ENDO-PRINCIPALLY PROJECTIVE MODULES

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Abstract. Let $R$ be an arbitrary ring with identity and $M$ a right $R$-module with $S = \text{End}_R(M)$. In this paper, we introduce a class of modules that is a generalization of principally projective (or simply p.p.) rings and Baer modules. The module $M$ is called endo-principally projective (or simply endo-p.p.) if for any $m \in M$, $l_S(m) = Se$ for some $e^2 = e \in S$. For an endo-p.p. module $M$, we prove that $M$ is endo-rigid (resp., endo-reduced, endo-symmetric, endo-semicommutative) if and only if the endomorphism ring $S$ is rigid (resp., reduced, symmetric, semicommutative), and we also prove that the module $M$ is endo-rigid if and only if $M$ is endo-reduced if and only if $M$ is endo-symmetric if and only if $M$ is endo-semicommutative. Among others, we show that if $M$ is abelian, then every direct summand of an endo-p.p. module is also endo-p.p.

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1. Introduction

Throughout this paper $R$ denotes an associative ring with identity, and modules will be unitary right $R$-modules. For a module $M$, $S = \text{End}_R(M)$ denotes the ring of right $R$-module endomorphisms of $M$. Then $M$ is a left $S$-module, a right $R$-module and an $(S,R)$-bimodule. In this work, for any of the rings $T$ and $R$ and any $(T,R)$-bimodule $M$, $r_R(\cdot)$ and $l_M(\cdot)$ denote the right annihilator of a subset of $M$ in $R$ and the left annihilator of a subset of $R$ in $M$, respectively. Similarly, $l_T(\cdot)$ and $r_M(\cdot)$ will be the left annihilator of a subset of $M$ in $T$ and the right annihilator of a subset of $T$ in $M$, respectively. A ring is reduced if it has no nonzero nilpotent elements. Recently, the reduced ring concept has been extended to modules by Lee and Zhou in [12], that is, a module $M$ is called reduced if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A ring $R$ is called semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module $M$ is called endo-semicommutative if for any
$f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$, this class of modules is called $S$-semicommutative in [3]. Baer rings [10] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring $R$ is said to be quasi-Baer [11] if the right annihilator of each right ideal of $R$ is generated (as a right ideal) by an idempotent. A ring $R$ is called right principally quasi-Baer [5] if the right annihilator of a principal right ideal of $R$ is generated by an idempotent. According to Rizvi and Roman [17], $M$ is called a Baer (resp. quasi-Baer) module if for all $R$-submodules (resp. fully invariant $R$-submodules) $N$ of $M$, $l_S(N) = Se$ with $e^2 = e \in S$. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers modulo $n$ and the $\mathbb{Z}$-module of integers modulo $n$.

2. Endo-Principally Projective Modules

Principally projective rings are introduced by Hattori [9] to study the torsion theory, that is, a ring $R$ is called left (right) p.p. if every principal left (right) ideal is projective. The concept of left (right) p.p. rings has been comprehensively studied in the literature. In [12], Lee and Zhou introduced p.p. modules as follows: an $R$-module $M$ is called p.p. if for any $m \in M$, $r_R(m) = eR$, where $e^2 = e \in R$. According to Baser and Harmanci [4], a module $M$ is called principally quasi-Baer if for any $m \in M$, $r_R(mR) = eR$, where $e^2 = e \in R$. Motivated by these and the aforementioned definitions of Rizvi and Roman we give the following definition.

**Definition 2.1.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. The module $M$ is called endo-p.p. if for any $m \in M$, $l_S(m) = Se$ for some $e^2 = e \in S$.

Note that a ring $R$ is called right (or left) p.p. if every principal right (or left) ideal of $R$ is a projective right (or left) $R$-module. Then, it is obvious that the module $R$ is endo-p.p. if and only if the ring $R$ is left p.p. It is clear that all Baer and quasi-Baer modules are endo-p.p.

