LOCAL CLOSURE FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper, $(X, \tau, \mathcal{I})$ denotes an ideal topological space. Analogously to the local function $\mathcal{L}$, we define an operator $\Gamma(A)(\mathcal{I}, \tau)$ called the local closure function of $A$ with respect to $\mathcal{I}$ and $\tau$ as follows: $\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : A \cap Cl(U) \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$. We investigate properties of $\Gamma(A)(\mathcal{I}, \tau)$. Moreover, by using $\Gamma(A)(\mathcal{I}, \tau)$, we introduce an operator $\Psi: \mathcal{P}(X) \rightarrow \tau$ satisfying $\Psi(A) = X - \Gamma(X - A)$ for each $A \in \mathcal{P}(X)$. We set $\sigma = \{A \subseteq X : A \subseteq \Psi(A)\}$ and $\sigma_0 = \{A \subseteq X : A \subseteq Int(Cl(\Psi(A)))\}$ and show that $\tau_0 \subseteq \sigma \subseteq \sigma_0$.

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1. Introduction and preliminaries

Let $(X, \tau)$ be a topological space with no separation properties assumed. For a subset $A$ of a topological space $(X, \tau)$, $Cl(A)$ and $Int(A)$ denote the closure and the interior of $A$ in $(X, \tau)$, respectively. An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a non-empty collection of subsets of $X$ which satisfies the following properties:

1. $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
2. $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every open set } U \text{ containing } x\}$ is called the local function of $A$ with respect to $\mathcal{I}$ and $\tau$ (see [1], [2]). We simply write $A^*$ instead of $A^*(\mathcal{I}, \tau)$ in case there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than $\tau$, generating by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau \text{ and } J \in \mathcal{I}\}$. It is known in Example 3.6 of [2] that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by $\tau^*$. Recall that $A$ is said to be $\ast$-dense in itself (resp. $\ast^*$-closed, $\ast$-perfect) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of $A$ in $(X, \tau^*)$, respectively. In 1968, Veličko [3] introduced the class of $\theta$-open sets. A set $A$ is said to be $\theta$-open [3] if every point of $A$

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has an open neighborhood whose closure is contained in \( A \). The \( \theta \)-interior \( \overline{\set{x}} \) of \( A \) in \( X \) is the union of all \( \theta \)-open subsets of \( A \) and is denoted by \( \text{Int}_\theta(A) \).

Naturally, the complement of a \( \theta \)-open set is said to be \( \theta \)-closed. Equivalently, \( \text{Cl}_\theta(A) = \{ x \in X : \text{Cl}(U) \cap A \neq \emptyset \text{ for every } U \in \tau(x) \} \) and a set \( A \) is \( \theta \)-closed if and only if \( A = \text{Cl}_\theta(A) \). Note that all \( \theta \)-open sets form a topology on \( X \) which is coarser than \( \tau \), and is denoted by \( \tau_\theta \) and that a space \((X, \tau)\) is regular if and only if \( \tau = \tau_\theta \). Note also that the \( \theta \)-closure of a given set need not be a \( \theta \)-closed set.

In this paper, analogously to the local function \( A^*(I, \tau) \), we define an operator \( \Gamma(A)(I, \tau) \) called the local closure function of \( A \) with respect to \( I \) and \( \tau \) as follows: \( \Gamma(A)(I, \tau) = \{ x \in X : A \cap \text{Cl}(U) \notin I \text{ for every } U \in \tau(x) \} \). We investigate properties of \( \Gamma(A)(I, \tau) \). Moreover, we introduce an operator \( \Psi_I : \mathcal{P}(X) \to \tau \) satisfying \( \Psi_I(A) = X - \Gamma(X - A) \) for each \( A \in \mathcal{P}(X) \). We set \( \sigma = \{ A \subseteq X : A \subseteq \Psi_I(A) \} \) and \( \sigma_0 = \{ A \subseteq X : A \subseteq \text{Int}((\text{Cl}(\Psi_I(A)))) \} \) and show that \( \tau_\theta \subseteq \sigma \subseteq \sigma_0 \).

2. Local closure functions

**Definition 2.1.** Let \((X, \tau, I)\) be an ideal topological space. For a subset \( A \) of \( X \), we define the following operator: \( \Gamma(A)(I, \tau) = \{ x \in X : A \cap \text{Cl}(U) \notin I \text{ for every } U \in \tau(x) \} \), where \( \tau(x) = \{ U \in \tau : x \in U \} \). In case there is no confusion \( \Gamma(A)(I, \tau) \) is briefly denoted by \( \Gamma(A) \) and is called the local closure function of \( A \) with respect to \( I \) and \( \tau \).

**Lemma 2.2.** Let \((X, \tau, I)\) be an ideal topological space. Then \( A^*(I, \tau) \subseteq \Gamma(A)(I, \tau) \) for every subset \( A \) of \( X \).

*Proof.\* Let \( x \in A^*(I, \tau) \). Then, \( A \cap U \notin I \) for every open set \( U \) containing \( x \).

Since \( A \cap U \subseteq A \cap \text{Cl}(U) \), we have \( A \cap \text{Cl}(U) \notin I \) and hence \( x \in \Gamma(A)(I, \tau) \). \( \square \)

**Example 2.3.** Let \( X = \{ a, b, c, d \} \), \( \tau = \{ \phi, X, \{ a, c \}, \{ d \}, \{ a, c, d \} \} \), and \( I = \{ \phi, \{ c \} \} \). Let \( A = \{ b, c, d \} \).

Then \( \Gamma(A) = \{ a, b, c, d \} \) and \( A^* = \{ b, d \} \).

**Example 2.4.** Let \((X, \tau)\) be the real numbers with the left-ray topology, i.e. \( \tau = \{ (-\infty, a) : a \in X \} \cup \{ X, \phi \} \). Let \( I_f \) be the ideal of all finite subsets of \( X \). Let \( A = [0, 1] \). Then \( \Gamma(A) = \{ x \in X : A \cap \text{Cl}(U) = A \notin I_f \text{ for every } U \in \tau(x) \} = X \) and \(-1 \notin A^* \) which shows \( A^* \subseteq \Gamma(A) \).

