ON THE CRITICAL GROUP OF A FAMILY OF GRAPHS

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Abstract. The critical group is a subtle isomorphism invariant of the graph and closely connected with the graph Laplacian matrix. In this paper, the abstract structure of the critical group of a family of graphs $H_n$, $n \geq 3$ is determined.

AMS Mathematics Subject Classification (2010): 05C25, 15A18, 05C50

Key words and phrases: Graph, Laplacian matrix, critical group, invariant factor, Smith normal form, tree number

1. Introduction

Let $G$ be a finite multi-graph with $n$ vertices. Let $A(G)$ and $D(G)$ be the adjacency and degree matrices of the graph $G$. Then, the Laplacian matrix $L(G)$ is defined as $L(G) = D(G) - A(G)$. The critical group of a graph $G$ is closely related with the Laplacian matrix $L(G)$ as follows: thinking of $L(G)$ as a linear map $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$, its cokernel has the form $\text{coker}(G) = \frac{\mathbb{Z}^n}{L(G)\mathbb{Z}^n} \cong \mathbb{Z} \oplus K(G)$, where $K(G)$ is the critical group on $G$ in the sense of isomorphism and the order of the critical group of a graph is equal to the number of spanning trees of the graph $[3,4,10,11,14]$.

Let $v_r$ be a vertex (called a root) of a graph $G$ with $n$ vertices. The critical group $K(G)$ of $G$ is also the quotient group $\mathbb{Z}^n$ by the subgroup spanned by the $n$ generators $\Delta_1, \ldots, \Delta_{r-1}, x_r, \Delta_{r+1}, \ldots, \Delta_n$, where $\Delta_i = d_i x_i - \sum_{v_j \text{ adjacent to } v_i} a_{ij} x_j$ and $x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$, whose unique nonzero entry 1 is in the position $i$, where $i = 1, 2, \ldots, n$. That is $K(G) = \text{span}(\Delta_1, \ldots, \Delta_{r-1}, x_r, \Delta_{r+1}, \ldots, \Delta_n)$. Notice that $K(G)$ is independent of the choice of $v_r$; for more details see $[8]$.

The explicit determination of the structure of $K(G)$ in a given family of graphs is not always easy, and a series of paper whose goal is to explicitly determine the structure of the group $K(G)$ has appeared in the last ten year, see for example $[1,2,3,4,5,6,13,15,20]$.

We construct the family of graphs $H_n$ by considering a cycle $C_{6n} : v_0, v_1, v_2, v_3, \ldots, v_{6n-1}, v_0$, where $n \geq 3$ and a new vertex $v$ adjacent to $n$ vertices $v_0, v_3, v_6, v_9, \ldots, v_{6n-2}$ of $C_{6n}$. This graph has order $6n + 1$ and size $7n$. The aim of this paper is to compute the structure of the critical group of this family of graphs $H_n$, $n \geq 2$ by determine its Smith normal form.
2. System of relations for the cokernel of the Laplacian of $H_n$

In this section, we will first show that there are at most two generators for the critical group $K(H_n)$ of the graph $H_n$ and reduce the relation matrix to the special matrix $B_n$. Then, we will give some properties of the sequences concerning the entries of this matrix $B_n$.

Now, we work on the system of relations of the cokernel of the Laplacian of $H_n$. Let $x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{6n}$, whose unique nonzero 1 is in the position corresponding to the vertex $v_i$. Here we have chosen the vertex $v$ as the root, such that $x_v = 0$. The relations of $\text{coker} L(H_n)$ give rise to the following system of equations:

\begin{align}
3x_{i-1} - x_i - x_{i-2} &= 0; \quad i \equiv 2 \pmod{6} \\
2x_{i-1} - x_i - x_{i-2} &= 0; \quad \text{otherwise}
\end{align}

**Lemma 2.1.** There are two sequences $(a_i)$ and $(b_i)$ of integral numbers such that

\begin{equation}
 x_i = a_i x_2 - b_i x_1, \quad 3 \leq i \leq 6n.
\end{equation}