**Example 2.2.** Let $R$ be a Prüfer domain (i.e., a ring with an identity, no zero divisors and all finitely generated ideals are projective) and $M$ the right $R$-module $R \oplus R$. By ([11], page 17) $S = \text{End}_R(M)$ is isomorphic to the ring of $2 \times 2$ matrices over $R$, and it is a Baer ring. Hence $M$ is Baer and so it is an endo-p.p. module.

Since $R \cong \text{End}_R(R)$, the following example shows that endo-p.p. modules may not be quasi-Baer or Baer.

**Example 2.3.** ([3], Example 8.2) Consider the ring $S = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Let $T = \{(a_n)_{n=1}^{\infty} | a_n \text{ is eventually constant}\}$ and $I = \{(a_n)_{n=1}^{\infty} | a_n = 0 \text{ eventually}\}$. Then

$$R = \begin{bmatrix} T/I & T/I \\ 0 & T \end{bmatrix}$$

is a left p.p. ring which is neither right p.p. nor right principally quasi-Baer. It follows that $R$ is an endo-p.p. module but not quasi-Baer or Baer.
Lemma 2.4. If every cyclic submodule of $M$ is a direct summand, then $M$ is endo-p.p.

Proof. Let $m \in M$. We prove $l_S(m) = Sf$ for some $f^2 = f \in S$. By hypothesis, $M = mR \oplus K$ for some submodule $K \leq M$. Let $e$ denote the projection of $M$ onto $mR$. It is easy routine to show that $l_S(m) = S(1 - e).

Note that the endomorphism ring of an endo-p.p. module may not be a right p.p. ring in general. For if $M$ is an endo-p.p. module and $\varphi \in S$, then we have two cases. $\text{Ker}\varphi = 0$ or $\text{Ker}\varphi \neq 0$. If $\text{Ker}\varphi = 0$, then for any $f \in r_S(\varphi)$, $\varphi f = 0$ implies $f = 0$. Hence $r_S(\varphi) = 0$. Assume that $\text{Ker}\varphi \neq 0$. There exists a nonzero $m \in M$ such that $\varphi m = 0$. By hypothesis, $\varphi \in l_S(m) = Se$ for some $e^2 = e \in S$. In this case $\varphi = \varphi e$ and so $r_S(\varphi) \leq (1 - e)S$. The following example shows that this inclusion is strict.

Example 2.5. Let $Q$ be the ring and $N$ the $Q$-module constructed by Osofsky in [13]. Since $Q$ is commutative, we can just as well think of $N$ as of a right $Q$-module. Let $S = \text{End}_Q(N)$. By Lemma 2.4, $N$ is an endo-p.p. module. Identify $S$ with the ring $\left[ \begin{array}{cc} Q & 0 \\ Q/I & Q/I \end{array} \right]$ in the obvious way, and consider $\varphi = \left[ \begin{array}{cc} 0 & 0 \\ 1 + I & 0 \end{array} \right] \in S$. Then $r_S(\varphi) = \left[ \begin{array}{cc} I & 0 \\ Q/I & Q/I \end{array} \right]$. This is not a direct summand of $S$ because $I$ is not a direct summand of $Q$. Therefore, $S$ is not a right p.p. ring.

A ring $R$ is called abelian if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Abelian modules are introduced in the context of categories by Roos in [13] and studied by Goodearl and Boyle [8], Rizvi and Roman [15]. A module $M$ is called abelian if for any $f \in S$, $e^2 = e \in S$, $m \in M$, we have $fem = efm$. Note that $M$ is an abelian module if and only if $S$ is an abelian ring. Recall that $M$ is called a duo module [13] if every submodule $N$ of $M$ is fully invariant, i.e., $f(N) \leq N$ for all $f \in S$. Note that for a duo module $M$, if $e$ is an idempotent and $f$ is an element in $S$, then $(1 - e)fem = 0 = ef(1 - e)m$ for every $m \in M$. Thus every duo module is abelian.