**Lemma 2.5.** Let \((X, \tau)\) be a topological space and \( A \) be a subset of \( X \). Then

1. If \( A \) is open, then \( \text{Cl}(A) = \text{Cl}_\theta(A) \).

2. If \( A \) is closed, then \( \text{Int}(A) = \text{Int}_\theta(A) \).

**Theorem 2.6.** Let \((X, \tau)\) be a topological space, \( I \) and \( J \) be two ideals on \( X \), and let \( A \) and \( B \) be subsets of \( X \). Then the following properties hold:

1. If \( A \subseteq B \), then \( \Gamma(A) \subseteq \Gamma(B) \).
2. If $\mathcal{I} \subseteq \mathcal{J}$, then $\Gamma(A)(\mathcal{I}) \supseteq \Gamma(A)(\mathcal{J})$.

3. $\Gamma(A) = Cl(\Gamma(A)) \subseteq Cl_{\theta}(A)$ and $\Gamma(A)$ is closed.

4. If $A \subseteq \Gamma(A)$ and $\Gamma(A)$ is open, then $\Gamma(A) = Cl_{\theta}(A)$.

5. If $A \in \mathcal{I}$, then $\Gamma(A) = \emptyset$.

Proof. (1) Suppose that $x \notin \Gamma(B)$. Then there exists $U \in \tau(x)$ such that $B \cap Cl(U) \in \mathcal{I}$. Since $A \cap Cl(U) \subseteq B \cap Cl(U)$, $A \cap Cl(U) \in \mathcal{I}$. Hence $x \notin \Gamma(A)$. Thus $X \setminus \Gamma(B) \subseteq X \setminus \Gamma(A)$ or $\Gamma(A) \subseteq \Gamma(B)$.

(2) Suppose that $x \in \Gamma(A)(\mathcal{J})$. There exists $U \in \tau(x)$ such that $A \cap Cl(U) \in \mathcal{J}$. Since $\mathcal{I} \subseteq \mathcal{J}$, $A \cap Cl(U) \in \mathcal{J}$ and $x \notin \Gamma(A)(\mathcal{J})$. Therefore, $\Gamma(A)(\mathcal{J}) \subseteq \Gamma(A)(\mathcal{I})$.

(3) We have $\Gamma(A) \subseteq Cl(\Gamma(A))$ in general. Let $x \in Cl(\Gamma(A))$. Then $\Gamma(A) \cap U \neq \emptyset$ for every $U \in \tau(x)$. Therefore, there exists some $y \in \Gamma(A) \cap U$ and $U \in \tau(y)$. Since $y \in \Gamma(A)$, $A \cap Cl(U) \notin \mathcal{I}$ and hence $x \notin \Gamma(A)$. Hence we have $Cl(\Gamma(A)) \subseteq \Gamma(A)$ and hence $\Gamma(A) = Cl(\Gamma(A))$.

(4) For any subset $A$ of $X$, by (3) we have $\Gamma(A) = Cl(\Gamma(A)) \subseteq Cl_{\theta}(A)$.

Since $A \subseteq \Gamma(A)$ and $\Gamma(A)$ is open, by Lemma 2.5, $Cl(\Gamma(A)) \subseteq Cl_{\theta}(\Gamma(A)) = Cl(\Gamma(A)) = \Gamma(A) \subseteq Cl_{\theta}(A)$ and hence $\Gamma(A) = Cl_{\theta}(A)$.

(5) Suppose that $x \in \Gamma(A)$. Then for any $U \in \tau(x)$, $A \cap Cl(U) \notin \mathcal{I}$. But since $A \in \mathcal{I}$, $A \cap Cl(U) \in \mathcal{I}$ for every $U \in \tau(x)$. This is a contradiction. Hence $\Gamma(A) = \emptyset$. $\square$

Lemma 2.7. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $U \in \tau_{\theta}$, then $U \cap \Gamma(A) = U \cap \Gamma(U \cap A) \subseteq \Gamma(U \cap A)$ for any subset $A$ of $X$.

Proof. Suppose that $U \in \tau_{\theta}$ and $x \in U \cap \Gamma(A)$. Then $x \in U$ and $x \in \Gamma(A)$. Since $U \in \tau_{\theta}$, then there exists $W \in \tau$ such that $x \in W \subseteq Cl(W) \subseteq U$. Let $V$ be any open set containing $x$. Then $V \cap W \in \tau(x)$ and $Cl(V \cap W) \cap A \notin \mathcal{I}$ and hence $Cl(V) \cap (U \cap A) \notin \mathcal{I}$. This shows that $x \in \Gamma(U \cap A)$ and hence we obtain $U \cap \Gamma(A) \subseteq \Gamma(U \cap A)$. Moreover, $U \cap \Gamma(A) \subseteq U \cap \Gamma(U \cap A)$ and by Theorem 2.6, $\Gamma(U \cap A) \subseteq \Gamma(A)$ and $U \cap \Gamma(U \cap A) \subseteq U \cap \Gamma(A)$. Therefore, $U \cap \Gamma(A) = U \cap \Gamma(U \cap A)$. $\square$

Theorem 2.8. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A, B$ any subsets of $X$. Then the following properties hold:

1. $\Gamma(\emptyset) = \emptyset$.

2. $\Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B)$.

Proof. (1) The proof is obvious.

