CANONICAL CONNECTIONS ON PARA-KENMOTSU MANIFOLDS

Adara M. Blaga

Abstract. In the context of para-Kenmotsu geometry properties of the \( \varphi \)-conjugate connections of some canonical linear connections (Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections) are established, underlining the relations between them and between their structure and virtual tensors. The case of projectively and dual-projectively equivalent connections is also treated. In particular, it is proved that the structure tensor is invariant under dual-projective transformations.

AMS Mathematics Subject Classification (2010): 53C21; 53C25; 53C44
Key words and phrases: linear connections; para-Kenmotsu structure

1. Introduction

The present paper is dedicated to a brief study of some canonical connections defined on a para-Kenmotsu manifold: Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections with a special view towards \( \varphi \)-conjugation. The structure and the virtual tensors attached to these connections are considered and in the last section, the invariance of the structure tensor under dual-projective transformations is proved. Note that the para-Kenmotsu structure was introduced by J. Welyczko in [13] for 3-dimensional normal almost paracontact metric structures. A similar notion called \( P \)-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [12].

Let \( M \) be a \((2n+1)\)-dimensional smooth manifold, \( \varphi \) a tensor field of \((1,1)\)-type, \( \xi \) a vector field, \( \eta \) a 1-form and \( g \) a pseudo-Riemannian metric on \( M \) of signature \((n+1,n)\).

Definition 1.1. [13] We say that \((\varphi, \xi, \eta, g)\) defines an almost paracontact metric structure on \( M \) if:

\[
\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad \varphi^2 = I_{\xi(M)} - \eta \otimes \xi, \quad g(\varphi \cdot, \varphi \cdot) = -g + \eta \otimes \eta
\]

and \( \varphi \) induces on the \(2n\)-dimensional distribution \( D := \ker \eta \) an almost paracomplex structure \( P \) i.e. \( P^2 = I_{\xi(M)} \) and the eigensubbundles \( D^+, D^- \), corresponding to the eigenvalues \( 1, -1 \) of \( P \) respectively, have equal dimension \( n \); hence \( D = D^+ \oplus D^- \).

\[\text{1} The author acknowledges the support by the research grant PN-II-ID-PCE-2011-3-0921.
\[\text{2} \] Department of Mathematics, West University of Timișoara,
e-mail: adarablaga@yahoo.com
In this case, \((M, \varphi, \xi, \eta, g)\) is called \emph{almost paracontact metric manifold}, \(\varphi\) the structure endomorphism, \(\xi\) the characteristic vector field, \(\eta\) the paracontact form and \(g\) \emph{compatible metric}.

Examples of almost paracontact metric structures can be found in [8] and [6]. From the definition it follows that \(\eta\) is the \(g\)-dual of the unitary vector field \(\xi\):
\[\eta(X) = g(X, \xi)\]  
and \(\varphi\) is a \(g\)-skew-symmetric operator:
\[g(\varphi X, Y) = -g(X, \varphi Y).\]

Remark that the canonical distribution \(D\) is \(\varphi\)-invariant since \(D = \text{Im}\varphi\). Moreover, \(\xi\) is orthogonal to \(D\) and therefore the tangent bundle splits orthogonally:
\[TM = D \oplus \langle \xi \rangle.\]

An analogue of the Kenmotsu manifold [9] in paracontact geometry will be further considered.

**Definition 1.2.** We say that the almost paracontact metric structure \((\varphi, \xi, \eta, g)\) is \emph{para-Kenmotsu} if the Levi-Civita connection \(\nabla\) of \(g\) satisfies \((\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X\), for any \(X, Y \in \mathcal{X}(M)\).

**Example 1.3.** Let \(M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}\) where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Set
\[\varphi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -\frac{\partial}{\partial z}, \quad \eta := -dz, \quad g := dx \otimes dx - dy \otimes dy + dz \otimes dz.\]

Then \((\varphi, \xi, \eta, g)\) defines a para-Kenmotsu structure on \(\mathbb{R}^3\).

Properties of this structure will be given in the next Proposition.