Theorem 2.6. Consider the following conditions for an $R$-module $M$.

1. $M$ is an endo-p.p. module.
2. The left annihilator in $S$ of every finitely generated $R$-submodule of $M$ is generated (as a left ideal) by an idempotent.

Then (2) $\Rightarrow$ (1). If $M$ is duo, also (1) $\Rightarrow$ (2).

Proof. (2) $\Rightarrow$ (1) Clear by definitions.

(1) $\Rightarrow$ (2) Assume that $M$ is a duo module and let $N$ be a finitely generated $R$-submodule of $M$. By induction we may assume $N = m_1R + m_2R$. So $l_S(m_1R) = Se_1$ and $l_S(m_2R) = Se_2$ where $e_1^2 = e_1$, $e_2^2 = e_2 \in S$. Then $l_S(N) = (Se_1) \cap (Se_2)$. Clearly, $l_S(N) \subseteq Se_1e_2$. Let $ge_1e_2 \in Se_1e_2$. Since $m_1R$ is fully invariant, $ge_1e_2N = ge_1e_2m_1R \leq ge_1m_1R = 0$. Hence $Se_1e_2 \subseteq l_S(N)$. Thus $l_S(N) = Se_1e_2$. Similarly, $l_S(N) = Se_2e_1$. And we have $Se_1e_2 = Se_2e_1$. So $e_1e_2 = fe_2e_1$ for some $f \in S$. Hence

$$e_1e_2 = e_1e_2e_1.$$
Similarly,

(2) \(e_2e_1 = e_2e_1e_2\)

Replacing (2) in (1) we obtain that \(e_1e_2\) is an idempotent. This completes the proof.

Proposition 2.7. Let \(M\) be an abelian module and \(N\) a direct summand of \(M\) with \(S' = \text{End}_R(N)\). If \(M\) is an endo-p.p. module, then \(N\) is also endo-p.p.

Proof. Let \(N\) be a direct summand of \(M\) and \(n \in N\). There exists \(e_2' = e \in S\) with \(l_S(n) = Se\). Since \(N\) is a direct summand of \(M\) and \(M\) is abelian, \(N\) is a fully invariant submodule of \(M\). It follows that \(eN \leq N\). Then the restriction \(e' = e|_N\) belongs to \(S'\). We claim that \(l_{S'}(n) = S'e'\). Let \(f \in l_{S'}(n)\). We extend \(f\) to \(g = f \oplus 0 \in S\). Then \(g \in l_S(n)\) and so \(g = ge\). Hence \(f = g|_N = (ge)|_N = fe' \in S'e'\). Thus \(l_{S'}(n) \subseteq S'e'\). The reverse inclusion is clear.

Let \(M\) be an \(R\)-module with \(S = \text{End}_R(M)\). The module \(M\) is called endo-principally quasi-Baer if for any \(m \in M\), \(l_S(Sm) = Se\) for some \(e^2 = e \in S\), this class of modules is called principally quasi-Baer in [20]. Then the following lemma is obvious.

Lemma 2.8. Consider the following conditions for an \(R\)-module \(M\).

(1) \(M\) is a Baer module.
(2) \(M\) is a quasi-Baer module.
(3) \(M\) is an endo-p.p. module.
(4) \(M\) is an endo-principally quasi-Baer module.

Then (1) \(\Rightarrow\) (2) \(\Rightarrow\) (4). If \(M\) is an endo-semicommutative module, then (2) \(\Rightarrow\)
(1), (2) \(\Rightarrow\) (3) and (3) \(\iff\) (4).

3. Applications

If \(R\) is a ring, then some properties of \(R\)-modules do not characterize the ring \(R\), namely there are reduced \(R\)-modules but \(R\) need not be reduced and there are abelian \(R\)-modules but \(R\) need not be an abelian ring. Because of that endo-reduced modules, endo-rigid modules, endo-symmetric modules, and endo-semicommutative modules are studied by the present authors in recent papers (see [2]). Our next endeavor is to investigate relationships between endo-reduced, endo-rigid, endo-symmetric, endo-semicommutative and abelian modules by using endo-p.p. modules.