(2) It follows from Theorem 2.6 that $\Gamma(A \cup B) \supseteq \Gamma(A) \cup \Gamma(B)$. To prove the reverse inclusion, let $x \notin \Gamma(A) \cup \Gamma(B)$. Then $x$ belongs neither to $\Gamma(A)$ nor to $\Gamma(B)$. Therefore there exist $U_{x}, V_{x} \in \tau(x)$ such that $Cl(U_{x}) \cap A \in \mathcal{I}$ and
\( Cl(V_x) \cap B \in \mathcal{I} \). Since \( \mathcal{I} \) is additive, \((Cl(U_x) \cap A) \cup (Cl(V_x) \cap B) \in \mathcal{I}\). Moreover, since \( \mathcal{I} \) is hereditary and

\[
(Cl(U_x) \cap A) \cup (Cl(V_x) \cap B) = [(Cl(U_x) \cap A) \cup Cl(V_x)] \cap [(Cl(U_x) \cap A) \cup B] \\
= (Cl(U_x) \cup Cl(V_x)) \cap (A \cup Cl(V_x)) \cap (Cl(U_x) \cup B) \cap (A \cup B) \\
\supseteq Cl(U_x \cap V_x) \cap (A \cup B),
\]

\( Cl(U_x \cap V_x) \cap (A \cup B) \in \mathcal{I} \). Since \( U_x \cap V_x \in \tau(x), x \notin \Gamma(A \cup B) \). Hence \((X \setminus \Gamma(A)) \cap (X \setminus \Gamma(B)) \subseteq X \setminus \Gamma(A \cup B) \) or \( \Gamma(A \cup B) \subseteq \Gamma(A) \cup \Gamma(B) \). Hence we obtain \( \Gamma(A) \cup \Gamma(B) = \Gamma(A \cup B) \).

**Lemma 2.9.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A, B\) be subsets of \(X\). Then \(\Gamma(A) - \Gamma(B) = \Gamma(A - B) - \Gamma(B)\).

**Proof.** We have by Theorem 2.8, \(\Gamma(A) = \Gamma[(A - B) \cup (A \cap B)] = \Gamma(A - B) \cup \Gamma(A \cap B) \subseteq \Gamma(A - B) \cup \Gamma(B)\). Thus \(\Gamma(A) - \Gamma(B) \subseteq \Gamma(A - B) \cup \Gamma(B)\). By Theorem 2.9, \(\Gamma(A - B) \cup \Gamma(B) \subseteq \Gamma(A - B) - \Gamma(B)\). Hence \(\Gamma(A) - \Gamma(B) = \Gamma(A - B) - \Gamma(B)\).

**Corollary 2.10.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A, B\) be subsets of \(X\) with \(B \in \mathcal{I}\). Then \(\Gamma(A \cup B) = \Gamma(A) = \Gamma(A - B)\).

**Proof.** Since \(B \in \mathcal{I}\), by Theorem 2.8, \(\Gamma(B) = \emptyset\). By Lemma 2.8, \(\Gamma(A) = \Gamma(A - B)\) and by Theorem 2.3, \(\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B) = \Gamma(A)\)

### 3. Closure compatibility of topological spaces

**Definition 3.1.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. We say the \(\tau\) is compatible with the ideal \(\mathcal{I}\), denoted \(\tau \sim \mathcal{I}\), if the following holds for every \(A \subseteq X\), if for every \(x \in A\) there exists \(U \in \tau(x)\) such that \(U \cap A \in \mathcal{I}\), then \(A \in \mathcal{I}\).

**Definition 3.2.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. We say the \(\tau\) is closure compatible with the ideal \(\mathcal{I}\), denoted \(\tau \sim_{\text{cl}} \mathcal{I}\), if the following holds for every \(A \subseteq X\), if for every \(x \in A\) there exists \(U \in \tau(x)\) such that \(Cl(U) \cap A \in \mathcal{I}\), then \(A \in \mathcal{I}\).

**Remark 3.3.** If \(\tau\) is compatible with \(\mathcal{I}\), then \(\tau\) is closure compatible with \(\mathcal{I}\).

**Theorem 3.4.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space, the following properties are equivalent:

1. \(\tau \sim_{\text{cl}} \mathcal{I}\);
2. If a subset \(A\) of \(X\) has a cover of open sets each of whose closure intersection with \(A\) is in \(\mathcal{I}\), then \(A \in \mathcal{I}\);
3. For every \(A \subseteq X\), \(A \cap \Gamma(A) = \emptyset\) implies that \(A \in \mathcal{I}\);
4. For every \( A \subseteq X \), \( A \setminus \Gamma(A) \in \mathcal{I} \);

5. For every \( A \subseteq X \), if \( A \) contains no nonempty subset \( B \) with \( B \subseteq \Gamma(B) \), then \( A \in \mathcal{I} \).

**Proof.** (1) \( \Rightarrow \) (2): The proof is obvious.

(2) \( \Rightarrow \) (3): Let \( A \subseteq X \) and \( x \in A \). Then \( x \notin \Gamma(A) \) and there exists \( V_x \in \tau(x) \) such that \( Cl(V_x) \cap A \in \mathcal{I} \). Therefore, we have \( A \subseteq \bigcup \{ V_x : x \in A \} \) and \( V_x \in \tau(x) \) and by (2), \( A \in \mathcal{I} \).

(3) \( \Rightarrow \) (4): For any \( A \subseteq X \), \( A - \Gamma(A) \subseteq A \) and \( (A - \Gamma(A)) \cap \Gamma(A - \Gamma(A)) \subseteq (A - \Gamma(A)) \cap \Gamma(A) = \emptyset \). By (3), \( A - \Gamma(A) \in \mathcal{I} \).

(4) \( \Rightarrow \) (5): By (4), for every \( A \subseteq X \), \( A - \Gamma(A) \in \mathcal{I} \). Let \( A - \Gamma(A) = J \in \mathcal{I} \), then \( A = J \cup (A \cap \Gamma(A)) \) and by Theorem \ref{thm:local_closure_functions}(2) and Theorem \ref{thm:local_closure_functions}(5), \( \Gamma(A) = \Gamma(J) \cup \Gamma(A \cap \Gamma(A)) = \Gamma(A \cap \Gamma(A)) \). Therefore, we have \( A \cap \Gamma(A) = A \cap (A \cap \Gamma(A)) \subseteq (A \cap \Gamma(A)) \) and \( A \cap \Gamma(A) \subseteq A \). By the assumption \( A \cap \Gamma(A) = \emptyset \) and hence \( A = A - \Gamma(A) \in \mathcal{I} \).