**Proposition 1.4.** On a para-Kenmotsu manifold \((M, \varphi, \xi, \eta, g)\), the following relations hold:
\[\nabla \xi = I_{\mathcal{X}(M)} - \eta \otimes \xi\]  
\[\eta(\nabla_X \xi) = 0,\]
\[R_{\nabla}(X, Y)\xi = \eta(X)Y - \eta(Y)X,\]
\[\eta(R_{\nabla}(X, Y)W) = -\eta(X)g(Y, W) + \eta(Y)g(X, W),\]
\[\nabla \eta = g - \eta \otimes \eta,\]
\[L_\xi \varphi = 0, \quad L_\xi \eta = 0, \quad L_\xi g = 2(g - \eta \otimes \eta),\]
where \(R_{\nabla}\) is the Riemann curvature tensor field of the Levi-Civita connection \(\nabla\) associated to \(g\). Moreover, \(\eta\) is closed, the distribution \(D\) is involutive and the Nijenhuis tensor field of \(\varphi\) vanishes identically.
2. Canonical connections on \((M, \varphi, \xi, \eta, g)\)

Let \((M, \varphi, \xi, \eta, g)\) be a para-Kenmotsu manifold. From [1], [11], [7], [14] we get the expressions and the relations with the para-Kenmotsu structure of the canonical connections on \(M\) we are interested in.

1. **Levi-Civita connection** \(\nabla\) satisfies [1]:

\[
\nabla \varphi = g(\varphi, \cdot) \otimes \xi - \varphi \otimes \eta, \quad \nabla \xi = I_{X(M)} - \eta \otimes \xi, \quad \nabla \eta = g - \eta \otimes \eta, \quad \nabla g = 0,
\]

its torsion and curvature being given by:

\[
T_{\nabla} = 0
\]

\[
\eta(R_{\nabla}(X,Y)W) = -\eta(X)g(Y, W) + \eta(Y)g(X, W).
\]

2. **Schouten-van Kampen connection** \(\tilde{\nabla}\) equals to [11]:

\[
\tilde{\nabla} := \nabla - I_{X(M)} \otimes \eta + g \otimes \xi
\]

and satisfies [11]:

\[
\tilde{\nabla} \varphi = 0, \quad \tilde{\nabla} \xi = 0, \quad \tilde{\nabla} \eta = 0, \quad \tilde{\nabla} g = 0,
\]

its torsion and curvature being given by:

\[
T_{\tilde{\nabla}} = \eta \otimes I_{X(M)} - I_{X(M)} \otimes \eta
\]

\[
R_{\tilde{\nabla}}(X,Y)W = R_{\nabla}(X,Y)W - g(W, X)Y + g(Y, W)X - \eta(W)g(X, Y)\xi.
\]

3. **Golab connection** \(\nabla^G\) equals to [7]:

\[
\nabla^G := \nabla - \eta \otimes \varphi
\]

and satisfies [2]:

\[
\nabla^G \varphi = \nabla \varphi, \quad \nabla^G \xi = \nabla \xi, \quad \nabla^G \eta = \nabla \eta, \quad \nabla^G g = \nabla g = 0,
\]

its torsion and curvature being given by:

\[
T_{\nabla^G} = \varphi \otimes \eta - \eta \otimes \varphi
\]

\[
R_{\nabla^G}(X,Y)W = R_{\nabla}(X,Y)W + g(T, W)\xi - g(\xi, W)T, \quad \text{where } T := -T_{\nabla^G}(X,Y).
\]

4. **Zamkovoy canonical paracontact connection** \(\nabla^Z\) equals to [14]:

\[
\nabla^Z_X Y := \nabla_X Y + \eta(X)\varphi Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)Y \cdot \xi
\]
equivalent to:
(2.13) \[ \nabla^Z = \nabla - I_{\mathcal{X}(M)} \otimes \eta + g \otimes \xi + \eta \otimes \varphi \]
and satisfies [23]:
(2.14) \[ \nabla^Z \varphi = 0, \ \nabla^Z \xi = 0, \ \nabla^Z \eta = 0, \ \nabla^Z g = 0, \]
its torsion and curvature being given by:
(2.15) \[ T_{\nabla^Z} = \eta \otimes (\varphi + I_{\mathcal{X}(M)}) - (\varphi + I_{\mathcal{X}(M)}) \otimes \eta (= -T_{\nabla^\varphi}) \]
(2.16) \[ R_{\nabla^Z} (X,Y)W = R_{\nabla}(X,Y)W + g(Y, W)X - g(X, W)Y. \]

3. \( \varphi \)-conjugate connections

In this section we shall consider the \( \varphi \)-conjugate connections of the four canonical connections presented above on a para-Kenmotsu manifold.

