ON UPPER AND LOWER CONTRA-$\omega$-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we define contra-$\omega$-continuous multifunctions between topological spaces and obtain some characterizations and some basic properties of such multifunctions.

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1. Introduction

Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunctions. A. Al-Omari et. al. introduced the concept of contra-$\omega$-continuous functions between topological spaces. In this paper, we define contra-$\omega$-continuous multifunctions and obtain some characterizations and some basic properties of such multifunctions.

2. Preliminaries

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a space $X$. For a subset $A$ of $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of $A$ with respect to $\tau$ and the interior of $A$ with respect to $\tau$, respectively. Recently, as generalization of closed sets, the notion of $\omega$-closed sets were introduced and studied by Hdeib. A point $x \in X$ is called a condensation point of $A$ if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. $A$ is said to be $\omega$-closed if it contains all its condensation points. The complement of an $\omega$-closed set is said to be $\omega$-open. It is well known that a subset $W$ of a space $(X, \tau)$ is $\omega$-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \setminus W$ is countable. The

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family of all ω-open subsets of a topological space \((X, \tau)\), denoted by \(\omega O(X)\), forms a topology on \(X\) finer than \(\tau\). The family of all ω-closed subsets of a topological space \((X, \tau)\) is denoted by \(\omega C(X)\). The ω-closure and the ω-interior, that can be defined in the same way as \(\text{Cl}(A)\) and \(\text{Int}(A)\), respectively, will be denoted by \(\omega \text{Cl}(A)\) and \(\omega \text{Int}(A)\), respectively. We set \(\omega O(X, x) = \{A: A \in \omega O(X) \text{ and } x \in A\}\) and \(\omega C(X, x) = \{A: A \in \omega C(X) \text{ and } x \in A\}\).

By a multifunction \(F: (X, \tau) \rightarrow (Y, \sigma)\), following [3], we shall denote the upper and lower inverse of a set \(B\) of \(Y\) by \(F^+(B)\) and \(F^-(B)\), respectively, that is, \(F^+(B) = \{x \in X : F(x) \subseteq B\}\) and \(F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}\). In particular, \(F^-(Y) = \{x \in X : y \in F(x)\}\) for each point \(y \in Y\) and for each \(A \subset X\), \(F(A) = \bigcup_{x \in A} F(x)\). Then \(F\) is said to be surjection if \(F(X) = Y\) and injection if \(x \neq y\) implies \(F(x) \cap F(y) = \emptyset\).

**Definition 2.1.** A multifunction \(F: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \([13]\):

(i) upper \(\omega\)-continuous if for each point \(x \in X\) and each open set \(V\) containing \(F(x)\), there exists \(U \in \omega O(X, x)\) such that \(F(U) \subset V\);

(ii) lower \(\omega\)-continuous if for each point \(x \in X\) and each open set \(V\) such that \(F(x) \cap V \neq \emptyset\), there exists \(U \in \omega O(X, x)\) such that \(U \subset F^-(V)\).

**Definition 2.2.** A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \([2]\) contra-\(\omega\)-continuous if for each point \(x \in X\) and each open set \(V\) containing \(f(x)\), there exists \(U \in \omega O(X, x)\) such that \(f(U) \subset V\).

3. On upper and lower contra-\(\omega\)-continuous multifunctions

**Definition 3.1.** A multifunction \(F: (X, \tau) \rightarrow (Y, \sigma)\) is said to be:

(i) upper contra-\(\omega\)-continuous if for each point \(x \in X\) and each closed set \(V\) containing \(F(x)\), there exists \(U \in \omega O(X, x)\) such that \(F(U) \subset V\);

(ii) lower contra-\(\omega\)-continuous if for each point \(x \in X\) and each closed set \(V\) such that \(F(x) \cap V \neq \emptyset\), there exists \(U \in \omega O(X, x)\) such that \(U \subset F^-(V)\).