(5) \( \Rightarrow \) (1): Let \( A \subseteq X \) and assume that for every \( x \in A \), there exists \( U \in \tau(x) \) such that \( Cl(U) \cap A \in \mathcal{I} \). Then \( A \cap \Gamma(A) = \emptyset \). Suppose that \( A \) contains \( B \) such that \( B \subseteq \Gamma(B) \). Then \( B = B \setminus \Gamma(B) \subseteq A \cap \Gamma(A) = \emptyset \). Therefore, \( A \) contains no nonempty subset \( B \) with \( B \subseteq \Gamma(B) \). Hence \( A \in \mathcal{I} \). \( \square \)

**Theorem 3.5.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. If \( \tau \) is closure compatible with \( \mathcal{I} \), then the following equivalent properties hold:

1. For every \( A \subseteq X \), \( A \cap \Gamma(A) = \emptyset \) implies that \( \Gamma(A) = \emptyset \);

2. For every \( A \subseteq X \), \( \Gamma(A - \Gamma(A)) = \emptyset \);

3. For every \( A \subseteq X \), \( \Gamma(A \cap \Gamma(A)) = \Gamma(A) \).

**Proof.** First, we show that (1) holds if \( \tau \) is closure compatible with \( \mathcal{I} \). Let \( A \) be any subset of \( X \) and \( A \cap \Gamma(A) = \emptyset \). By Theorem \ref{thm:local_closure_functions}, \( A \in \mathcal{I} \) and by Theorem \ref{thm:local_closure_functions}(5), \( \Gamma(A) = \emptyset \).

(1) \( \Rightarrow \) (2): Assume that for every \( A \subseteq X \), \( A \cap \Gamma(A) = \emptyset \) implies that \( \Gamma(A) = \emptyset \). Let \( B = A - \Gamma(A) \), then

\[
B \cap \Gamma(B) = (A - \Gamma(A)) \cap \Gamma(A - \Gamma(A)) = (A \cap (X - \Gamma(A))) \cap \Gamma(A \cap (X - \Gamma(A))) \subseteq [A \cap (X - \Gamma(A))] \cap [\Gamma(A) \cap (\Gamma(X - \Gamma(A)))] = \emptyset.
\]

By (1), we have \( \Gamma(B) = \emptyset \). Hence \( \Gamma(A - \Gamma(A)) = \emptyset \).

(2) \( \Rightarrow \) (3): Assume for every \( A \subseteq X \), \( \Gamma(A - \Gamma(A)) = \emptyset \).

\[
A = (A - \Gamma(A)) \cup (A \cap \Gamma(A)) \quad \Gamma(A) = \Gamma[(A - \Gamma(A)) \cup (A \cap \Gamma(A))] = \Gamma(A - \Gamma(A)) \cup \Gamma(A \cap \Gamma(A)) = \Gamma(A \cap \Gamma(A)).
\]

(3) \( \Rightarrow \) (1): Assume for every \( A \subseteq X \), \( A \cap \Gamma(A) = \emptyset \) and \( \Gamma(A \cap \Gamma(A)) = \Gamma(A) \). This implies that \( \emptyset = \Gamma(\emptyset) = \Gamma(A) \). \( \square \)
Theorem 3.6. Let \((X, \tau, \mathcal{I})\) be an ideal topological space, then the following properties are equivalent:

1. \(\text{Cl}(\tau) \cap \mathcal{I} = \emptyset\), where \(\text{Cl}(\tau) = \{\text{Cl}(V) : V \in \tau\}\);
2. If \(I \in \mathcal{I}\), then \(\text{Int}_{\theta}(I) = \emptyset\);
3. For every clopen \(G\), \(G \subseteq \Gamma(G)\);
4. \(X = \Gamma(X)\).

Proof. \((1) \Rightarrow (2)\): Let \(\text{Cl}(\tau) \cap \mathcal{I} = \emptyset\) and \(I \in \mathcal{I}\). Suppose that \(x \in \text{Int}_{\theta}(I)\). Then there exists \(U \in \tau\) such that \(x \in U \subseteq \text{Cl}(U) \subseteq I\). Since \(I \in \mathcal{I}\) and hence \(\emptyset \neq \{x\} \subseteq \text{Cl}(U) \in \text{Cl}(\tau) \cap \mathcal{I}\). This is contrary to \(\text{Cl}(\tau) \cap \mathcal{I} = \emptyset\). Therefore, \(\text{Int}_{\theta}(I) = \emptyset\).

\((2) \Rightarrow (3)\): Let \(x \in G\). Assume \(x \notin \Gamma(G)\), then there exists \(U_x \in \tau(x)\) such that \(G \cap \text{Cl}(U_x) \in \mathcal{I}\) and hence \(G \cap U_x \in \mathcal{I}\). Since \(G\) is clopen, by (2) and Lemma 3.5, \(x \in G \cap U_x = \text{Int}(G \cap U_x) \subseteq \text{Int}(G \cap \text{Cl}(U_x)) = \text{Int}_{\theta}(G \cap \text{Cl}(U_x)) = \emptyset\). This is a contradiction. Hence \(x \in \Gamma(G)\) and \(G \subseteq \Gamma(G)\).

\((3) \Rightarrow (4)\): Since \(X\) is clopen, then \(X = \Gamma(X)\).

\((4) \Rightarrow (1)\): \(X = \Gamma(X) = \{x \in X : \text{Cl}(U) \cap X = \text{Cl}(U) \notin \mathcal{I} \text{ for each open set } U \text{ containing } x\}\). Hence \(\text{Cl}(\tau) \cap \mathcal{I} = \emptyset\).

\(\square\)

Theorem 3.7. Let \((X, \tau, \mathcal{I})\) be an ideal topological space, \(\tau\) be closure compatible with \(\mathcal{I}\). Then for every \(G \in \tau_{\theta}\) and any subset \(A\) of \(X\), \(\text{Cl}(\Gamma(G \cap A)) = \Gamma(G \cap A) \subseteq \Gamma(G \cap \Gamma(A)) \subseteq \text{Cl}_{\theta}(G \cap \Gamma(A))\).