Recall that for an arbitrary linear connection \( \nabla \), the \( \varphi \)-conjugate connection of \( \nabla \) is defined by:
(3.1) \[ \nabla^{(\varphi)} := \nabla + \varphi \circ \nabla \varphi, \]
that is, \( \nabla^{(\varphi)} X Y = \varphi(\nabla_X \varphi Y) + \eta(\nabla_X Y)\xi \), for any \( X, Y \in \mathfrak{X}(M) \). Applying the \( \varphi \)-conjugation by \( n \) times, \( n \in \mathbb{N} \), we can prove, by mathematical induction, that:
(3.2) \[ \nabla^{n(\varphi)} X Y = \varphi^n(\nabla_X \varphi^n Y) + \eta(\nabla_X Y)\xi. \]

Therefore:
(3.3) \[ \nabla^{n(\varphi)} X Y - \nabla^{(\varphi)} X Y = \varphi^n(\nabla_X \varphi^n Y) - \varphi(\nabla_X Y) \]
for any \( n \in \mathbb{N} \) and any \( X, Y \in \mathfrak{X}(M) \).

Appearing in the theory of Courant algebroids, the \( \varphi \)-torsion of a linear connection can be expressed in terms of torsion of the \( \varphi \)-conjugate connection, namely:

**Proposition 3.1.** Let \( (M, \varphi, \xi, \eta, g) \) be a para-Kenmotsu manifold, \( \nabla \) a linear connection on \( M \) and \( \nabla^{(\varphi)} \) its \( \varphi \)-conjugate connection. Then:
(3.4) \[ T_{(\nabla^{\varphi})}(X,Y) = -T_{\nabla^{(\varphi)}}(X,Y) + T_{\nabla}(X,Y) + \varphi(T_{\nabla}(\varphi X,Y)) + \varphi(T_{\nabla}(X,\varphi Y)) - \varphi^2(T_{\nabla}(X,Y)), \]
for any \( X, Y \in \mathfrak{X}(M) \).
Proof. For any $X, Y \in \mathfrak{X}(M)$, the $\varphi$-torsion of $\nabla$ is defined:

$$T_{(\nabla, \varphi)}(X, Y) := \varphi(\nabla_\varphi Y - \nabla_\varphi X) - [\varphi X, \varphi Y].$$

Observe that:

$$T_{\nabla}(X, Y) - T_{\nabla(\varphi)}(X, Y) = \nabla^{(\varphi)}_X Y - \nabla_X Y - \nabla^{(\varphi)}_Y X + \nabla_{\nabla Y} X =$$

$$(\varphi \circ \nabla_\varphi)(Y, X) - (\varphi \circ \nabla_\varphi)(X, Y) =$$

$$= \varphi(\nabla_\varphi Y \varphi X - \nabla_X \varphi Y) - \varphi^2(\nabla_\varphi Y X - \nabla_X Y) =$$

$$= \varphi(T_{\nabla}(Y, \varphi X) + \nabla_\varphi X Y + [Y, \varphi X] - T_{\nabla}(X, \varphi Y) - \nabla_\varphi Y X - [X, \varphi Y]) +$$

$$+ \varphi^2(T_{\nabla}(X, Y) + [X, Y]) =$$

$$= \varphi(\nabla_\varphi X Y - \nabla_\varphi Y X) - \varphi[X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] +$$

$$+ \varphi(T_{\nabla}(Y, \varphi X) - T_{\nabla}(X, \varphi Y)) + \varphi^2(T_{\nabla}(X, Y)) :=$$

$$:= T_{(\nabla, \varphi)}(X, Y) + N_\varphi(X, Y) + \varphi(T_{\nabla}(Y, \varphi X)) - \varphi(T_{\nabla}(X, \varphi Y)) + \varphi^2(T_{\nabla}(X, Y)) =$$

$$= T_{(\nabla, \varphi)}(X, Y) - \varphi(T_{\nabla}(\varphi X, Y)) - \varphi(T_{\nabla}(X, \varphi Y)) + \varphi^2(T_{\nabla}(X, Y)).$$