The following examples show that the concepts of upper \(\omega\)-continuity (resp. lower \(\omega\)-continuity) and upper contra-\(\omega\)-continuity (resp. lower contra-\(\omega\)-continuity) are independent of each other.

**Example 3.2.** Let \(X = \mathbb{R}\) with the topology \(\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - Q\}\). Define a multifunction \(F: (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)\) as follows:

\[
F(x) = \begin{cases} 
Q & \text{if } x \in \mathbb{R} - Q \\
\mathbb{R} - Q & \text{if } x \in Q.
\end{cases}
\]

Then \(F\) is upper contra-\(\omega\)-continuous but is not upper \(\omega\)-continuous.

**Example 3.3.** Let \(X = \mathbb{R}\) with the topology \(\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - Q\}\). Define a multifunction \(F: (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)\) as follows:

\[
F(x) = \begin{cases} 
Q & \text{if } x \in Q \\
\mathbb{R} - Q & \text{if } x \in \mathbb{R} - Q.
\end{cases}
\]

Then \(F\) is upper \(\omega\)-continuous but is not upper contra-\(\omega\)-continuous.
In a similar form, we can find examples in order to show that lower contra-\(\omega\)-continuity and lower \(\omega\)-continuity are independent of each other.

**Theorem 3.4.** The following statements are equivalent for a multifunction \(F : (X, \tau) \to (Y, \sigma)\):

(i) \(F\) is upper contra-\(\omega\)-continuous;

(ii) \(F^+(V) \in \omega O(X)\) for every closed subset \(V\) of \(Y\);

(iii) \(F^-(V) \in \omega C(X)\) for every open subset \(V\) of \(Y\);

(iv) for each \(x \in X\) and each closed set \(K\) containing \(F(x)\), there exists \(U \in \omega O(X, x)\) such that if \(y \in U\), then \(F(y) \subseteq K\).

**Proof.** (i) \(\Leftrightarrow\) (ii): Let \(V\) be a closed subset in \(Y\) and \(x \in F^+(V)\). Since \(F\) is upper contra-\(\omega\)-continuous, there exists \(U \in \omega O(X, x)\) such that \(F(U) \subseteq V\). Hence, \(F^+(V)\) is \(\omega\)-open in \(X\). The converse is similar.

(ii) \(\Leftrightarrow\) (iii): It follows from the fact that \(F^+(Y \setminus V) = X \setminus F^-(V)\) for every subset \(V\) of \(Y\).

(iii) \(\Leftrightarrow\) (iv): This is obvious. \(\square\)

**Theorem 3.5.** The following statements are equivalent for a multifunction \(F : (X, \tau) \to (Y, \sigma)\):

(i) \(F\) is lower contra-\(\omega\)-continuous;

(ii) \(F^-(V) \in \omega O(X)\) for every closed subset \(V\) of \(Y\);

(iii) \(F^+(K) \in \omega C(X)\) for every open subset \(K\) of \(Y\);

(iv) for each \(x \in X\) and each closed set \(K\) such that \(F(x) \cap K \neq \emptyset\), there exists \(U \in \omega O(X, x)\) such that if \(y \in U\), then \(F(y) \subseteq K\).

**Proof.** The proof is similar to that of Theorem 3.4. \(\square\)

**Corollary 3.6.** \([3]\) The following statements are equivalent for a function \(f : X \to Y\):

(i) \(f\) is contra-\(\omega\)-continuous;

(ii) \(f^{-1}(V) \in \omega O(X)\) for every closed subset \(V\) of \(Y\);

(iii) \(f^{-1}(U) \in \omega C(X)\) for every open subset \(U\) of \(Y\);

(iv) for each \(x \in X\) and each closed set \(K\) containing \(f(x)\), there exists \(U \in \omega O(X, x)\) such that \(f(U) \subseteq K\).

**Definition 3.7.** A topological space \((X, \tau)\) is said to be semi-regular \([11]\) if for each open set \(U\) of \(X\) and for each point \(x \in U\), there exists a regular open set \(V\) such that \(x \in V \subseteq U\).