Proof. By Theorem 3.3 and Theorem 2.0, we have \(\Gamma(G \cap A) = \Gamma((G \cap A) \cap \Gamma(G \cap A)) \subseteq \Gamma(G \cap \Gamma(A))\). Moreover, by Theorem 2.0, \(\text{Cl}(\Gamma(G \cap A)) = \Gamma(G \cap A) \subseteq \Gamma(G \cap \Gamma(A)) \subseteq \text{Cl}_{\theta}(G \cap \Gamma(A))\).

\(\square\)

4. \(\Psi_{\Gamma}\)-operator

Definition 4.1. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. An operator \(\Psi_{\Gamma} : \mathcal{P}(X) \to \tau\) is defined as follows: for every \(A \subseteq X\), \(\Psi_{\Gamma}(A) = \{x \in X : \text{there exists } U \in \tau(x) \text{ such that } \text{Cl}(U) - A \in \mathcal{I}\}\) and observe that \(\Psi_{\Gamma}(A) = X - \Gamma(X - A)\).

Several basic facts concerning the behavior of the operator \(\Psi_{\Gamma}\) are included in the following theorem.

Theorem 4.2. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then the following properties hold:

1. If \(A \subseteq X\), then \(\Psi_{\Gamma}(A)\) is open.
2. If \(A \subseteq B\), then \(\Psi_{\Gamma}(A) \subseteq \Psi_{\Gamma}(B)\).
3. If \(A, B \in \mathcal{P}(X)\), then \(\Psi_{\Gamma}(A \cap B) = \Psi_{\Gamma}(A) \cap \Psi_{\Gamma}(B)\).
4. If \(A \subseteq X\), then \(\Psi_{\Gamma}(A) = \Psi_{\Gamma}(\Psi_{\Gamma}(A))\) if and only if \(\Gamma(X - A) = \Gamma(\Gamma(X - A))\).
5. If \( A \in \mathcal{I} \), then \( \Psi_\Gamma(A) = X - \Gamma(X) \).

6. If \( A \subseteq X \), \( I \in \mathcal{I} \), then \( \Psi_\Gamma(A - I) = \Psi_\Gamma(A) \).

7. If \( A \subseteq X \), \( I \in \mathcal{I} \), then \( \Psi_\Gamma(A \cup I) = \Psi_\Gamma(A) \).

8. If \((A - B) \cup (B - A) \in \mathcal{I} \), then \( \Psi_\Gamma(A) = \Psi_\Gamma(B) \).

Proof. (1) This follows from Theorem 2.6 (3).

(2) This follows from Theorem 2.6 (1).

(3)

\[
\Psi_\Gamma(A \cap B) = X - \Gamma(X - (A \cap B))
\]

\[
= X - \Gamma((X - A) \cup (X - B))
\]

\[
= X - \Gamma(X - A) \cup \Gamma(X - B)
\]

\[
= [X - \Gamma(X - A) \cap [X - \Gamma(X - B)]
\]

\[
= \Psi_\Gamma(A) \cap \Psi_\Gamma(B).
\]

(4) This follows from the facts:

1. \( \Psi_\Gamma(A) = X - \Gamma(X - A) \).

2. \( \Psi_\Gamma(\Psi_\Gamma(A)) = X - \Gamma(X - (X - \Gamma(X - A))) = X - \Gamma(\Gamma(X - A)) \).

(5) By Corollary 2.10 we obtain that \( \Gamma(X - A) = \Gamma(X) \) if \( A \in \mathcal{I} \).

(6) This follows from Corollary 2.10 and \( \Psi_\Gamma(A - I) = X - \Gamma[X - (A - I)] = X - \Gamma[(X - A) \cup I] = X - \Gamma(X - A) = \Psi_\Gamma(A) \).

(7) This follows from Corollary 2.10 and \( \Psi_\Gamma(A \cup I) = X - \Gamma[X - (A \cup I)] = X - \Gamma[(X - A) - I] = X - \Gamma(X - A) = \Psi_\Gamma(A) \).

(8) Assume \((A - B) \cup (B - A) \in \mathcal{I} \). Let \( A - B = I \) and \( B - A = J \). Observe that \( I, J \in \mathcal{I} \) by heredity. Also observe that \( B = (A - I) \cup J \). Thus \( \Psi_\Gamma(A) = \Psi_\Gamma(A - I) = \Psi[(A - I) \cup J] = \Psi_\Gamma(B) \) by (6) and (7). \( \square \)

**Corollary 4.3.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then \( U \subseteq \Psi_\Gamma(U) \) for every \( \theta \)-open set \( U \subseteq X \).

Proof. We know that \( \Psi_\Gamma(U) = X - \Gamma(X - U) \). Now \( \Gamma(X - U) \subseteq Cl_\theta(X - U) = X - U \), since \( X - U \) is \( \theta \)-closed. Therefore, \( U = X - (X - U) \subseteq X - \Gamma(X - U) = \Psi_\Gamma(U) \). \( \square \)

Now we give an example of a set \( A \) which is not \( \theta \)-open but satisfies \( A \subseteq \Psi_\Gamma(A) \).

**Example 4.4.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a, c\}, \{d\}, \{a, c, d\}\} \), and \( \mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\} \). Let \( A = \{a\} \). Then \( \Psi_\Gamma(\{a\}) = X - \Gamma(X - \{a\}) = X - \Gamma(\{b, c, d\}) = X - \{b, d\} = \{a, c\} \). Therefore, \( A \subseteq \Psi_\Gamma(A) \), but \( A \) is not \( \theta \)-open.