Consider now the $\varphi$-conjugate connection of the Levi-Civita connection $\nabla$:

$$(\nabla^{(\varphi)}) := \nabla + \varphi \circ \nabla$$

which equals to:

$$(\nabla^{(\varphi)})_X Y = \nabla_X Y - \eta(Y)X + \eta(X)\eta(Y)\xi.$$
Proof. 1.

\( (\nabla^{(\varphi)}_X \varphi) Y := \nabla^{(\varphi)}_X \varphi Y - \varphi(\nabla^{(\varphi)}_X Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y) + \eta(Y) \varphi X := (\nabla_X \varphi) Y + \eta(Y) \varphi X = g(\varphi X, Y) \xi; \)

2.

\( \nabla^{(\varphi)}_X \xi = \nabla_X \xi - X + \eta(X) \xi; \)

3.

\( (\nabla^{(\varphi)}_X \eta) Y := X(\eta(Y)) - \eta(\nabla^{(\varphi)}_X Y) = X(\eta(Y)) - \eta(\nabla_X Y) := (\nabla_X \eta) Y; \)

4.

\( (\nabla^{(\varphi)}_X g)(Y, W) := X(g(Y, W)) - g(\nabla^{(\varphi)}_X Y, W) - g(Y, \nabla^{(\varphi)}_X W) = \eta(Y) g(X, W) + \eta(W) g(X, Y) - 2 \eta(X) \eta(Y) \eta(W); \)

5.

\( T_{\nabla^{(\varphi)}}(X, Y) := \nabla^{(\varphi)}_X Y - \nabla^{(\varphi)}_Y X - [X, Y] = \eta(X) Y - \eta(Y) X; \)

6.

\( R_{\nabla^{(\varphi)}}(X, Y) W := \nabla^{(\varphi)}_X \nabla^{(\varphi)}_Y W - \nabla^{(\varphi)}_Y \nabla^{(\varphi)}_X W - \nabla^{(\varphi)}_{[X,Y]} W. \)

We obtain:

\[ \nabla^{(\varphi)}_X \nabla^{(\varphi)}_Y W = \nabla^{(\varphi)}_X (\nabla_Y W) - \eta(W) \nabla^{(\varphi)}_X Y - X(\eta(W)) Y + \eta(Y) \eta(W) \nabla^{(\varphi)}_X \xi + X(\eta(Y)) \eta(W) \xi + X(\eta(W)) \eta(Y) \xi = \nabla_X \nabla_Y W - \eta(\nabla_Y W) X + \eta(X) \eta(\nabla_Y W) \xi - \eta(W) \nabla_X Y - \eta(Y) \eta(W) X - \eta(X) \eta(Y) \eta(W) \xi - X(\eta(W)) Y + X(\eta(Y)) \eta(W) \xi + X(\eta(W)) \eta(Y) \xi. \]

Similarly:

\[ \nabla^{(\varphi)}_Y \nabla^{(\varphi)}_X W = \nabla_Y \nabla_X W - \eta(\nabla_X W) Y + \eta(Y) \eta(\nabla_X W) \xi - \eta(W) \nabla_Y X - \eta(X) \eta(W) Y - \eta(X) \eta(Y) \eta(W) \xi - Y(\eta(W)) X + Y(\eta(X)) \eta(W) \xi + Y(\eta(W)) \eta(X) \xi. \]

Also:

\[ \nabla^{(\varphi)}_{[X,Y]} W = \nabla_{[X,Y]} W - \eta(W) [X, Y] + \eta([X, Y]) \eta(W) \xi. \]

