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THE SPRAY AND ANTISPRAY THEORY IN THE SUBSPACES OF MIRON'S $Osc^k M$[1]

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Abstract. The spray theory in $Osc^k M$ was introduced by R. Miron and Gh. Atanasiu in [7], [8] R. Miron with coauthors in [6], [9], [10] gave a comprehensive theory of higher order geometry and the spray theory. In [2] I. Čomić reported the relation between J structure, Liouville vector fields and the S-vector field as a more general basis than used by R. Miron and with the different variable $y^{k\alpha} = \dfrac{d^k x^\alpha}{dt^k}$ ($y^{k\alpha} = \dfrac{1}{k!}\dfrac{d^k x^\alpha}{dt^k}$ in Miron's papers). Here, the adapted basis is changed in such a way that the mentioned relations have a new, simpler and more elegant form. The combinatorial aspect was also used and the notion of antispray is introduced. Using the specially adapted bases the spray and antispray theories in the subspaces of $Osc^k M$ were established.

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1. Adapted bases in $T(Osc^k M)$ and $T^*(Osc^k M)$

$E = Osc^k M$ is a $(k+1)n-$dimensional C^∞ manifold, where M is a basic manifold of class C^∞ and dimension n, such that every point $u \in E$ in a local chart (V, ψ) has the coordinates $(y^{0a}, y^{1a}, y^{2a}, ..., y^{ka}) = (y^{Aa})$, $A = 0, 1, 2, ..., k$, $a = 1, 2, ..., n$, where $y^{0a} \in M$ is the point of some curve $c(t) : \mathbb{R} \to M$, and the coordinates are connected by the relation $y^{Aa} = \frac{d}{dt} y^{(A-1)a}$, $A = 1, 2, \ldots, k$. It is convenient that capitals Latin run over $\{0, 1, 2, \ldots, k\}$ and small Latin letters run over $\{1, 2, \ldots, n\}$. The following abbreviations will be used:

$$\partial_{Aa} = \frac{\partial}{\partial y^{Aa}}, \qquad B^{a'}_a = \partial_{0a} y^{0a'}, \qquad B^a_{a'} = \partial_{0a'} y^{0a}.$$

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A pseudogroup of the allowable coordinate transformations on E is given by

(1.1)
$$\begin{aligned}
y^{0a'} &= y^{0a'}(y^{0a}) = y^{0a'}(y^{01}, y^{02}, \dots, y^{0n})\\
y^{1a'} &= (\partial_{0a} y^{0a'}) y^{1a}\\
y^{2a'} &= (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a}\\
&\vdots\\
y^{ka'} &= (\partial_{0a} y^{(k-1)a'}) y^{1a} + (\partial_{1a} y^{(k-1)a'}) y^{2a} + \dots + (\partial_{(k-1)a} y^{(k-1)a'}) y^{ka},
\end{aligned}$$

with an additional condition that $y^{0a} \to y^{0a'}$ is a regular transformation on M, i.e. $det[B_a^{a'}] \neq 0$, where $B_a^{a'} = \left[\dfrac{\partial y^{0a'}}{\partial y^{0a}}\right]_{n\times n}$, [1]. The Jacobian of (1.1) is $[B_{(a)}^{(a')}]$, and its main diagonal matrix $Diag[B_{(a)}^{(a')}]$:

$$[B_{(a)}^{(a')}] = \begin{bmatrix} \partial_{0a} y^{0a'} & 0 & \dots & 0\\ \partial_{0a} y^{1a'} & \partial_{1a} y^{1a'} & \dots & 0\\ \vdots & \vdots & & \vdots\\ \partial_{0a} y^{ka'} & \partial_{1a} y^{ka'} & \dots & \partial_{ka} y^{ka'} \end{bmatrix},$$

$$Diag[B_{(a)}^{(a')}] = Diag(\partial_{0a} y^{0a'}, \partial_{1a} y^{1a'}, \dots, \partial_{ka} y^{ka'}) = B_a^{a'} I_{(k+1)\times(k+1)}.$$

