorem 4.2.** Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ defines a totally geodesic foliation if and only if

1. $\overline{g}(h^s(X, fZ), \mathcal{J}Y) = -\overline{g}(\nabla_X^sFZ, \mathcal{J}Y)$,
2. $D^s(X, \mathcal{J}N)$ has no component in $\mathcal{J}(D_1)$,

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(\text{ltr}(TM))$.

**Proof.** Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. It is easy to see that the distribution $D_1$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\nabla$ is a metric connection, using (2.7), (2.11) and (2.20), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \mathcal{J}Z, \mathcal{J}Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(h^s(X, fZ) + \nabla_X^sFZ, \mathcal{J}Y)$. Now, from (2.7), (2.11) and (2.20), for all $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(\text{ltr}(TM))$, we get $\overline{g}(\nabla_X Y, N) = -\overline{g}(\mathcal{J}Y, \nabla_X \mathcal{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\mathcal{J}Y, D^s(X, \mathcal{J}N))$. Thus, the theorem is completed.

**Theorem 4.3.** Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_2$ defines a totally geodesic foliation if and only if

1. $\overline{g}(fY, A_{\mathcal{J}Z}X) = \overline{g}(FY, \nabla_X ^s \mathcal{J}Z)$,
2. $\overline{g}(fY, A_{\mathcal{J}N}X) = \overline{g}(FY, D^s(X, \mathcal{J}N))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(\text{ltr}(TM))$.

**Proof.** Let $M$ be a screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution $D_2$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\nabla$ is a metric connection, using (2.7), (2.11) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\mathcal{J}Y, \nabla_X \mathcal{J}Z)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(fY, A_{\mathcal{J}Z}X) - \overline{g}(FY, \nabla_X \mathcal{J}Z)$. In view of (2.7), (2.11) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(\text{ltr}(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\mathcal{J}Y, \nabla_X \mathcal{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\mathcal{J}Y, A_{\mathcal{J}N}X) - \overline{g}(FY, D^s(X, \mathcal{J}N))$. Thus, we obtain the required results.
Theorem 4.4. Let $M$ be a mixed geodesic screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ defines a totally geodesic foliation if and only if $\nabla_X Z$ and $D^s(X, J_\xi)$ have no components in $\mathcal{J}(D_1)$, for all $X \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $\xi \in \Gamma(ltr(TM))$.

Proof. Let $M$ be a mixed geodesic screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution $D_1$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\nabla$ is a metric connection, using (2.1.14), (2.1.19) and (2.2.1), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we get $\nabla(\nabla_X Y, Z) = -\nabla(\nabla_X Z, J_\xi Y)$, which implies $\nabla(\nabla_X Y, Z) = -\nabla(\nabla_X Z, J_\xi Y)$. Now, from (2.1.14), (2.2.1) and (2.2.1), for any $X, Y \in \Gamma(D_1)$ and $\xi \in \Gamma(ltr(TM))$, we obtain $\nabla(\nabla_X Y, N) = -\nabla(\nabla_X Y, D^s(X, J_\xi))$. This concludes the theorem.

Theorem 4.5. Let $M$ be a mixed geodesic screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection $\nabla$ on $S(TM)$ is a metric connection if and only if

(i) $\nabla(fW, A^s_{\xi} Z) = \nabla(FW, h^s(Z, J_\xi))$,

(ii) $h^s(X, J_\xi)$ has no component in $\mathcal{J}(D_1)$, for all $X \in \Gamma(D_1)$, $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(RadTM)$.

Proof. Let $M$ be a mixed geodesic screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $h^l(X, Z) = 0$, for all $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. Since $\nabla$ is a metric connection, using (2.1.14), (2.2.1) and (2.2.1), for any $X, Y \in \Gamma(D_1)$ and $\xi \in \Gamma(RadTM)$, we obtain $\nabla(h^l(X, Y), \xi) = -g(\nabla_X J_\xi Y, \xi) + \nabla(\nabla_X J_\xi Y, X)$, which implies $\nabla(h^l(X, Y), \xi) = -g(\nabla_X J_\xi Y, \xi)$. In view of (2.1.14), (2.2.1) and (2.2.1), for any $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(RadTM)$, we get $\nabla(h^l(Z, W), \xi) = -\nabla(fW, \nabla_Z J_\xi) - \nabla(FW, h^s(Z, J_\xi))$, thus $\nabla(h^l(Z, W), \xi) = \nabla(fW, A^s_{\xi} Z) - \nabla(FW, h^s(Z, J_\xi))$. This completes the proof.

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