It follows:

\[ R_{\nabla^{(\varphi)}}(X, Y) W = R_X(X, Y) W + (d\eta)(X, Y) \eta(W) \xi + g(W, \nabla_X \xi) \eta(Y) \xi - g(W, \nabla_Y \xi) \eta(X) \xi - \eta(\nabla_Y W) X + \eta(\nabla_X W) Y + Y(\eta(W)) X - X(\eta(W)) Y - \eta(Y) \eta(W) X + \eta(X) \eta(W) Y = \]
Canonical connections on para-Kenmotsu manifolds

\[\nabla(X; Y)W + g(W, X)\eta(Y)\xi - g(W, Y)\eta(X)\xi - g(W, X)Y + g(W, Y)X + \eta(X)\eta(W)Y - \eta(Y)\eta(W)X = \nabla(X; Y)W - g(W, X)(Y - \eta(Y)\xi) + g(Y, W)(X - \eta(X)\xi) + \eta(W)[\eta(X)Y - \eta(Y)X].\]

Concerning the Schouten-van Kampen, the Golab and the Zamkovoy canonical paracontact connections, we can state:

**Proposition 3.3.** On a para-Kenmotsu manifold \( (M, \varphi, \xi, \eta, g) \), the \( \varphi \)-conjugate connections of \( \tilde{\nabla} \), \( \nabla^G \) and \( \nabla^Z \) respectively, are given by:

\[
(3.7) \quad \tilde{\nabla}^\varphi = \tilde{\nabla}, \quad (\nabla^G)^\varphi = \nabla^\varphi - \eta \otimes \varphi, \quad (\nabla^Z)^\varphi = \nabla^Z.
\]

**Proof.** They follow from relations (2.4), (2.8), (2.12) and (3.1). \qed

**Remark 3.4.** For \( n \in \mathbb{N} \), applying the \( \varphi \)-conjugation by \( n \) times, we obtain:

\[
(\nabla^\varphi)^n = \nabla^n, \quad (\tilde{\nabla}^\varphi)^n = \tilde{\nabla}^n, \quad (\nabla^G)^n(\varphi) = (\nabla^G)^n, \quad (\nabla^Z)^n(\varphi) = \nabla^Z.
\]

Indeed, for \( \nabla \), \( \tilde{\nabla} \) and \( \nabla^Z \) follows immediately from the previous Proposition. For the Golab connection, notice that \( \varphi^{2n+1} = \varphi \), so:

\[
(\nabla^G)^n(\varphi)Y = \varphi^n(\nabla^G_X\varphi^nY) + \eta(\nabla^G_XY)\xi = \varphi^n(\nabla_X\varphi^nY) + \eta(\nabla_XY)\xi - \eta(X)\varphi^{2n+1}Y = \nabla^n_XY - \eta(X)\varphi Y = (\nabla^\varphi - \eta \otimes \varphi)(X, Y) = (\nabla^G)^\varphi.
\]

4. Relating \( \nabla \), \( \tilde{\nabla} \), \( \nabla^G \) and \( \nabla^Z \). A view towards the structure and the virtual tensors

Remark that the Golab connection \( \nabla^G \) is obtained by perturbing the Levi-Civita connection \( \nabla \) with \( \eta \otimes \varphi \), so the two connections coincide on \( \mathcal{D} \). The same thing happens for the Schouten-van Kampen connection \( \tilde{\nabla} \) and the Zamkovoy canonical paracontact connection \( \nabla^Z \):  

\[
(4.1) \quad \nabla^G = \nabla - \eta \otimes \varphi, \quad \tilde{\nabla} = \nabla^Z - \eta \otimes \varphi.
\]

Therefore:

\[
(4.2) \quad \nabla + \tilde{\nabla} = \nabla^G + \nabla^Z.
\]

Also, from relations (2.4), (2.8) and (2.12) follow that \( (\nabla, \nabla^G) \) and \( (\tilde{\nabla}, \nabla^Z) \) behave similarly with respect to \( (\varphi, \xi, \eta, g) \):

\[
(4.3) \quad \nabla \varphi = \nabla^G \varphi, \quad \nabla \xi = \nabla^G \xi, \quad \nabla \eta = \nabla^G \eta, \quad \nabla g = \nabla^G g = 0;
\]
\( \nabla \varphi = \nabla^Z \varphi = 0, \quad \nabla \xi = \nabla^Z \xi = 0, \quad \nabla \eta = \nabla^Z \eta = 0, \quad \nabla g = \nabla^Z g = 0. \)