$T(E)$ is the tangent bundle. If $u \in E$ is an arbitrary fixed point then $T_u(E)$ is a tangent space with a natural basis $\bar{B} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}, ..., \partial_{ka}\} = \{\partial_{Aa}\}$. Its dual space $T_u{}^*(Osc^k M)$ has a natural basis $\bar{B}^* = \{dy^{0a}, dy^{1a}, dy^{2a}, ..., dy^{ka}\} = \{dy^{Aa}\}$. These bases are mutually dual, $[dy^{(a)}][\partial_{(b)}] = \delta_b^a I$. Their elements are not transforming as d-tensors, but $[\partial_{(a)}] = [\partial_{(a')}][B_{(a)}^{(a')}]$, $[dy^{(a')}] = [B_{(a)}^{(a')}][dy^{(a)}]$.

Abbreviated notation follows:

$$[\partial_{(a)}] = [\partial_{0a}\ \partial_{1a}\ \dots\ \partial_{ka}], \quad [dy^{(a)}] = \left[dy^{0a}\ dy^{1a}\ \dots\ dy^{ka}\right]^T.$$

The elements of $\bar{B}$ are adapted by nonlinear connection coefficients N (and the elements of $\bar{B}^*$ by $M = N^{-1}$):

(1.2)
$$[N_{(a)}^{(b)}] = \begin{bmatrix} \delta_a^b & 0 & 0 & \dots & 0\\ -N_{0a}^{1b} & \delta_a^b & 0 & & 0\\ -N_{0a}^{2b} & -N_{1a}^{2b} & \delta_a^b & & 0\\ \vdots & & & & \\ -N_{0a}^{kb} & -N_{1a}^{kb} & -N_{2a}^{kb} & & \delta_a^b \end{bmatrix},$$

$$[M_{(c)}^{(b)}] = \begin{bmatrix} \delta_c^b & 0 & 0 & \dots & 0\\ M_{0c}^{1b} & \delta_c^b & 0 & \dots & 0\\ M_{0c}^{2b} & M_{1c}^{2b} & \delta_c^b & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ M_{0c}^{kb} & M_{1c}^{kb} & M_{2c}^{kb} & \dots & \delta_c^b \end{bmatrix},$$

which satisfy conditions:

$$(1.3)\quad \begin{aligned} N^{(A+B)b'}_{Aa'}\partial_{0a}y^{0a'} &= \sum_{C=1}^{B} N^{(A+C)c}_{Aa}\partial_{(A+C)c}y^{(A+B)b'} - \partial_{Aa}y^{(A+B)b'} \\ \Longleftrightarrow [N^{(b')}_{(a')}]B^{a'}_{a}I &= [B^{(b')}_{(c)}][N^{(c)}_{(a)}], \end{aligned}$$

$$(1.4)\quad \begin{aligned} M^{(A+B)b}_{Aa}\partial_{0b}y^{0a'} &= \sum_{C=0}^{B-1} M^{(A+B)a'}_{(A+C)c'}\partial_{Aa}y^{(A+C)c'} + \partial_{Aa}y^{(A+B)a'}, \\ \Longleftrightarrow B^{a'}_{b}I[M^{(b)}_{(a)}] &= [M^{(a')}_{(c')}][B^{(c')}_{(a)}], \end{aligned}$$

$$(1.5)\quad \begin{aligned} N^{(A+B)b}_{Aa} &= M^{(A+B)b}_{Aa} - \sum_{C=1}^{B-1} M^{(A+B)b}_{(A+C)c}N^{(A+C)c}_{Aa}. \\ \Longleftrightarrow [M^{(b)}_{(c)}][N^{(c)}_{(a)}] &= \delta^b_a I. \end{aligned}$$

The adapted bases for $T(Osc^kM)$ and $T^*(Osc^kM)$, determined by N and M respectively are

$$B = \{\delta_{0a}, \delta_{1a}, ..., \delta_{ka}\} = \{\delta_{Aa}\} \text{ and } B^* = \{\delta y^{0b}, \delta y^{1b}, ..., \delta y^{kb}\} = \{\delta y^{Bb}\}.$$