**Example 4.5.** Let \((X, \tau)\) be the real numbers with the left-ray topology, i.e. \( \tau = \{(-\infty, a) : a \in X\} \cup \{X, \emptyset\} \). Let \( \mathcal{I}_f \) be the ideal of all finite subsets of \( X \). Let \( A = X - \{0, 1\} \). Then \( \Psi_\Gamma(\{A\}) = X - \Gamma(\{0, 1\}) = X \). Therefore, \( A \subseteq \Psi_\Gamma(A) \), but \( A \) is not \( \theta \)-open.
\textbf{Theorem 4.6.} Let \((X, \tau, I)\) be an ideal topological space and \(A \subseteq X\). Then the following properties hold:

1. \(\Psi_\Gamma(A) = \bigcup \{U \in \tau : \text{Cl}(U) - A \in I\}\).
2. \(\Psi_\Gamma(A) \supseteq \bigcup \{U \in \tau : (\text{Cl}(U) - A) \cup (A - \text{Cl}(U)) \in I\}\).

\textit{Proof.} (1) This follows immediately from the definition of \(\Psi_\Gamma\)-operator.
(2) Since \(I\) is heredity, it is obvious that \(\bigcup \{U \in \tau : (\text{Cl}(U) - A) \cup (A - \text{Cl}(U)) \in I\} \subseteq \bigcup \{U \in \tau : \text{Cl}(U) - A \in I\} = \Psi_\Gamma(A)\) for every \(A \subseteq X\). \(\square\)

\textbf{Theorem 4.7.} Let \((X, \tau, I)\) be an ideal topological space. If \(\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}\). Then \(\sigma\) is a topology for \(X\).

\textit{Proof.} Let \(\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}\). Since \(\phi \in I\), by Theorem 4.5 \(\Gamma(\phi) = \phi\) and \(\Psi_\Gamma(X) = X - \Gamma(X - X) = X - \Gamma(\phi) = X\). Moreover, \(\Psi_\Gamma(\phi) = X - \Gamma(X - \phi) = X - X = \phi\). Therefore, we obtain that \(\phi \subseteq \Psi_\Gamma(\phi)\) and \(X \subseteq \Psi_\Gamma(X) = X\), and thus \(\phi\) and \(X \in \sigma\). Now if \(A, B \in \sigma\), then by Theorem 4.6 \(A \cap B \subseteq \Psi_\Gamma(A) \cap \Psi_\Gamma(B) = \Psi_\Gamma(A \cap B)\) which implies that \(A \cap B \in \sigma\). If \(\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma\), then \(A_\alpha \subseteq \Psi_\Gamma(A_\alpha) \subseteq \Psi_\Gamma(\bigcup A_\alpha)\). This shows that \(\sigma\) is a topology. \(\square\)

\textbf{Lemma 4.8.} If either \(A \in \tau\) or \(B \in \tau\), then \(\text{Int}(\text{Cl}(A \cap B)) = \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(B))\).

\textit{Proof.} This is an immediate consequence of Lemma 3.5 of [3]. \(\square\)

\textbf{Theorem 4.9.} Let \(\sigma_0 = \{A \subseteq X : A \subseteq \text{Int}(\text{Cl}(\Psi_\Gamma(A)))\}\), then \(\sigma_0\) is a topology for \(X\).

\textit{Proof.} By Theorem 4.5, for any subset \(A\) of \(X\), \(\Psi_\Gamma(A)\) is open and \(\sigma \subseteq \sigma_0\). Therefore, \(\emptyset, X \in \sigma_0\). Let \(A, B \in \sigma_0\). Then by Lemma 4.5 and Theorem 4.7, we have \(A \cap B \subseteq \text{Int}(\text{Cl}(\Psi_\Gamma(A))) \cap \text{Int}(\text{Cl}(\Psi_\Gamma(B))) = \text{Int}(\text{Cl}(\Psi_\Gamma(A) \cap \Psi_\Gamma(B))) = \text{Int}(\text{Cl}(\Psi_\Gamma(A \cap B)))\). Therefore, \(A \cap B \in \sigma_0\). Let \(A_\alpha \in \sigma_0\) for each \(\alpha \in \Delta\). By Theorem 4.7, for each \(\alpha \in \Delta\), \(A_\alpha \subseteq \text{Int}(\text{Cl}(\Psi_\Gamma(A_\alpha))) \subseteq \text{Int}(\text{Cl}(\Psi_\Gamma(\bigcup A_\alpha)))\) and hence \(\bigcup A_\alpha \subseteq \text{Int}(\text{Cl}(\Psi_\Gamma(\bigcup A_\alpha)))\). Hence \(\bigcup A_\alpha \in \sigma_0\). This shows that \(\sigma_0\) is a topology for \(X\). \(\square\)

By Theorem 4.5 and Corollary 4.8 the following relations holds:

\[\theta\text{-open} \longrightarrow \text{open} \]  
\[\sigma\text{-open} \longrightarrow \sigma_0\text{-open}\]

\textbf{Remark 4.10.} 1. In Example 4.7, \(A\) is \(\sigma\)-open but it is not open. Therefore, every \(\sigma_0\)-open set is not open.
2. Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \) and \( \mathcal{I} = \{\phi, \{a\}\} \) be an ideal on \( X \). We observe that \( \{a\} \) is open but it is not \( \sigma_0 \)-open sets, since \( \Psi_\Gamma(\{a\}) = X - \Gamma(\{b, c\}) = X - X = \phi \). Also, \( \{c\} \) is not open but it is \( \sigma \)-open set, since \( \Psi_\Gamma(\{c\}) = X - \Gamma(\{a, b\}) = X - \{b\} = \{a, c\} \).

3. **Question:** Is there an example which shows that \( \sigma \subsetneq \sigma_0 \)?

**Theorem 4.11.** Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space. Then \( \tau \sim_\Gamma \mathcal{I} \) if and only if \( \Psi_\Gamma(A) - A \in \mathcal{I} \) for every \( A \subseteq X \).