Other geometrical structures associated to a pair of a tensor field \( \varphi \) and a linear connection \( \nabla \) are the structure and the virtual tensors defined as follows:

\[ C^\varphi(X, Y) := \frac{1}{2} \left[ (\nabla_{\varphi X} \varphi)Y + (\nabla_X \varphi) \varphi Y \right] \]

and respectively:

\[ B^\varphi(X, Y) := \frac{1}{2} \left[ (\nabla_{\varphi X} \varphi)Y - (\nabla_X \varphi) \varphi Y \right]. \]

These tensors have been introduced in [10] for almost complex structures. They also appear in [3] and, in [4], for almost product structures.

In our context, we have:

**Proposition 4.1.** On a para-Kenmotsu manifold \((M, \varphi, \xi, \eta, g)\), the structure and the virtual tensors associated to \( \nabla, \nabla^G \) and \( \nabla^Z \) satisfy:

\[ C^\varphi(X, Y) = C^\varphi_{(\nabla^G)}(X, Y) = -\frac{1}{2} \eta(Y) \varphi^2 X, \]

\[ B^\varphi(X, Y) = B^\varphi_{(\nabla^G)}(X, Y) = -\frac{1}{2} \eta(Y) [X + \eta(X) \xi] + g(X, Y) \xi, \]

\[ C^\varphi_{(\nabla)} = C^\varphi_{(\nabla^Z)} = B^\varphi_{(\nabla)} = B^\varphi_{(\nabla^Z)} = 0. \]

**Proof.** These relations follow if we replace the expressions \( (\nabla_X \varphi)Y = (\nabla^G_X \varphi)Y = g(\varphi X, Y) \xi - \eta(Y) \varphi X \) and \( \nabla \varphi = \nabla^Z \varphi = 0 \) in (11) and (10). \( \square \)

As a consequence:

**Corollary 4.2.** Under the hypotheses above, we have:

\[ C^\varphi - B^\varphi = C^\varphi_{(\nabla^G)} - B^\varphi_{(\nabla^G)} = -\frac{1}{2} L \xi g \otimes \xi. \]

Concerning their \( \varphi \)-conjugate connections, we can state:

**Proposition 4.3.** On a para-Kenmotsu manifold \((M, \varphi, \xi, \eta, g)\), the structure tensors of all the \( \varphi \)-conjugate connections of \( \nabla, \nabla^G \) and \( \nabla^Z \) vanish identically and the virtual tensors satisfy:

\[ B^\varphi_{(\nabla^{(\varphi)})} = B^\varphi_{(\nabla^G)^{(\varphi)}} = -g(\varphi, \varphi) \otimes \xi, \quad B^\varphi_{(\nabla^{(\varphi)})} = B^\varphi_{(\nabla^Z)^{(\varphi)}} = 0. \]

**Proof.** Remark first:

\( \nabla^{(\varphi)} \varphi = \nabla \varphi + \varphi \otimes \eta, \quad (\nabla^G)^{(\varphi)} \varphi = \nabla^G \varphi + \varphi \otimes \eta, \)
The canonical connections on para-Kenmotsu manifolds

\[ \nabla(\varphi) = \tilde{\nabla}, \quad (\nabla^Z(\varphi)) = \nabla^Z \varphi \]

and use the fact that \( \nabla \varphi = \nabla^G \varphi = g(\varphi, \cdot) \otimes \xi - \varphi \otimes \eta \) and \( \tilde{\nabla} \varphi = \nabla^Z \varphi = 0 \). Then:

\[ C^\varphi_{\nabla(\varphi)}(X, Y) = C^\varphi_{\nabla^G}(X, Y) + \frac{1}{2} \eta(Y) \varphi^2 X = 0, \]

\[ C^\varphi_{(\nabla^G)(\varphi)}(X, Y) = C^\varphi_{\nabla^G}(X, Y) + \frac{1}{2} \eta(Y) \varphi^2 X = 0 \]

and

\[ C^\varphi_{(\nabla)(\varphi)} = C^\varphi_{\nabla^G} = C^\varphi_{Z} = C^\varphi_{(\nabla^Z)(\varphi)}. \]