The conditions (1.3)–(1.5) provide that their elements transform as d-tensors, and they are mutually dual (these facts are proved in [4] with slightly different notation):

$$(1.6)\quad [\delta y^{(a)}][\delta_{(b)}] = \delta^a_b I,\ [\delta_{(a)}] = [\delta_{(a')}]B^{a'}_{a}I,\ [\delta y^{(a')}] = B^{a'}_{a}I[\delta y^{(a)}].$$

2. Specially adapted bases and the J structure

Conditions (1.3), (1.4) and (1.5) are also satisfied for a special choice of nonlinear connection coefficients (this is proved in [2]):

$$(2.1)\quad \begin{gathered} N^{(A+B)a}_{Ab} = \binom{A+B}{A} N^{Ba}_{0b},\quad M^{(A+B)a}_{Ab} = \binom{A+B}{A} M^{Bb}_{0b}, \\ A = 0,1,\dots,k,\quad B = 0,1,\dots,k-A. \end{gathered}$$

Definition 2.1. The specially adapted basis B of $T(E)$ is $[\delta_{(a)}] = [\partial_{(b)}][N^{(b)}_{(a)}]$, where

$$[N^{(b)}_{(a)}] = \begin{bmatrix} \binom{0}{0}\delta^b_a & 0 & 0 & \dots & 0 \\ -\binom{1}{0}N^{1b}_{0a} & \binom{1}{1}\delta^b_a & 0 & \dots & 0 \\ -\binom{2}{0}N^{2b}_{0a} & -\binom{2}{1}N^{1b}_{0a} & \binom{2}{2}\delta^b_a & \dots & 0 \\ \vdots & & & & \vdots \\ -\binom{k}{0}N^{kb}_{0a} & -\binom{k}{1}N^{(k-1)b}_{0a} & -\binom{k}{2}N^{(k-2)b}_{0a} & \dots & \binom{k}{k}\delta^b_a \end{bmatrix}.$$

Definition 2.2. The specially adapted basis B^* of $T^*(E)$ is $[\delta y^{(a)}] = [M^{(a)}_{(b)}][dy^{(b)}]$, where

$$[M^{(a)}_{(b)}] = \begin{bmatrix} \binom{0}{0}\delta^a_b & 0 & 0 & \dots & 0 \\ \binom{1}{0}M^{1a}_{0b} & \binom{1}{1}\delta^a_b & 0 & \dots & 0 \\ \binom{2}{0}M^{2a}_{0b} & \binom{2}{1}M^{1a}_{0b} & \binom{2}{2}\delta^a_b & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{k}{0}M^{ka}_{0b} & \binom{k}{1}M^{(k-1)a}_{0b} & \binom{k}{2}M^{(k-2)a}_{0b} & \dots & \binom{k}{k}\delta^a_b \end{bmatrix}.$$

Specially adapted bases $B = \{\delta_{0a}, \delta_{1a}, \dots, \delta_{ka}\}$ and $B^* = \{\delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka}\}$ get the properties (1.6), and are comprehensive with k-tangent structure J.

Definition 2.3. The k-tangent structure J is an $\mathcal{R}$-linear mapping $J : \chi(E) \to \chi(E)$ defined in the natural basis of $T(E)$:

$$\begin{aligned} &J\partial_{0a} = \partial_{1a}, J\partial_{1a} = 2\partial_{2a}, \dots, \\ &\dots J\partial_{\alpha a} = (\alpha+1)\partial_{(\alpha+1)a}, \dots, J\partial_{(k-1)a} = k\partial_{ka}, J\partial_{ka} = 0. \end{aligned} \tag{2.2}$$

The J structure is a nilpotent linear mapping with the index $(k+1)$, $J^{k+1} = 0$.