Lemma 3.1. Let \(M\) be an \(R\)-module. If \(M\) is an endo-semicommutative module, then \(S\) is a semicommutative ring. The converse holds if \(M\) is an endo-p.p. module.

Proof. The first statement is from [2, Proposition 2.20]. Conversely, assume that \(M\) is an endo-p.p. module and \(S\) is a semicommutative ring. Let \(fm = 0\) for \(f \in S\) and \(m \in M\). Since \(M\) is an endo-p.p. module, there exists \(e^2 = e \in S\) such that \(l_S(m) = Se\). Since \(fm = 0\), \(f \in l_S(m) = Se\) and then \(fg \in Seg\) for
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all \( g \in S \). By assumption, \( S \) is an abelian ring and so \( e \) is central in \( S \). Then \( eg = ge \) for all \( g \in S \). Hence \( fg \in Sge \subseteq Se = l_S(m) \). Thus \( fgm = 0 \) for all \( g \in S \). This completes the proof.

**Lemma 3.2.** If a module \( M \) is endo-semicommutative, then \( M \) is abelian. The converse holds if \( M \) is an endo-p.p. module.

**Proof.** One way is clear because \( S \) semicommutative implies \( S \) abelian and so \( M \) is abelian. Suppose that \( M \) is an abelian and endo-p.p. module. Let \( f \in S \), \( m \in M \) with \( fm = 0 \). Then \( f \in l_S(m) \). Since \( M \) is an endo-p.p. module, there exists an idempotent \( e \) in \( S \) such that \( l_S(m) = Se \) and so \( Sem = 0 \) and \( fe = f \). By supposition, \( eSm = 0 \). Then \( feSm = fSm = 0 \).

Recall that an \( R \)-module \( M \) is called endo-reduced if \( fm = 0 \) implies that \( Imf \cap Sm = 0 \) for each \( f \in S \), \( m \in M \), this class of modules is called reduced in \([2]\). Following the definition of a reduced module in \([12] \) and \([16] \), \( M \) is endo-reduced if and only if \( f^2m = 0 \) implies \( fSm = 0 \) for each \( f \in S \), \( m \in M \). Also, an \( R \)-module \( M \) is called endo-rigid \([2]\) if for any \( f \in S \) and \( m \in M \), \( f^2m = 0 \) implies \( fm = 0 \). In this direction we have the following result.

**Lemma 3.3.** If \( M \) is an endo-reduced module, then \( S \) is a reduced ring. The converse holds if \( M \) is an endo-p.p. module.

**Proof.** The first statement is from \([2] \), Lemma 2.11 and Proposition 2.14. Conversely, assume that \( M \) is an endo-p.p. module and \( S \) is a reduced ring. Then in particular \( S \) is an abelian ring. Let \( fm = 0 \) for \( f \in S \) and \( m \in M \), and \( f^2m' = gm' \in fM \cap Sm' \). We may find an idempotent \( e \) in \( S \) such that \( f \in l_S(m) = Se \). By assumption, \( e \) is central in \( S \). So \( f = fe = ef \). Multiplying \( f^2m' = gm' \) from the left by \( e \), we have \( f^2m' = egm = gem = 0 \). Hence \( fM \cap Sm = 0 \). Thus \( M \) is endo-reduced.

**Lemma 3.4.** If a module \( M \) is endo-reduced, then it is endo-semicommutative. The converse is true if \( M \) is endo-p.p.

**Proof.** Similar to the proof of Lemma 3.3.

**Lemma 3.5.** If \( M \) is an endo-rigid module, then \( S \) is a reduced ring. The converse holds if \( M \) is an endo-p.p. module.