**Proof.** **Necessity.** Assume \( \tau \sim_\Gamma \mathcal{I} \) and let \( A \subseteq X \). Observe that \( x \in \Psi_\Gamma(A) - A \) if and only if \( x \not\in A \) and \( x \not\in \Gamma(X - A) \) if and only if \( x \not\in A \) and there exists \( U_x \in \tau(x) \) such that \( Cl(U_x) - A \in \mathcal{I} \) if and only if \( U_x \in \tau(x) \) such that \( x \in Cl(U_x) - A \in \mathcal{I} \). Now, for each \( x \in \Psi_\Gamma(A) - A \) and \( U_x \in \tau(x) \), \( Cl(U_x) \cap (\Psi_\Gamma(A) - A) \in \mathcal{I} \) by heredity and hence \( \Psi_\Gamma(A) - A \in \mathcal{I} \) by assumption that \( \tau \sim_\Gamma \mathcal{I} \).

**Sufficiency.** Let \( A \subseteq X \) and assume that for each \( x \in A \) there exists \( U_x \in \tau(x) \) such that \( Cl(U_x) \cap A \in \mathcal{I} \). Observe that \( \Psi_\Gamma(X - A) - (X - A) = A - \Gamma(A) = \{x : \text{there exists } U_x \in \tau(x) \text{ such that } x \in Cl(U_x) \cap A \in \mathcal{I}\} \). Thus we have \( A \subseteq \Psi_\Gamma(X - A) - (X - A) \in \mathcal{I} \) and hence \( A \in \mathcal{I} \) by heredity of \( \mathcal{I} \).

**Proposition 4.12.** Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space with \( \tau \sim_\Gamma \mathcal{I} \), \( A \subseteq X \). If \( N \) is a nonempty open subset of \( \Gamma(A) \cap \Psi_\Gamma(A) \), then \( N - A \in \mathcal{I} \) and \( Cl(N) \cap A \not\in \mathcal{I} \).

**Proof.** If \( N \subseteq \Gamma(A) \cap \Psi_\Gamma(A) \), then \( N - A \subseteq \Psi_\Gamma(A) - A \in \mathcal{I} \) by Theorem 4.11 and hence \( N - A \in \mathcal{I} \) by heredity. Since \( N \subseteq \tau - \{\phi\} \) and \( N \subseteq \Gamma(A) \), we have \( Cl(N) \cap A \not\in \mathcal{I} \) by the definition of \( \Gamma(A) \).

In [3], Newcomb defines \( A = B \mod \mathcal{I} \) if \( (A - B) \cup (B - A) \in \mathcal{I} \) and observes that \( = \mod \mathcal{I} \) is an equivalence relation. By Theorem 1.11, we have that if \( A = B \mod \mathcal{I} \), then \( \Psi_\Gamma(A) = \Psi_\Gamma(B) \).

**Definition 4.13.** Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space. A subset \( A \) of \( X \) is called a Baire set with respect to \( \tau \) and \( \mathcal{I} \), denoted \( A \in \mathcal{B}_r(X, \tau, \mathcal{I}) \), if there exists a \( \theta \)-open set \( U \) such that \( A = U \mod \mathcal{I} \).

**Lemma 4.14.** Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space with \( \tau \sim_\Gamma \mathcal{I} \). If \( U, V \in \tau_\theta \) and \( \Psi_\Gamma(U) = \Psi_\Gamma(V) \), then \( U = V \mod \mathcal{I} \).

**Proof.** Since \( U \in \tau_\theta \), by Corollary 1.13 we have \( U \subseteq \Psi_\Gamma(U) \) and hence \( U - V \subseteq \Psi_\Gamma(U) - V = \Psi_\Gamma(V) - V \in \mathcal{I} \) by Theorem 1.11. Therefore, \( U - V \in \mathcal{I} \). Similarly, \( V - U \in \mathcal{I} \). Now, \((U - V) \cup (V - U) \in \mathcal{I} \) by additivity. Hence \( U = V \mod \mathcal{I} \).

**Theorem 4.15.** Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space with \( \tau \sim_\Gamma \mathcal{I} \). If \( A, B \in \mathcal{B}_r(X, \tau, \mathcal{I}) \), and \( \Psi_\Gamma(A) = \Psi_\Gamma(B) \), then \( A = B \mod \mathcal{I} \).
Proof. Let $U, V \in \tau_0$ be such that $A = U \bmod I$ and $B = V \bmod I$. Now \( \Psi_\Gamma(A) = \Psi_\Gamma(U) \) and \( \Psi_\Gamma(B) = \Psi_\Gamma(V) \) by Theorem 4.14(8). Since \( \Psi_\Gamma(A) = \Psi_\Gamma(B) \) implies that \( \Psi_\Gamma(U) = \Psi_\Gamma(V) \) and hence $U = V \bmod I$ by Lemma 4.14. Hence $A = B \bmod I$ by transitivity. \( \square \)

Proposition 4.16. Let \((X, \tau, I)\) be an ideal topological space.

1. If $B \in \mathcal{B}_r(X, \tau, I) - I$, then there exists $A \in \tau_0 - \{\phi\}$ such that $B = A \bmod I$.

2. Let $\text{Cl}(\tau) \cap I = \phi$, then $B \in \mathcal{B}_r(X, \tau, I) - I$ if and only if there exists $A \in \tau_0 - \{\phi\}$ such that $B = A \bmod I$.

Proof. (1) Assume $B \in \mathcal{B}_r(X, \tau, I) - I$, then $B \in \mathcal{B}_r(X, \tau, I)$. Hence there exists $A \in \tau_0$ such that $B = A \bmod I$. If $A = \phi$, then we have $B = \phi \bmod I$. This implies that $B \in I$ which is a contradiction.

(2) Assume there exists $A \in \tau_0 - \{\phi\}$ such that $B = A \bmod I$, hence by Definition 4.14, $B \in \mathcal{B}_r(X, \tau, I)$. Then $A = (B - J) \cup I$, where $J = B - A, I = A - B \in I$. If $B \in I$, then $A \in I$ by heredity and additivity. Since $A \in \tau_0 - \{\phi\}, A \neq \phi$ and there exists $U \in \tau$ such that $\phi \neq U \subseteq \text{Cl}(U) \subseteq A$. Since $A \in I$, $\text{Cl}(U) \in I$ and hence $\text{Cl}(U) \in \text{Cl}(\tau) \cap I$. This contradicts that $\text{Cl}(\tau) \cap I = \phi$. \( \square \)

Proposition 4.17. Let \((X, \tau, I)\) be an ideal topological space with $\tau \cap I = \phi$. If $B \in \mathcal{B}_r(X, \tau, I) - I$, then $\Psi_\Gamma(B) \cap \text{Int}_\theta(\Gamma(B)) \neq \phi$.