Introducing, the tensor of the type $(1,1)$:

$$[J^{(a)}_{(b)}] = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1\delta^a_b & 0 & 0 & \dots & 0 & 0 \\ 0 & 2\delta^a_b & 0 & \dots & 0 & 0 \\ 0 & 0 & 3\delta^a_b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k\delta^a_b & 0 \end{bmatrix},$$

The J structure in the natural bases $\bar{B}$ and $\bar{B}^*$ can be written in the tensor form

(2.3)

$$\begin{aligned} J &= [\partial_{0a}\partial_{1a}\dots\partial_{ka}][J^{(a)}_{(b)}] \otimes \begin{bmatrix} dy^{0a} \\ dy^{1a} \\ \vdots \\ dy^{ka} \end{bmatrix} = [\partial_{(a)}][J^{(a)}_{(b)}] \otimes [dy^{(b)}] = \\ &= \partial_{1a} \otimes dy^{0a} + 2\partial_{2a} \otimes dy^{1a} + 3\partial_{3a} \otimes dy^{2a} + \dots + k\partial_{ka} \otimes dy^{(k-1)a}. \end{aligned}$$

Theorem 2.1. *The k-tangent structure J defined by (2.3) acts on the dual tangent space $T^*(E)$, transforming elements of the natural basis $\bar{B}^*$ in the following way:*

$$dy^{0b}J = 0, \quad dy^{1b}J = dy^{0b}, \quad dy^{2b}J = 2dy^{1b}, \ \dots \ dy^{kb}J = kdy^{(k-1)b}. \tag{2.4}$$

Theorem 2.2. *The k-tangent structure J in the special adapted bases B and B^* is expressed by*

$$(2.5)\quad \begin{aligned} J &= [\delta_{0a}\delta_{1a}\ldots\delta_{ka}][J^{(a)}_{(b)}]\otimes\begin{bmatrix}\delta y^{0a}\\ \delta y^{1a}\\ \vdots\\ \delta y^{ka}\end{bmatrix} = [\delta_{(a)}][J^{(a)}_{(b)}]\otimes[\delta y^{(b)}] = \\ &= \delta_{1a}\otimes\delta y^{0a} + 2\delta_{2a}\otimes\delta y^{1a} + 3\delta_{3a}\otimes\delta y^{2a} + \cdots + k\delta_{ka}\otimes\delta y^{(k-1)a}. \end{aligned}$$

Theorem 2.3. *The k-tangent structure J transforms elements of the specially adapted basis $B = \{\delta_{0a}, \delta_{1a}, \ldots, \delta_{ka}\}$ in the following way*

$$(2.6)\quad \begin{gathered} J\delta_{0a} = \delta_{1a}, J\delta_{1a} = 2\delta_{2a}, \ldots, J\delta_{Aa} = (A+1)\delta_{(A+1)a}, \ldots \\ \ldots, J\delta_{(k-1)a} = k\delta_{ka}, J\delta_{ka} = 0. \end{gathered}$$

Proof. Using (2.5) it is obvious that

$$J[\delta_{0a}\,\delta_{1a}\ldots\delta_{ka}] = [\delta_{0b}\,\delta_{1b}\ldots\delta_{kb}][J^{(b)}_{(c)}]\delta^c_a I = [\delta_{1a}\,2\delta_{2a}\ldots k\delta_{ka}\,0]$$

□

Theorem 2.4. *The k-tangent structure J acts on the elements of the specially adapted basis B^* in the following way*

$$(2.7)\quad \delta y^{0b}J = 0, \delta y^{1b}J = \delta y^{0b}, \delta y^{2b}J = 2\delta y^{1b}, \ldots, \delta y^{kb}J = k\delta y^{(k-1)b}.$$

Proof. From (2.5) it follows

$$\begin{bmatrix}\delta y^{0a}\\ \delta y^{1a}\\ \delta y^{2a}\\ \vdots\\ \delta y^{ka}\end{bmatrix} J = \delta^a_b I[J^{(b)}_{(c)}]\begin{bmatrix}\delta y^{0c}\\ \delta y^{1c}\\ \delta y^{2c}\\ \vdots\\ \delta y^{kc}\end{bmatrix} = \begin{bmatrix}0\\ \delta y^{0a}\\ 2\delta y^{1a}\\ \vdots\\ k\delta y^{(k-1)a}\end{bmatrix}.$$

□

Previous theorems prove that the J structure in the natural bases and specially adapted bases has the same components.