**Proof.** The first statement is from \([2] \) Lemma 2.20. Conversely, assume that \( M \) is an endo-p.p. module and \( S \) is a reduced ring. Let \( f^2m = 0 \) for \( f \in S \) and \( m \in M \). Since \( M \) is an endo-p.p. module, there exists \( e^2 = e \in S \) such that \( f \in l_S(fm) = Se \). Then \( efm = 0 \) and \( f = fe \). By assumption, \( S \) is an abelian ring and so \( e \) is central in \( S \). Then \( fm = fem = efm = 0 \). Hence \( M \) is an endo-rigid module.

We now give a relation between endo-reduced modules and endo-rigid modules.

**Lemma 3.6.** If \( M \) is an endo-reduced module, then \( M \) is an endo-rigid module. The converse holds if \( M \) is endo-p.p.
Proof. The first statement is from \[2, \text{Lemma 2.14}\]. Conversely, let \( M \) be an endo-p.p. and endo-rigid module. Assume that \( fm = 0 \) for \( f \in S \) and \( m \in M \). Then there exists \( e^2 = e \in S \) such that \( f \in l_S(mR) = Se \). By Lemma 3.8, \( e \) is central in \( S \) and \( fe = ef = f \) and \( em = 0 \). Let \( fm' = gm \in fM \cap Sm \). Then \( efm' = fm' = gem = 0 \). Therefore \( M \) is endo-reduced. □

According to Lambek, a ring \( R \) is called symmetric \[\text{[1]}\] if whenever \( a, b, c \in R \) satisfy \( abc = 0 \) implies \( cab = 0 \). A module \( M \) is called symmetric (\[\text{[1]}\] and \[\text{[2]}\]) if whenever \( a, b \in R, m \in M \) satisfy \( mab = 0 \), we have \( mba = 0 \). Symmetric \( R \)-modules are also studied in \[\text{[1]}\] and \[\text{[2]}\]. In our case, we have the following.

**Definition 3.7.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). The module \( M \) is called endo-symmetric if for any \( m \in M \) and \( f, g \in S \), \( fgm = 0 \) implies \( gfm = 0 \).

**Lemma 3.8.** If \( M \) is an endo-symmetric module, then \( S \) is a symmetric ring. The converse holds if \( M \) is an endo-p.p. module.

**Proof.** Let \( f, g, h \in S \) and assume \( fgh = 0 \). Then \( fghm = 0 \) for all \( m \in M \). By hypothesis, \( hfgm = 0 \) for all \( m \in M \). Hence \( hfg = 0 \). Conversely, assume that \( M \) is an endo-p.p. module and \( S \) is a symmetric ring. Let \( fgm = 0 \). There exists \( e^2 = e \in S \) such that \( f \in l_S(gm) = Se \). Then \( f = fe \) and \( gem = 0 \). Similarly, there exists an idempotent \( e_1 \in S \) such that \( eg \in l_S(m) = Se_1 \). Hence \( eg = ege_1 \) and \( e_1m = 0 \). By hypothesis, \( Se_1m = 0 \) implies \( e_1Sm = 0 \) and so \( ege_1Sm = egSm = 0 \). Thus \( 0 = egfm = g fem = gfm \). □

**Lemma 3.9.** If \( M \) is endo-symmetric, then \( M \) is endo-semicommutative. The converse is true if \( M \) is an endo-p.p. module.

**Proof.** Let \( f \in S \) and \( m \in M \) with \( fm = 0 \). Then for all \( g \in S \), \( gfm = 0 \) implies \( fgm = 0 \). So \( fSm = 0 \). Conversely, let \( f, g \in S \) and \( m \in M \) with \( fgm = 0 \). Then \( f \in l_S(gm) = Se \) for some \( e^2 = e \in S \). So \( f = fe \) and \( efm = 0 \). Since \( M \) is endo-semicommutative, \( egSm = 0 \). Therefore \( gfm = g fem = gem = efgm = 0 \) because \( e \) is central. □

**Lemma 3.10.** If \( M \) is an endo-reduced module, then \( M \) is endo-symmetric. The converse holds if \( M \) is an endo-p.p. module.