**Definition 3.8.** \([12]\) Let \((X, \tau)\) be a topological space and \(A\) a subset of \(X\) and \(x\) a point of \(X\). Then
(i) \( x \) is called \( \delta \)-cluster point of \( A \) if \( A \cap \text{Int(Cl}(U)) \neq \emptyset \), for each open set \( U \) containing \( x \).

(ii) the family of all \( \delta \)-cluster points of \( A \) is called the \( \delta \)-closure of \( A \) and is denoted by \( \text{Cl}_\delta(A) \).

(iii) \( A \) is said to be \( \delta \)-closed if \( \text{Cl}_\delta(A) = A \). The complement of a \( \delta \)-closed set is said to be a \( \delta \)-open set.

**Theorem 3.9.** For a multifunction \( F : (X, \tau) \to (Y, \sigma) \), where \( Y \) is semi-regular, the following are equivalent:

(i) \( F \) is upper contra-\( \omega \)-continuous;

(ii) \( F^+(\text{Cl}_\delta(B)) \in \omega O(X) \) for every subset \( B \) of \( Y \);

(iii) \( F^+(K) \in \omega O(X) \) for every \( \delta \)-closed subset \( K \) of \( Y \);

(iv) \( F^-(V) \in \omega C(X) \) for every \( \delta \)-open subset \( V \) of \( Y \).

**Proof.**  
(i) \( \Rightarrow \) (ii): Let \( B \) be any subset of \( Y \). Then \( \text{Cl}_\delta(B) \) is closed and by Theorem 3.4, \( F^+(\text{Cl}_\delta(B)) \in \omega O(X) \).  
(ii) \( \Rightarrow \) (iii): Let \( K \) be a \( \delta \)-closed set of \( Y \). Then \( \text{Cl}_\delta(K) = K \). By (ii), \( F^+(K) \) is \( \omega \)-open.  
(iii) \( \Rightarrow \) (iv): Let \( V \) be a \( \delta \)-open set of \( Y \). Then \( Y \setminus V \) is \( \delta \)-closed. By (iii), \( F^+(Y \setminus V) = X \setminus F^-(V) \) is \( \omega \)-open. Hence, \( F^-(V) \) is \( \omega \)-closed.  
(iv) \( \Rightarrow \) (i): Let \( V \) be any open set of \( Y \). Since \( Y \) is semi-regular, \( V \) is \( \delta \)-open. By (iv), \( F^-(V) \) is \( \omega \)-closed and by Theorem 3.4, \( F \) is upper contra-\( \omega \)-continuous.

**Theorem 3.10.** For a multifunction \( F : (X, \tau) \to (Y, \sigma) \), where \( Y \) is semi-regular, the following are equivalent:

(i) \( F \) is lower contra-\( \omega \)-continuous;

(ii) \( F^-(\text{Cl}_\delta(B)) \in \omega O(X) \) for every subset \( B \) of \( Y \);

(iii) \( F^-(K) \in \omega O(X) \) for every \( \delta \)-closed subset \( K \) of \( Y \);

(iv) \( F^+(V) \in \omega C(X) \) for every \( \delta \)-open subset \( V \) of \( Y \).

**Proof.** The proof is similar to that of Theorem 3.9.

**Remark 3.11.** By Theorems 3.9 and 3.10, we obtain the following new characterization for contra-\( \omega \)-continuous functions.