Moreover, the sequences have the following recurrence relations,

\[
\begin{cases}
 a_i = 3a_{i-1} - a_{i-2}, & i \equiv 2 \pmod{6} \\
 a_i = 2a_{i-1} - a_{i-2}, & \text{otherwise} \\
 b_i = 3b_{i-1} - b_{i-2}, & i \equiv 2 \pmod{6} \\
 b_i = 2b_{i-1} - b_{i-2}, & i \equiv 0, 3, 4, 5 \pmod{6} \\
 b_i = b_{i-1}, & i \equiv 1 \pmod{6}
\end{cases}
\]

**Proof.** We know from the system of equations (2.1 & 2.2) that the group $K(H_n)$ has at most 2 generators, i.e., each $x_i$ can be expressed in terms of $x_2$ and $x_1$. So, there are at least $3n - 2$ diagonal entries of Smith normal form of $L(H_n)$ that are equal to 1, however, the remaining invariant factors of $\text{coker}(H_n)$ hide inside the relations matrix induced by $x_2$ and $x_1$. Based on the structure of $H_n$ and from equation (2.1), we have

\begin{equation}
 B_n = \begin{pmatrix}
 a_{6n+1} & a_{6n} + 1 \\
 b_{6n+1} + 1 & b_{6n} + 3
\end{pmatrix}.
\end{equation}

From the above argument, one can reduce $L(H_n)$ up to the equivalence $I_{6n-2} \oplus (B_n)$ by performing some row and column operations. Now, we only need to evaluate Smith normal form of the matrix $B_n$.

3. Analysis of the coefficients of the Smith normal form of $B_n$

In this section we will try to find the Smith normal form of $B_n$ by calculating the diagonal entries. Let us define the following sequences of positive integers
with the initial conditions, $\sigma_0 = 0$, $\sigma_1 = 1$

$$
\begin{align*}
\sigma_m &= 8\sigma_{m-1} - \sigma_{m-2}, \\
\rho_m &= \sigma_m + \sigma_{m+1}, \\
r_m &= 241\sigma_{m-1} - 31\sigma_{m-2}, \\
s_m &= \sigma_{m+1} - \sigma_m, \\
t_m &= 6\sigma_m - \sigma_{m-1}.
\end{align*}
$$

The following proposition is very easy to prove by induction.

**Proposition 3.1.**

- $2 \nmid r_m \quad \forall \ m$,
- $2 \nmid s_m \quad \forall \ m$.

**Proposition 3.2.** The sequences $r_m$, $s_m$ and $t_m$ are relatively prime for each $m$ i.e.,

(3.1) \hspace{1cm} \gcd(r_m, s_m, t_m) = 1

**Proof.** On contrary, suppose that there exists a prime $p$ such that $p \mid r_m$, $p \mid s_m$ and $p \mid t_m$, then $p \mid \sigma_m$. Since $\sigma_m = 8\sigma_{m-1} - \sigma_{m-2} = s_m - t_m \Rightarrow p \mid \sigma_m$. Hence, we get $p \mid \sigma_{m-1} & p \mid \sigma_{m-2}$. Again, we have, $\sigma_{m-1} = 8\sigma_{m-2} - \sigma_{m-3} \Rightarrow p \mid \sigma_{m-3} \Rightarrow \cdots p \mid \sigma_{m-j} \cdots p \mid \sigma_1 = 1 \Rightarrow p = 1$, a contradiction, hence $(r_m, s_m, t_m) = 1$.

**The Odd Case**

**Lemma 3.3.** If $n = 2m + 1$, then we have the following relation,

$$
\begin{align*}
a_{6n+1} &= 6s_m\rho_m, \\
a_{6n} + 1 &= 6t_m\rho_m, \\
b_{6n+1} + 1 &= 6t_m\rho_m, \\
b_{6n} + 3 &= r_m\rho_m.
\end{align*}
$$

**Proof.** It is easy to prove by induction.

**Proposition 3.4.**

(3.2) \hspace{1cm} \gcd(a_{6n+1}, a_{6n} + 1, b_{6n+1} + 1, b_{6n} + 3) = \rho_m.