Also:

\[ B^\varphi_{\nabla(\varphi)}(X, Y) = B^\varphi_{\nabla^G}(X, Y) + \frac{1}{2} \eta(Y) \varphi^2 X = -g(\varphi X, \varphi Y) \xi, \]

\[ B^\varphi_{(\nabla^G)(\varphi)}(X, Y) = B^\varphi_{\nabla^G}(X, Y) + \frac{1}{2} \eta(Y) \varphi^2 X = -g(\varphi X, \varphi Y) \xi \]

and

\[ B^\varphi_{Z} = B^\varphi_{\nabla^G} = B^\varphi_{(\nabla^Z)(\varphi)}. \]

\[ B^\varphi_{\nabla(\varphi)} = B^\varphi_{\nabla^G} = B^\varphi_{Z} = B^\varphi_{(\nabla^Z)(\varphi)}. \]

\[ \nabla(\varphi) Y := \nabla_X(\varphi) Y - \varphi(\nabla_X(\varphi) Y) = (\nabla_X(\varphi) Y + \varphi((\nabla_X(\varphi) Y) - \varphi^2((\nabla_X(\varphi) Y), \]

therefore:

\[ C^\varphi_{\nabla(\varphi)}(X, Y) := \frac{1}{2}[(\nabla_X(\varphi) Y + (\nabla_X(\varphi) \varphi) Y] = \frac{1}{2}[(\nabla_X(\varphi) Y + (\nabla_X(\varphi) \varphi Y] + \]

\[ + \frac{1}{2} \varphi[(\nabla_X(\varphi) \varphi + (\nabla_X(\varphi) \varphi^2 Y] - \frac{1}{2} \varphi^2[(\nabla_X(\varphi) \varphi) Y + (\nabla_X(\varphi) \varphi Y] := \]

\[ := C^\varphi_{\nabla}(X, Y) + \varphi(C^\varphi_{\nabla}(X, \varphi Y)) - \varphi^2(C^\varphi_{\nabla}(X, Y)). \]

Similarly:

\[ B^\varphi_{\nabla(\varphi)}(X, Y) = B^\varphi_{\nabla}(X, Y) + \varphi(B^\varphi_{\nabla}(X, \varphi Y)) - \varphi^2(B^\varphi_{\nabla}(X, Y)). \]

5. Projectively and dual-projectively equivalent connections

In the last section we shall treat the case of projectively and dual-projectively equivalent connections studying their invariance under such transformations. Recall that two linear connections \( \tilde{\nabla} \) and \( \tilde{\nabla}' \) are called [3]:

i) \textit{projectively equivalent} if there exists a 1-form \( \eta \) such that:

\[ \tilde{\nabla}' = \tilde{\nabla} + \eta \otimes I_{X(M)} + I_{X(M)} \otimes \eta; \]

ii) \textit{dual-projectively equivalent} if there exists a 1-form \( \eta \) such that:

\[ \tilde{\nabla}' = \tilde{\nabla} - g \otimes \xi, \]
where $\xi$ is the $g$-dual vector field of $\eta$ and $g$ a pseudo-Riemannian metric.

Consider $\nabla$ and $\nabla'$ two linear projectively equivalent connections satisfying:

\begin{equation}
\nabla' = \nabla + \eta \otimes I_{\Omega} + I_{\Omega} \otimes \eta, \tag{5.3}
\end{equation}

for $\eta$ the paracontact form, and their $\varphi$-conjugate connections, $\nabla^{(\varphi)}$ and $(\nabla')^{(\varphi)}$:

\begin{align*}
(\nabla')^{(\varphi)}_X Y &= \nabla'_X Y + \varphi((\nabla'_X \varphi)Y) = \\
&= \nabla_X Y + \varphi((\nabla_X \varphi)Y) + \eta(X)Y + \eta(Y)X - \eta(Y)\varphi^2 X := \\
(\nabla^{(\varphi)}_X Y &= \nabla^{(\varphi)}_X Y + \eta(X)Y + \eta(X)\eta(Y)\xi. \tag{5.4}
\end{align*}

From a direct computation follows:

**Lemma 5.1.** If $(M, \varphi, \xi, \eta, g)$ is a para-Kenmotsu manifold, then:

1. $\nabla' \varphi = \nabla \varphi - \varphi \otimes \eta$;
2. $(\nabla')^{(\varphi)} \varphi = (\nabla^{(\varphi)}) \varphi$.