Proof. Assume $B \in \mathcal{B}_r(X, \tau, I) - I$, then by Proposition 4.14(1), there exists $A \in \tau_0 - \{\phi\}$ such that $B = A \bmod I$. By Theorem 4.14 and Lemma 4.14, $A = A \cap X = A \cap \Gamma(X) \subseteq \Gamma(A) \cap X = \Gamma(A)$. This implies that $\phi \neq A \subseteq \Gamma(A) = \Gamma((B - J) \cup I) = \Gamma(B)$, where $J = B - A, I = A - B \in I$ by Corollary 4.14. Since $A \in \tau_0, A \subseteq \text{Int}_\theta(\Gamma(B))$. Also, $\phi \neq A \subseteq \Psi_\Gamma(A) = \Psi_\Gamma(B)$ by Corollary 4.14 and Theorem 4.14(8). Consequently, we obtain $A \subseteq \Psi_\Gamma(B) \cap \text{Int}_\theta(\Gamma(B))$. \( \square \)

Given an ideal topological space \((X, \tau, I)\), let $U(X, \tau, I)$ denote \( \{A \subseteq X : \text{there exists } B \in \mathcal{B}_r(X, \tau, I) - I \text{ such that } B \subseteq A\} \).

Proposition 4.18. Let \((X, \tau, I)\) be an ideal topological space with $\tau \cap I = \phi$. If $\tau = \tau_0$, then the following statements are equivalent:

1. $A \in U(X, \tau, I)$;
2. $\Psi_\Gamma(A) \cap \text{Int}_\theta(\Gamma(A)) \neq \phi$;
3. $\Psi_\Gamma(A) \cap \Gamma(A) \neq \phi$;
4. $\Psi_\Gamma(A) \neq \phi$;
5. $\text{Int}_\theta(A) \neq \phi$;
6. There exists $N \in \tau - \{\phi\}$ such that $N - A \in I$ and $N \cap A \notin I$. 

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Proof. (1) ⇒ (2): Let \( B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I} \) such that \( B \subseteq A \). Then \( \text{Int}_\theta(\Gamma(B)) \subseteq \text{Int}_\theta(\Gamma(A)) \) and \( \Psi_\Gamma(B) \subseteq \Psi_\Gamma(A) \) and hence \( \text{Int}_\theta(\Gamma(B)) \cap \Psi_\Gamma(B) \subseteq \text{Int}_\theta(\Gamma(A)) \cap \Psi_\Gamma(A) \). By Proposition, we have \( \Psi_\Gamma(A) \cap \text{Int}_\theta(\Gamma(A)) \neq \phi \).

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (5): If \( \Psi_\Gamma(A) \neq \phi \), then there exists \( U \in \tau - \{\phi\} \) such that \( U - A \in \mathcal{I} \). Since \( U \notin \mathcal{I} \) and \( U = (U - A) \cup (U \cap A) \), we have \( U \cap A \notin \mathcal{I} \). By Theorem, \( \phi \neq (U \cap A) \subseteq \Psi_\Gamma(U) \cap A = \Psi_\Gamma((U - A) \cup (U \cap A)) \cap A = \Psi_\Gamma(U \cap A) \cap A \subseteq \Psi_\Gamma(A) \cap A = \text{Int}_\theta(A) \). Hence \( \text{Int}_\theta(A) \neq \phi \).

(5) ⇒ (6): If \( \text{Int}_\theta(A) \neq \phi \), then by Theorem 3.1 there exists \( N \in \tau - \{\phi\} \) and \( I \in \mathcal{I} \) such that \( \phi \neq N - I \subseteq A \). We have \( N - A \in \mathcal{I} \), \( N = (N - A) \cup (N \cap A) \) and \( N \notin \mathcal{I} \). This implies that \( N \cap A \notin \mathcal{I} \).

(6) ⇒ (1): Let \( B = N \cap A \notin \mathcal{I} \) with \( N \in \tau_\theta - \{\phi\} \) and \( N - A \in \mathcal{I} \). Then \( B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I} \) since \( B \notin \mathcal{I} \) and \( (B - N) \cup (N - B) = N - A \in \mathcal{I} \).

\[ \text{Theorem 4.19.} \] Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space with \( \tau \sim_\Gamma \mathcal{I} \), where \( Cl(\tau) \cap \mathcal{I} = \phi \). Then for \( A \subseteq X, \Psi_\Gamma(A) \subseteq \Gamma(A) \).

Proof. Suppose \( x \in \Psi_\Gamma(A) \) and \( x \notin \Gamma(A) \). Then there exists a nonempty neighborhood \( U_x \in \tau(x) \) such that \( Cl(U_x) \cap A \in \mathcal{I} \). Since \( x \notin \Gamma(A) \), by Theorem, \( x \notin \{ U \in \tau : Cl(U) \cap A \in \mathcal{I} \} \) and there exists \( V \in \tau(x) \) and \( Cl(V) - A \in \mathcal{I} \). Now we have \( U_x \cap V \in \tau(x) \), \( Cl(U_x \cap V) \cap A \in \mathcal{I} \) and \( Cl(U_x \cap V) - A \in \mathcal{I} \) by heredity. Hence by finite additivity we have \( Cl(U_x \cap V) \cap A \cup (Cl(U_x \cap V) - A) = Cl(U_x \cap V) \in \mathcal{I} \). Since \( (U_x \cap V) \in \tau(x) \), this is contrary to \( Cl(\tau) \cap \mathcal{I} = \phi \). Therefore, \( x \in \Gamma(A) \). This implies that \( \Psi_\Gamma(A) \subseteq \Gamma(A) \).

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