**Proof.** By Lemma 3.3 and then Proposition 3.2, we have the desired result.

**Proposition 3.5.** If $n = 2m + 1$, then

$$
\det B_n = 3\rho_m^2
$$

where $B_n$ is defined in equation (2.4).
Proof.

\[
\det B_n = 6\rho_m^2 (s_m r_m - 6t_m)
= 6\rho_m^2 \left[ (\sigma_m - \sigma_{m-1})^2 - 3\sigma_m \sigma_{m-1} \right]
= 6\rho_m^2 \left[ (\sigma_{m-1} - \sigma_{m-2})^2 - 2\sigma_{m-1} \sigma_{m-2} \right]
\vdots
= 6\rho_m^2 \left[ (\sigma_2 - \sigma_1)^2 - 2\sigma_2 \sigma_1 \right]
= 6\rho_m^2
\]

**Theorem 3.6.** If \( n = 2m+1 \), then the critical group of \( \mathcal{H}_n \) is the direct product of two cyclic groups i.e.,

\[
K(\mathcal{H}_n) = \mathbb{Z}_{\rho_m} \oplus \mathbb{Z}_{6\rho_m}
\]

**Proof.** Since the matrix \( B_n \) has Smith normal form as \( \text{diag}(s_{11}, s_{22}) \) and \( s_{11} \) equals to the greatest common divisor of all the entries of \( B_n \). So, by Proposition 3.4, we have

\[
(3.3) \quad s_{11} = \rho_m.
\]

Also \( s_{11}s_{22} = \det B_n \) and then by Proposition 3.3, we have

\[
(3.4) \quad s_{11}s_{22} = 6\rho_m^2.
\]

Combining (3.3) and (3.4), we obtain

\[
(3.5) \quad s_{22} = \rho_m,
\]

which completes the proof.

**The Even Case**

If \( n = 2m \), and consider the following sequence of positive integers with initial conditions, \( \rho_0 = -1 \), \( \rho_1 = 1 \),

\[
\rho_m = 8\rho_{m-1} - \rho_{m-2}, \quad \rho_m = \sigma_m + \sigma_{m-1},
\]

\[
\lambda_m = \frac{1}{2} [241\rho_{m-1} - 31\rho_{m-2}],
\]

\[
\mu_m = \frac{1}{2} [7\rho_m - \rho_{m-1}],
\]

\[
\nu_m = 6\rho_m - \rho_{m-1}.
\]

**Lemma 3.7.** If \( n = 2m \), then we have the following relation,

\[
a_{6n+1} = 12\mu_m \sigma_m
\]

\[
a_{6n + 1} = 6\nu_m \sigma_m
\]

\[
b_{6n+1 + 1} = 6\nu_m \sigma_m
\]

\[
b_{6n + 3} = 2\lambda_m \sigma_m.
\]
Proof. It is easy to prove by induction.

**Proposition 3.8.** The sequences $\mu_m$, $\nu_m$ and $\lambda_m$ are relatively prime for each $m$, i.e.

\[ \gcd(\mu_m, \nu_m \lambda_m) = 1 \]

*Proof. One can prove this proposition by similar arguments as in the proof of Proposition 3.2.*

**Proposition 3.9.**

\[ \gcd(a_{6n+1}, a_{6n} + 1, b_{6n+1} + 1, b_{6n} + 3) = 2\sigma_m \]

*Proof. By Lemma 3.4 and then Proposition 3.2, we have the desired result.*

**Proposition 3.10.** If $n = 2m$, then

\[ \det B_n = 21\sigma_m^2, \]

where $B_n$ is defined in equation 2.4.