**Corollary 3.12.** For a function \( f : X \to Y \), where \( Y \) is semi-regular, the following are equivalent:

(i) \( f \) is contra-\( \omega \)-continuous;

(ii) \( f^{-1}(\text{Cl}_\delta(B)) \in \omega O(X) \) for every subset \( B \) of \( Y \);

(iii) \( f^{-1}(K) \in \omega O(X) \) for every \( \delta \)-closed subset \( K \) of \( Y \);

(iv) \( f^{-1}(V) \in \omega C(X) \) for every \( \delta \)-open subset \( V \) of \( Y \).
Definition 3.13. A subset $K$ of a space $X$ is said to be strongly $S$-closed \[\] (resp. $\omega$-compact \[\]) relative to $X$ if every cover of $K$ by closed (resp. $\omega$-open) sets of $X$ has a finite subcover. A space $X$ is said to be strongly $S$-closed (resp. $\omega$-compact) if $X$ is strongly $S$-closed (resp. $\omega$-compact) relative to $X$.

Theorem 3.14. Let $F : (X, \tau) \to (Y, \sigma)$ be an upper contra-$\omega$-continuous surjective multifunction and $F(x)$ is strongly $S$-closed relative to $Y$ for each $x \in X$. If $A$ is a $\omega$-continuous relative to $X$, then $F(A)$ is strongly $S$-closed relative to $Y$.

Proof. Let $\{V_i : i \in \Delta\}$ be any cover of $F(A)$ by closed sets of $Y$. For each $x \in A$, there exists a finite subset $\Delta(x)$ of $\Delta$ such that $F(x) \subset \bigcup\{V_i : i \in \Delta(x)\}$. Put $V(x) = \bigcup\{V_i : i \in \Delta(x)\}$. Then $F(x) \subset V(x)$ and there exists $U(x) \in \omega O(X, x)$ such that $F(U(x)) \subset V(x)$. Since $\{U(x) : x \in A\}$ is a cover of $A$ by $\omega$-open sets in $X$, there exists a finite number of points of $A$, say, $x_1, x_2, \ldots, x_n$ such that $A \subset \bigcup\{U(x_i) : 1 = 1, 2, \ldots, n\}$. Therefore, we obtain $F(A) \subset F(\bigcup_{i=1}^n U(x_i)) \subset \bigcup_{i=1}^n F(U(x_i)) \subset \bigcup_{i=1}^n V(x_i) \subset \bigcup_{i=1}^n \bigcup_{i=\Delta(x_i)} V_i$. This shows that $F(A)$ is strongly $S$-closed relative to $Y$. \[\]

Corollary 3.15. Let $F : (X, \tau) \to (Y, \sigma)$ be an upper contra-$\omega$-continuous surjective multifunction and $F(x)$ is $\omega$-compact relative to $Y$ for each $x \in X$. If $X$ is $\omega$-compact, then $Y$ is strongly $S$-closed.

Corollary 3.16. If $f : (X, \tau) \to (Y, \sigma)$ is contra-$\omega$-continuous surjective and $A$ is $\omega$-compact relative to $X$, then $f(A)$ is strongly $S$-closed relative to $Y$.

Lemma 3.17. Let $A$ and $B$ be subsets of a topological space $(X, \tau)$.

(i) If $A \in \omega O(X)$ and $B \in \tau$, then $A \cap B \in \omega O(B)$;

(ii) If $A \in \omega O(B)$ and $B \in \omega O(X)$, then $A \in \omega O(X)$.

Theorem 3.18. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $U$ an open subset of $X$. If $F$ is an upper contra-$\omega$-continuous (resp. lower contra-$\omega$-continuous), then $F|_U : U \to Y$ is an upper contra-$\omega$-continuous (resp. lower contra-$\omega$-continuous) multifunction.

Proof. Let $V$ be any closed set of $Y$. Let $x \in U$ and $x \in F|_U^{-1}(V)$. Since $F$ is lower contra-$\omega$-continuous multifunction, there exists a $\omega$-open set $G$ containing $x$ such that $G \subset F^{-1}(V)$. Then $x \in G \cap U \in \omega O(A)$ and $G \cap U \subset F|_U^{-1}(V)$ . This shows that $F|_U$ is a lower contra-$\omega$-continuous. The proof of the upper contra-$\omega$-continuous of $F|_U$ is similar. \[\]

Corollary 3.19. If $f : (X, \tau) \to (Y, \sigma)$ is contra-$\omega$-continuous and $U \in \tau$, then $f|_U : U \to Y$ is contra-$\omega$-continuous.