**Proof.** The first statement is from \[2, \text{Lemma 2.18}\]. Conversely, let \( f \in S \) and \( m \in M \) with \( f^2m = 0 \). Then \( f \in l_S(fm) = Se \) for some \( e^2 = e \in S \). So \( f = fe \) and \( efm = 0 \). By Lemma 3.8, \( M \) is endo-semicommutative, and so \( efm = 0 \). Then \( fgm = fegm = efgm = 0 \) for any \( g \in S \). Therefore \( fSm = 0 \). □

The next example shows that the reverse implication of the first statement in Lemma 3.10 is not true in general, i.e., there exists an endo-symmetric module which is neither endo-reduced nor endo-p.p. and nor endo-rigid.

**Example 3.11.** Consider a ring
\[ R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\} \]

and a right \( R \)-module
\[ M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}. \]

Let \( f \in S \) and \( f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix} \). Multiplying the latter by \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) we have \( f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad+bc \end{bmatrix} \). Similarly, let \( g \in S \) and \( g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix} \). Then \( g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix} \).

For any \( \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M \), \( f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad' + bc' \end{bmatrix} \). Then it is easy to check that for any \( \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M \),
\[ fg \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f \begin{bmatrix} 0 & 0 \\ ac & ac' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ac'd + bc'c \end{bmatrix} \]
and,
\[ gf \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & acd' + ac'd + bcc' \end{bmatrix} \]

Hence \( fg = gf \) for all \( f, g \in S \). Therefore \( S \) is commutative and so \( M \) is endo-symmetric. Define \( f \in S \) by \( f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \), where \( \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M \). Then \( f \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \) and \( f^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0 \). Hence \( M \) is neither endo-reduced nor endo-rigid. If \( m = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \), then \( l_S(m) \neq 0 \) since the endomorphism \( f \) defined preceding belongs to \( l_S(m) \). \( M \) is indecomposable as a right \( R \)-module, therefore \( S \) does not have any idempotents other than zero and identity. Hence \( l_S(m) \) can not be generated by an idempotent as a left ideal of \( S \).

We now summarize the relations between endo-rigid, endo-reduced, endo-symmetric and endo-semicommutative modules and their endomorphism rings by using endo-p.p. modules.

**Theorem 3.12.** If \( M \) is an endo-p.p. module, then we have the following.

1. \( M \) is an endo-rigid module if and only if \( S \) is a reduced ring.
2. \( M \) is an endo-reduced module if and only if \( S \) is a reduced ring.
3. \( M \) is an endo-symmetric module if and only if \( S \) is a symmetric ring.
4. \( M \) is an endo-semicommutative module if and only if \( S \) is a semicommutative ring.

We wind up the paper with some observations concerning relationships between endo-reduced modules, endo-rigid modules, endo-symmetric modules, endo-semicommutative modules and abelian modules by using endo-p.p. modules.

**Theorem 3.13.** If $M$ is an endo-p.p. module, then the following conditions are equivalent.
1. $M$ is an endo-rigid module.
2. $M$ is an endo-reduced module.
3. $M$ is an endo-symmetric module.
4. $M$ is an endo-semicommutative module.
5. $M$ is an abelian module.

Proof. (1) $\iff$ (2) Lemma 3.6, (2) $\iff$ (3) Lemma 3.10, (3) $\iff$ (4) Lemma 3.9, (4) $\iff$ (5) Lemma 3.2.

We obtain the following well-known result as a direct consequence.

**Corollary 3.14.** If $R$ is a right p.p. ring, then the following conditions are equivalent.
1. $R$ is a reduced ring.
2. $R$ is a symmetric ring.
3. $R$ is a semicommutative ring.
4. $R$ is an abelian ring.

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**References**


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