*Proof.*

\[
\begin{align*}
\det B_n &= (a_{6n+1})(b_{6n} + 3) - (a_{6n} + 1)^2 \\
&= 6\sigma_m^2[(\rho_m - \rho_m-1)^2 - 6\rho_m\rho_{m-1}] \\
&= 6\sigma_m^2[(\rho_m-1 - \rho_m-2)^2 - 6\rho_m-1\rho_m-2] \\
&\vdots \\
&= 6\sigma_m^2[(\rho_2 - \rho_1)^2 - 3\rho_2\rho_1] \\
&= 60\sigma_m^2
\end{align*}
\]

**Theorem 3.11.** If $n = 2m$, then the critical group of $\mathcal{H}_n$ is the direct product of two cyclic groups, i.e.

\[ K(\mathcal{H}_n) = \mathbb{Z}_{2\sigma_m} \oplus \mathbb{Z}_{30\sigma_m} \]

*Proof. By Proposition 3.3 and Proposition 3.10, we have the desired result.*

**Proposition 3.12.** For each $m, n \geq 1$ we have

\[ \sigma_{m+n} = \sigma_{m+1}\sigma_n - \sigma_m\sigma_{n-1} \quad \text{and} \quad \rho_{m+n} = \sigma_{m+1}\rho_n - \sigma_m\rho_{n-1} \]

*Proof. Set $A := \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}$ we have that

\[ A^m := \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}^m = \begin{pmatrix} \sigma_{m+1} & -\sigma_m \\ \sigma_m & -\sigma_m \end{pmatrix} \]
Since $A^{m+n-1} = A^m A^{n-1}$, we have
\[
\begin{pmatrix}
\sigma_{m+n} & -\sigma_{m+n-1} \\
\sigma_{m+n-1} & -\sigma_{m+n-2}
\end{pmatrix} = \begin{pmatrix}
\sigma_{m+1} & -\sigma_m \\
\sigma_m & -\sigma_{m-1}
\end{pmatrix} \begin{pmatrix}
\sigma_n & -\sigma_{n-1} \\
\sigma_{n-1} & -\sigma_{n-2}
\end{pmatrix}
\]
Comparing the top left entry in the left-hand side with the corresponding in the right-hand side gives the first equality. For the second identity, we use the inductive hypothesis implies that $\sigma_m$ divides $\sigma_{m+1}$. Moreover, we have that $\det(A) = \det(A + I)$.

**Theorem 3.13.**

- If $a$ and $b$ are both even and $a \mid b$, then $\sigma_a \mid \sigma_b$
- If $a$ and $b$ are both odd and $a \mid b$, then $\rho_a \mid \rho_b$
- If $a$ is odd and $b$ is even and $a \mid b$, then $\rho_a \mid \sigma_b$

Moreover, we have that $\det(A_a) \mid \det(A_b)$.

**Proof.** For the first statement, we prove by induction on $t$ that $\sigma_a$ divides $\sigma_{at}$. This is true if $t = 0$. If $\sigma_a$ divides $\sigma_{at}$, hence we have $\sigma_{a(t+1)} = \sigma_{at+1} \sigma_a - \sigma_{at} \sigma_a$. The inductive hypothesis implies that $\sigma_a$ divides second term, hence it also divides $\sigma_{a(t+1)}$.

Now we prove that $2a+1$ divides $2b+1$, then $\rho_a$ divides $\rho_b$. First notice that, by Proposition 3.12, we have $\sigma_{2m+1} = \sigma_{m+1}^2 - \sigma_m^2$. Let $2b+1 = (2a+1)(2t+1)$. We prove by induction on $t$ that $\rho_a$ divides $\rho_b = \rho_{2at+at+1}$. This is true if $t = 0$, if $\rho_a$ divides $\rho_{2at+at+1}$, we have $\rho_{2a(t+1)+a+(t+1)} = \rho_{(2a+1)(2at+at+1)} = \sigma_{2a+2} \rho_{2at+at+1} - \sigma_{2a+1} \rho_{2at+at+1}$. The first term is a multiple of $\rho_a$ by induction hypothesis. Moreover, we have $\sigma_{2a+1} = \sigma_{a+1}^2 - \sigma_a^2 = (\sigma_{a+1} + \sigma_a)(\sigma_{a+1} - \sigma_a) = \rho_a(\sigma_{a+1} - \sigma_a)$. Hence the second term is a multiple of $\rho_a$.