Theorem 3.20. Let $\{U_i : i \in \Delta\}$ be an open cover of a topological space $X$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper contra-$\omega$-continuous if and only if the restriction $F|_{U_i} : U_i \to Y$ is upper contra-$\omega$-continuous for each $i \in \Delta$. \[\]
Proof. Suppose that $F$ is upper contra-$\omega$-continuous. Let $i \in \Delta$ and $x \in U_i$ and $V$ be a closed set of $Y$ containing $F|_{U_i}(x)$. Since $F$ is upper contra-$\omega$-continuous and $F(x) = F|_{U_i}(x)$, there exists $G \in \omega O(X, x)$ such that $F(G) \subset V$. Set $U = G \cap U_i$, then $x \in U \in \omega O(U_i, x)$ and $F|_{U_i}(U) = F(U) \subset V$. Therefore, $F|_{U_i}$ is upper contra-$\omega$-continuous. Conversely, let $x \in X$ and $V \in \omega O(Y)$ containing $F(x)$. There exists $i \in \Delta$ such that $x \in U_i$. Since $F|_{U_i}$ is upper contra-$\omega$-continuous and $F(x) = F|_{U_i}(x)$, there exists $U \in \omega O(U_i, x)$ such that $F|_{U_i}(U) \subset V$. Then we have $U \in \omega O(X, x)$ and $F(U) \subset V$. Therefore, $F$ is upper contra-$\omega$-continuous.

Theorem 3.21. Let $X$ and $X_j$ be topological spaces for $i \in I$. If a multifunction $F : X \to \prod_{i \in I} X_i$ is an upper (lower) contra-$\omega$-continuous multifunction, then $P_i \circ F$ is an upper (lower) contra-$\omega$-continuous multifunction for each $i \in I$, where $P_i : \prod_{i \in I} X_i \to X_i$ is the projection for each $i \in I$.

Proof. Let $H_i$ be a closed subset of $X_j$. We have $(P_i \circ F)^+(H_j) = F^+(P_j^+(H_j)) = F^+(H_j \times \prod_{i \neq j} X_i)$. Since $F$ is an upper contra-$\omega$-continuous multifunction, $F^+(H_j \times \prod_{i \neq j} X_i)$ is $\omega$-open in $X$. Hence, $P_i \circ F$ is an upper (lower) contra-$\omega$-continuous.

Corollary 3.22. Let $X$ and $X_i$ be topological spaces for $i \in I$. If a function $F : X \to \prod_{i \in I} X_i$ is a contra-$\omega$-continuous, then $P_i \circ F$ is a contra-$\omega$-continuous function for each $i \in I$, where $P_i : \prod_{i \in I} X_i \to X_i$ is the projection for each $i \in I$.

Definition 3.23. A topological space $X$ is said to be:

(i) $\omega$-normal [1] if each pair of nonempty disjoint closed sets can be separated by disjoint $\omega$-open sets.

(ii) ultranormal [10] if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 3.24. If $F : (X, \tau) \to (Y, \sigma)$ is an upper contra-$\omega$-continuous punctually closed multifunction and $Y$ is ultranormal, then $X$ is $\omega$-normal.

Proof. The proof follows from the definitions.

Corollary 3.25. If $f : (X, \tau) \to (Y, \sigma)$ is a contra-$\omega$-continuous closed multifunction and $Y$ is ultranormal, then $X$ is $\omega$-normal.

Definition 3.26. [2] Let $A$ be a subset of a space $X$. The $\omega$-frontier of $A$ denoted by $\omega Fr(A)$, is defined as follows: $\omega Fr(A) = \omega Cl(A) \cap \omega Cl(X \setminus A)$.