Let $[J^{(a)}_{(b)}]^T$ be a transposed matrix of $[J^{(a)}_{(b)}]$, i.e.

$$[J^{(a)}_{(b)}]^T = \begin{bmatrix}0 & \delta^a_b & 0 & \ldots & \\ 0 & 0 & 2\delta^a_b & \ldots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \ldots & k\delta^a_b\\ 0 & 0 & 0 & \ldots & 0\end{bmatrix}.$$

Definition 2.4. The k-tangent structure $\bar{J} = J^T$ is transposed one of J,

(2.8)
$$\begin{aligned}\bar{J} = J^T &= [dy^{0a} dy^{1a} \dots dy^{ka}][J^{(a)}_{(b)}]^T \otimes \begin{bmatrix} \partial_{0a} \\ \partial_{1a} \\ \vdots \\ \partial_{ka} \end{bmatrix} = [dy^{(b)}]^T [J^{(a)}_{(b)}]^T \otimes [\partial_{(a)}]^T \\ &= dy^{0a} \otimes \partial_{1a} + 2dy^{1a} \otimes \partial_{2a} + 3dy^{2a} \otimes \partial_{3a} + \cdots + k dy^{(k-1)a} \otimes \partial_{ka}.\end{aligned}$$

From the above definition and Theorem 2.3 one gets expression of $\bar{J}$ in the specially adapted bases:

Theorem 2.5. *In the specially adapted bases B and B^* the structure $\bar{J}$ has the form*

(2.9)
$$\begin{aligned}\bar{J} = J^T &= [\delta y^{0b}\, \delta y^{1b} \dots \delta y^{kb}][J^{(a)}_{(b)}]^T \otimes \begin{bmatrix} \delta_{0a} \\ \delta_{1a} \\ \vdots \\ \delta_{ka} \end{bmatrix} = [\delta y^{(b)}]^T [J^{(a)}_{(b)}]^T \otimes [\delta_{(a)}]^T \\ &= \delta y^{0a} \otimes \delta_{1a} + 2\delta y^{1a} \otimes \delta_{2a} + 3\delta y^{2a} \otimes \delta_{3a} + \cdots + k\delta y^{(k-1)a} \otimes \delta_{ka}.\end{aligned}$$

Theorems 2.1 - 2.5 give

Theorem 2.6. *The action of $\bar{J}$-structure on the natural and specially adapted bases of $T(E)$ and $T(E)^*$ is given by the following equations*

(2.10)
$$\begin{bmatrix} \partial_{0a} \\ \partial_{1a} \\ \partial_{2a} \\ \vdots \\ \partial_{ka} \end{bmatrix} \bar{J} = \begin{bmatrix} \partial_{0a}\bar{J} \\ \partial_{1a}\bar{J} \\ \partial_{2a}\bar{J} \\ \vdots \\ \partial_{ka}\bar{J} \end{bmatrix} = \begin{bmatrix} \partial_{1a} \\ 2\partial_{2a} \\ 3\partial_{3a} \\ \vdots \\ k\partial_{ka} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \delta_{0a} \\ \delta_{1a} \\ \delta_{2a} \\ \vdots \\ \delta_{ka} \end{bmatrix} \bar{J} = \begin{bmatrix} \delta_{0a}\bar{J} \\ \delta_{1a}\bar{J} \\ \delta_{2a}\bar{J} \\ \vdots \\ \delta_{ka}\bar{J} \end{bmatrix} = \begin{bmatrix} \delta_{1a} \\ 2\delta_{2a} \\ 3\delta_{3a} \\ \vdots \\ k\delta_{ka} \\ 0 \end{bmatrix},$$