Finally, we prove that if $2a+1$ divides $2b$, then $\rho_a$ divides $\sigma_{2a+1}$. Let $2b = (2a+1)2t$. We have to prove that $\rho_a$ divides $\sigma_{(2a+1)t}$. As we have already seen, $\rho_a$ divides $\sigma_{2a+1}$. Moreover, we have that $\sigma_{2a+1}$ is a divisor of $\sigma_{(2a+1)t} = \sigma_b$. By these facts and being $\det(A_a)$, we have the second statement.

By the statements verifying during the proof of Theorem 3.13, one can see that for $a$ dividing $b$, each entry of the matrix $(A_a)$, divides the corresponding on in the matrix $\det(A_b)$. This leads to the following theorem.

**Theorem 3.14.** If $a \mid b$, then the critical group of $\mathcal{H}_a$ is isomorphic to a subgroup of the critical group of $\mathcal{H}_b$.

4. The tree number

Let $G$ be a graph, then the tree number $k(G)$ is equal to the number of spanning trees of the graph $G$. In this section, we will give the closed formula for the number of spanning trees for the graph $\mathcal{H}_n$, we refer [3] for the terminologies.

**Proposition 4.1.** [5]

Let $G$ be a nearly regular graph of degree $r$ and $H$ be its subgraph obtained by removing the exceptional vertex, then
\[ k(G) = P_H(r), \]
where $P_H(t)$ is the characteristic polynomial of the graph $H$. 
Remark 4.2. The wheel graph $W_n$ can be obtained from a cycle $C_n$ by adding a new vertex connected by an edge to all vertices of $C_n$. Hence, $W_n$ is nearly regular graph and by proposition 4.1, we get

$$k(W_n) = P_{C_n}(r).$$

The characteristic polynomial of a cycle $C_n$ is given as

$$P_{C_n}(t) = 2T_n\left(\frac{t}{2}\right) - 2,$$

where

$$T_n(t) = \frac{n}{2} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \binom{n-m}{m} (2t)^{n-2m}$$

is the Chebyshev polynomial of the first kind. It is easy to see that it gives the same number of spanning trees of wheel graph given by N. Biggs in [2].

A very interesting application of the Proposition 4.1 is given as follows. The inner dual planner graph $G^{**}$ is the subgraph of the usual dual $G^*$ obtained by deleting the vertex corresponding to the infinite region of the original planer graph.

Let $G$ be a plane graph in which any finite region is bounded by a cycle of fixed length $r$. Then, $G^*$ is a nearly regular graph, so we have the following result.

**Proposition 4.3.** [5]

Let $G$ is a plane graph in which any bounded region is a cycle of length $r$, then

$$k(G) = P_{G^{**}}(r),$$

where $P_{G^{**}}(t)$ is the characteristic polynomial of the graph $G^{**}$.

**Theorem 4.4.** The tree number for the graph $\mathcal{H}_n$ is

$$k(\mathcal{H}_n) = P_{C_n}(8) = 2T_n(4) - 2,$$

where $T_n(t)$ is the Chebyshev polynomial of the first kind.

**Proof.** Since $\mathcal{H}_n$ is a plane graph in which any bounded region is bounded by a cycle of length 8 and total number of bounded regions is $n$. Hence, in this case the inner dual will be a cycle of length $n$ and its characteristic polynomial is defined in equation (4.1), and it follows the result.

**Acknowledgement**

The author is very thankful to the referee for his valuable suggestions for the improvement of this paper.
References


Received by the editors April 19, 2012