Theorem 3.27. The set of points $x$ of $X$ at which a multifunction $F : (X, \tau) \to (Y, \sigma)$ is not upper contra-$\omega$-continuous (resp. upper contra-$\omega$-continuous) is identical with the union of the $\omega$-frontiers of the upper (resp. lower) inverse images of closed sets containing (resp. meeting) $F(x)$.
Proof. Let \( x \) be a point of \( X \) at which \( F \) is not upper contra-\( \omega \)-continuous. Then there exists a closed set \( V \) of \( Y \) containing \( F(x) \) such that \( U \cap (X \setminus F^+(V)) \neq \emptyset \) for each \( U \in \omega O(X, x) \). Then \( x \in \omega Cl(X \setminus F^+(V)) \). Since \( x \in F^+(V) \), we have \( x \in \omega Cl(F^+(Y)) \) and hence \( x \in \omega Fr(F^+(A)) \). Conversely, let \( V \) be any closed set of \( Y \) containing \( F(x) \) and \( x \in \omega Fr(F^+(V)) \). Now, assume that \( F \) is upper contra-\( \omega \)-continuous at \( x \), then there exists \( U \in \omega O(X, x) \) such that \( F(U) \subset V \). Therefore, we obtain \( x \in U \subset \omega Int(F^+(V)) \). This contradicts that \( x \in \omega Fr(F^+(V)) \). Thus, \( F \) is not upper contra-\( \omega \)-continuous. The proof of the second case is similar. \( \square \)

Corollary 3.28. \([3]\) The set of all points \( x \) of \( X \) at which \( f : (X, \tau) \to (Y, \sigma) \) is not contra-\( \omega \)-continuous is identical with the union of the \( \omega \)-frontiers of the inverse images of closed sets of \( Y \) containing \( f(x) \).

Definition 3.29. A multifunction \( F : (X, \tau) \to (Y, \sigma) \) is said to have a contra \( \omega \)-closed graph if for each pair \( (x, y) \in (X \times Y) \setminus G(F) \) there exist \( U \in \omega O(X, x) \) and a closed set \( V \) of \( Y \) containing \( y \) such that \( (U \times V) \cap G(F) = \emptyset \).

Lemma 3.30. For a multifunction \( F : (X, \tau) \to (Y, \sigma) \), the following holds:

\[(i) \ G^+_F(A \times B) = A \cap F^+(B); \]
\[(ii) \ G^-_F(A \times B) = A \cap F^-(B) \]

for any subset \( A \) of \( X \) and \( B \) of \( Y \).

Theorem 3.31. Let \( F : (X, \tau) \to (Y, \sigma) \) be an \( u.\omega \)-c. multifunction from a space \( X \) into a \( T_2 \) space \( Y \). If \( F(x) \) is \( \alpha \)-paracompact for each \( x \in X \), then \( G(F) \) is \( \omega \)-closed.

Proof. Suppose that \( (x_0, y_0) \notin G(F) \). Then \( y_0 \notin F(x_0) \). Since \( Y \) is a \( T_2 \) space, for each \( y \in F(x_0) \) there exist disjoint open sets \( V(y) \) and \( W(y) \) containing \( y \) and \( y_0 \), respectively. The family \( \{V(y) : y \in F(x_0)\} \) is an open cover of \( F(x_0) \). Thus, by \( \alpha \)-paracompactness of \( F(x_0) \), there is a locally finite open cover \( \Delta = \{U_\beta : \beta \in I\} \) which refines \( \{V(y) : y \in F(x_0)\} \). Therefore, there exists an open neighborhood \( W_0 \) of \( y_0 \) such that \( W_0 \) intersects only finitely many members \( U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n} \) of \( \Delta \). Choose \( y_1, y_2, \ldots, y_n \in F(x_0) \) such that \( U_{\beta_i} \subset V(y_i) \) for each \( 1 \leq i \leq n \), and set \( W = W_0 \cap ( \bigcap_{i=1}^n W(y_i) ) \). Then \( W \) is an open neighborhood of \( y_0 \) such that \( W \cap ( \bigcup_{\beta \in I} V_\beta ) = \emptyset \). By the upper \( \omega \)-continuity of \( F \), there is a \( U \in \omega O(X, x_0) \) such that \( U \subset F^+(\bigcup_{\beta \in I} V_\beta) \). It follows that \( (U \times W) \cap G(F) = \emptyset \). Therefore, \( G(F) \) is \( \omega \)-closed. \( \square \)