(2.11)
$$\begin{aligned}\bar{J}[dy^{0a}\ dy^{1a}\ dy^{2a} \dots dy^{ka}] &= [\bar{J}dy^{0a}\ \bar{J}dy^{1a}\ \bar{J}dy^{2a} \dots \bar{J}dy^{ka}] \\ &= [0\ dy^{0a}\ 2dy^{1a}\ 3dy^{2a} \dots k dy^{(k-1)a}], \\ \bar{J}[\delta y^{0a}\ \delta y^{1a}\ \delta y^{2a} \dots \delta y^{ka}] &= [\bar{J}\delta y^{0a}\ \bar{J}\delta y^{1a}\ \bar{J}\delta y^{2a} \dots \bar{J}\delta y^{ka}] \\ &= [0\ dy^{0a}\ 2\delta y^{1a}\ 3\delta y^{2a} \dots k\delta y^{(k-1)a}].\end{aligned}$$

3. Liouville fields and the J structure

Definition 3.1. The vector fields $\Gamma_{(1)}, \Gamma_{(2)}, \dots, \Gamma_{(k)}$, which in the natural basis $\bar{B}$ of $T(E)$ have the form:

$$
\begin{aligned}
\Gamma_{(1)} &= \tbinom{k}{0} y^{1a}\partial_{ka},\\
\Gamma_{(2)} &= \tbinom{k}{1} y^{2a}\partial_{ka} + \tbinom{k-1}{0} y^{1a}\partial_{(k-1)a},\\
\Gamma_{(3)} &= \tbinom{k}{2} y^{3a}\partial_{ka} + \tbinom{k-1}{1} y^{2a}\partial_{(k-1)a} + \tbinom{k-2}{0} y^{1a}\partial_{(k-2)a},\\
&\vdots\\
\Gamma_{(k)} &= \tbinom{k}{k-1} y^{ka}\partial_{ka} + \tbinom{k-1}{k-2} y^{(k-1)a}\partial_{(k-1)a} + \dots + \tbinom{2}{1} y^{2a}\partial_{2a} + \tbinom{1}{0} y^{1a}\partial_{1a},
\end{aligned}
\tag{3.1}
$$

are the Liouville vector fields in $T(Osc^kM)$.

Theorem 3.1. *The Liouville vector fields determined by (3.1) are d-vector fields.*

Remark 3.1. It can be proved that $(k-(i-1))!\Gamma_{(i)}$ (determined by (3.1)) are exactly the Liouville vector fields $\Gamma^{(i)}$ given by R. Miron and Gh. Atanasiu in [7], [8].

Theorem 3.2. *The Liouville vector fields (3.1) in the specially adapted basis B (see Definition 2.1) have the form*

$$
\begin{aligned}
\Gamma_{(1)} &= \tbinom{k}{0}\frac{\delta y^{0a}}{dt}\delta_{ka},\\
\Gamma_{(2)} &= \tbinom{k}{1}\frac{\delta y^{1a}}{dt}\delta_{ka} + \tbinom{k-1}{0}\frac{\delta y^{0a}}{dt}\delta_{(k-1)a},\\
\Gamma_{(3)} &= \tbinom{k}{2}\frac{\delta y^{2a}}{dt}\delta_{ka} + \tbinom{k-1}{1}\frac{\delta y^{1a}}{dt}\delta_{(k-1)a} + \tbinom{k-2}{0}\frac{\delta y^{0a}}{dt}\delta_{(k-2)a},\\
&\vdots\\
\Gamma_{(k)} &= \tbinom{k}{k-1}\frac{\delta y^{(k-1)a}}{dt}\delta_{ka} + \tbinom{k-1}{k-2}\frac{\delta y^{(k-2)a}}{dt}\delta_{(k-1)a} + \dots + \tbinom{2}{1}\frac{\delta y^{1a}}{dt}\delta_{2a} + \tbinom{1}{0}\frac{\delta y^{0a}}{dt}\delta_{1a},
\end{aligned}
\tag{3.2}
$$