**Proposition 5.2.** Let $(M, \varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold and $\nabla$, $\nabla'$ two linear projectively equivalent connections satisfying (5.3). Then the structure and the virtual tensors of them and their $\varphi$-conjugate connections satisfy:

\begin{align*}
C^{(\varphi)}_{\nabla'} &= C^{(\varphi)}_\nabla - \frac{1}{2} \varphi^2 \otimes \eta, \quad C^{(\varphi)}_{(\nabla')^{(\varphi)}} = C^{(\varphi)}_{\nabla^{(\varphi)}}, \tag{5.5} \\
B^{(\varphi)}_{\nabla'} &= B^{(\varphi)}_\nabla - \frac{1}{2} \varphi^2 \otimes \eta, \quad B^{(\varphi)}_{(\nabla')^{(\varphi)}} = B^{(\varphi)}_{\nabla^{(\varphi)}}. \tag{5.6}
\end{align*}

**Proof.** Use the relations from Lemma 5.1 in the expressions of $C^{(\varphi)}$ and $B^{(\varphi)}$.

As a consequence:

**Corollary 5.3.** Under the hypotheses above, we have:

\begin{equation}
C^{(\varphi)}_{\nabla'} - C^{(\varphi)}_\nabla = B^{(\varphi)}_{\nabla'} - B^{(\varphi)}_\nabla = C^{(\varphi)}_\nabla. \tag{5.7}
\end{equation}

Take now $\nabla$ and $\nabla'$ two linear dual-projectively equivalent connections satisfying:

\begin{equation}
\nabla' = \nabla - g \otimes \xi, \tag{5.8}
\end{equation}

for $\xi$ the characteristic vector field, and their $\varphi$-conjugate connections, $\nabla^{(\varphi)}$ and $(\nabla')^{(\varphi)}$:

\begin{align*}
(\nabla')^{(\varphi)}_X Y &= \nabla'_X Y + \varphi((\nabla'_X \varphi)Y) = \nabla_X Y + \varphi((\nabla_X \varphi)Y) - g(X, Y)\xi := \\
&= \nabla^{(\varphi)}_X Y - g(X, Y)\xi = (\nabla^{(\varphi)})'_X Y. \tag{5.9}
\end{align*}

From a direct computation follows:
Lemma 5.4. If \((M, \varphi, \xi, \eta, g)\) is a para-Kenmotsu manifold, then:

1. \(\tilde{\nabla}' \varphi = \tilde{\nabla} \varphi + g(\varphi, \cdot) \otimes \xi\);
2. \((\tilde{\nabla}')^{(\varphi)} = \tilde{\nabla}^{(\varphi)} + g(\varphi, \cdot) \otimes \xi\).

Proposition 5.5. Let \((M, \varphi, \xi, \eta, g)\) be a para-Kenmotsu manifold and \(\nabla, \tilde{\nabla}'\) two linear dual-projectively equivalent connections satisfying (5.8). Then the structure and the virtual tensors of them and their \(\varphi\)-conjugate connections satisfy:

\[
C_{\tilde{\nabla}'}^\varphi = C_{\tilde{\nabla}}^\varphi, \quad C_{(\tilde{\nabla}')^{(\varphi)}}^\varphi = C_{\tilde{\nabla}^{(\varphi)}}^\varphi, 
\]

\[
B_{\tilde{\nabla}'}^\varphi = B_{\tilde{\nabla}}^\varphi - 2g(\varphi, \varphi) \otimes \xi, \quad B_{(\tilde{\nabla}')^{(\varphi)}}^\varphi = B_{\tilde{\nabla}^{(\varphi)}}^\varphi - 2g(\varphi, \varphi) \otimes \xi. 
\]

Proof. Use the relations from Lemma 5.4 in the expressions of \(C^\varphi\) and \(B^\varphi\). □

We can conclude:

Theorem 5.6. On a para-Kenmotsu manifold, the structure tensor is invariant under dual-projective transformations.

References