Theorem 3.32. Let \( F : (X, \tau) \to (Y, \sigma) \) be a multifunction from a space \( X \) into a \( \omega \)-compact space \( Y \). If \( G(F) \) is \( \omega \)-closed, then \( F \) is \( u.\omega \)-c.

Proof. Suppose that \( F \) is not \( u.\omega \)-c. Then there exists a nonempty closed subset \( C \) of \( Y \) such that \( F^-(C) \) is not \( \omega \)-closed in \( X \). We may assume \( F^-(C) \neq \emptyset \). Then there exists a point \( x_0 \in \omega Cl(F^-(C)) \setminus F^-(C) \). Hence for each point \( y \in C \), we have \( (x_0, y) \notin G(F) \). Since \( F \) has a \( \omega \)-closed graph, there are
Let $U(y)$ and $V(y)$ containing $x_0$ and $y$, respectively such that $(U(y) \times V(y)) \cap G(F) = \emptyset$. Then $\{Y \setminus C\} \cup \{V(y) : y \in C\}$ is a $\omega$-open cover of $Y$, and thus it has a subcover $\{Y \setminus C\} \cup \{V(y_i) : y_i \in C, 1 \leq i \leq n\}$. Let $U = \bigcap_{i=1}^{n} U(y_i)$ and $V = \bigcup_{i=1}^{n} V(y_i)$. It is easy to verify that $C \subset V$ and $(U \times V) \cap G(F) = \emptyset$. Since $U$ is a $\omega$-neighborhood of $x_0$, $U \cap F^-(C) \neq \emptyset$. It follows that $\emptyset \neq (U \times C) \cap G(F) \subset (U \times V) \cap G(F)$. This is a contradiction. Hence the proof is completed. \hfill $\Box$

**Corollary 3.33.** Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a multifunction into a $\omega$-compact $T_2$ space $Y$ such that $F(x)$ is $\omega$-closed for each $x \in X$. Then $F$ is u.$\omega$-c. if and only if it has a $\omega$-closed graph.

**Theorem 3.34.** Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be an u.$\omega$-c. multifunction into a $\omega$-$T_2$ space $Y$. If $F(x)$ is $\alpha$-paracompact for each $x \in X$, then $G(F)$ is $\omega$-closed.

**Proof.** The proof is clear. \hfill $\Box$

**Theorem 3.35.** Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a multifunction and $X$ be a connected space. If the graph multifunction of $F$ is upper contra-$\omega$-continuous (resp. lower contra-$\omega$-continuous), then $F$ is upper contra-$\omega$-continuous (resp. lower contra-$\omega$-continuous).

**Proof.** Let $x \in X$ and $V$ be any open subset of $Y$ containing $F(x)$. Since $X \times V$ is a $\omega$-open set of $X \times Y$ and $G_F(x) \subset X \times V$, there exists a $\omega$-open set $U$ containing $x$ such that $G_F(U) \subset X \times V$. By Lemma 3.30, we have $U \subset G^+_F(X \times V) = F^+(V)$ and $F(U) \subset V$. Thus, $F$ is u.$\omega$-c.. The proof of the l.$\omega$-c. of $F$ can be done using a similar argument. \hfill $\Box$

**References**


On upper and lower contra-$\omega$-continuous multifunctions

[10] Staum, R., The algebra of bounded continuous functions into a nonarchimedean

Amer. Math. Soc. 41 (1937), 374-381.

103-118.


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