Proof. The proof is a direct consequence of certain theorems given in [2]. □

Definition 3.2. The Liouville vector fields $\bar{\Gamma}_0, \bar{\Gamma}_1, \dots, \bar{\Gamma}_k$ are

$$\bar{\Gamma}_A = \Gamma_{(A+1)}dt, \qquad A = 0, 1, \dots, k-1,$$

$$\bar{\Gamma}_k = \tbinom{k}{k}\delta y^{ka}\delta_{ka} + \tbinom{k-1}{k-1}\delta y^{(k-1)a}\delta_{(k-1)a} + \dots + \tbinom{1}{1}\delta y^{1a}\delta_{1a} + \tbinom{0}{0}\delta y^{0a}\delta_{0a}.$$

Proposition 3.1. *The Liouville vector fields* $\bar{\Gamma}_A, \quad A = 0, 1, \dots, k$, *in the specially adapted bases are given by*

$$
\begin{aligned}
\bar{\Gamma}_0 = \quad & \Gamma_{(1)}dt &&= \tbinom{k}{0}\delta y^{0a}\delta_{ka},\\
\bar{\Gamma}_1 = \quad & \Gamma_{(2)}dt &&= \tbinom{k}{1}\delta y^{1a}\delta_{ka} + \tbinom{k-1}{0}\delta y^{0a}\delta_{(k-1)a},\\
\bar{\Gamma}_2 = \quad & \Gamma_{(3)}dt &&= \tbinom{k}{2}\delta y^{2a}\delta_{ka} + \tbinom{k-1}{1}\delta y^{1a}\delta_{(k-1)a} + \tbinom{k-2}{0}\delta y^{0a}\delta_{(k-2)a},\\
& \vdots &&\\
\bar{\Gamma}_{k-1} = \quad & \Gamma_{(k)}dt &&= \tbinom{k}{k-1}\delta y^{(k-1)a}\delta_{ka} + \tbinom{k-1}{k-2}\delta y^{(k-2)a}\delta_{(k-1)a} + \cdots\\
& &&\cdots + \tbinom{2}{1}\delta y^{1a}\delta_{2a} + \tbinom{1}{0}\delta y^{0a}\delta_{1a},\\
& &&= k\delta y^{(k-1)a}\delta_{ka} + \ldots + 3\delta y^{2a}\delta_{3a} + 2\delta y^{1a}\delta_{2a} + \delta y^{0a}\delta_{1a},\\
& \bar{\Gamma}_k &&= \delta y^{ka}\delta_{ka} + \delta y^{(k-1)a}\delta_{(k-1)a} + \cdots + \delta y^{1a}\delta_{1a} + \delta y^{0a}\delta_{0a}.
\end{aligned}
\tag{3.3}
$$

Theorem 3.3. *The following relation is valid*

$$[\bar{\Gamma}_0\ \bar{\Gamma}_1\ \bar{\Gamma}_2 \dots \bar{\Gamma}_k] = [\delta y^{0a}\ \delta y^{1a}\ \delta y^{2a} \dots \delta y^{ka}][\bar{\Gamma}_a], \tag{3.4}$$

where

$$
[\bar{\Gamma}_a] = \begin{bmatrix}
\binom{k}{0}\delta_{ka} & \binom{k-1}{0}\delta_{(k-1)a} & \cdots & \binom{1}{0}\delta_{1a} & \binom{0}{0}\delta_{0a}\\
0 & \binom{k}{1}\delta_{ka} & \cdots & \binom{2}{1}\delta_{2a} & \binom{1}{1}\delta_{1a}\\
\vdots & \vdots & & \vdots & \vdots\\
0 & 0 & \cdots & \binom{k}{k-1}\delta_{ka} & \binom{k-1}{k-1}\delta_{(k-1)a}\\
0 & 0 & \cdots & 0 & \binom{k}{k}\delta_{ka}
\end{bmatrix}
\tag{3.5}
$$

Definition 3.3. The Liouville 1-form fields $\Gamma_0, \Gamma_1, \dots, \Gamma_k$ are defined by

$$
\begin{bmatrix} \Gamma_0\\ \Gamma_1\\ \vdots\\ \Gamma_k \end{bmatrix} = [\Gamma_a] \begin{bmatrix} \delta y^{0a}\\ \delta y^{1a}\\ \vdots\\ \delta y^{ka} \end{bmatrix}, \qquad [\Gamma_a] = [\bar{\Gamma}_a]^T.
\tag{3.6}
$$

As

$$
[\Gamma_a] = \begin{bmatrix}
\binom{k}{0}\delta_{ka} & 0 & \cdots & 0 & 0\\
\binom{k-1}{0}\delta_{(k-1)a} & \binom{k}{1}\delta_{ka} & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots\\
\binom{1}{0}\delta_{1a} & \binom{2}{1}\delta_{2a} & \cdots & \binom{k}{k-1}\delta_{ka} & 0\\
\binom{0}{0}\delta_{0a} & \binom{1}{1}\delta_{1a} & \cdots & \binom{k-1}{k-1}\delta_{(k-1)a} & \binom{k}{k}\delta_{ka}
\end{bmatrix},
$$

the explicit forms of the Liouville 1-form fields are

(3.7)

$$\begin{aligned}
\Gamma_0 &= \tbinom{k}{0}\delta_{ka}\delta y^{0a},\\
\Gamma_1 &= \tbinom{k-1}{0}\delta_{(k-1)a}\delta y^{0a} + \tbinom{k}{1}\delta_{ka}\delta y^{1a},\\
\Gamma_2 &= \tbinom{k-2}{0}\delta_{(k-2)a}\delta y^{0a} + \tbinom{k-1}{1}\delta_{(k-1)a}\delta y^{1a} + \tbinom{k}{2}\delta_{ka}\delta y^{2a},\\
&\vdots\\
\Gamma_{k-1} &= \tbinom{1}{0}\delta_{1a}\delta y^{0a}+\tbinom{2}{1}\delta_{2a}\delta y^{1a}+\cdots+\tbinom{k-1}{k-2}\delta_{(k-1)a}\delta y^{(k-2)a}+\tbinom{k}{k-1}\delta_{ka}\delta y^{(k-1)a},\\
\Gamma_k &= \tbinom{0}{0}\delta_{0a}\delta y^{0a}+\tbinom{1}{1}\delta_{1a}\delta y^{1a}+\cdots+\tbinom{k-1}{k-1}\delta_{(k-1)a}\delta y^{(k-1)a}+\tbinom{k}{k}\delta_{ka}\delta y^{ka}.
\end{aligned}$$

Remark 3.2. It is easy to see that the term $\binom{k-A}{B}\delta y^{Ba}\delta_{(k-A)a}$ in the vector $\bar{\Gamma}_C$ corresponds to $\binom{k-A}{B}\delta_{(k-A)a}\delta y^{Ba}$ in the 1-form Γ_C.

Theorem 3.4. *The k-tangent structure J acts on the Liouville 1-form fields Γ_A in the following way:*

(3.8) $\quad J\Gamma_A = \Gamma_A J = (k-(A-1))\Gamma_{(A-1)},\ A=1,2,\dots,k,\ J\Gamma_0=\Gamma_0 J=0,$

i.e.

$$\begin{aligned}
&J\Gamma_0=\Gamma_0J=0, \quad J\Gamma_1=\Gamma_1J=k\Gamma_0, \qquad J\Gamma_2=\Gamma_2J=(k-1)\Gamma_1,\\
&\dots,\quad J\Gamma_{k-1}=\Gamma_{k-1}J=2\Gamma_{k-2} \quad J\Gamma_k=\Gamma_kJ=\Gamma_{k-1}.
\end{aligned}$$

Proof. Using Theorem 2.3 and equations (3.7) one gets

$$\begin{aligned}
J\Gamma_A &= (\delta_{1a}\otimes\delta y^{0a}+2\delta_{2a}\otimes\delta y^{1a}+3\delta_{3a}\otimes\delta y^{2a}+\ldots+k\delta_{ka}\otimes\delta y^{(k-1)a})\\
&\quad (\tbinom{k-A}{0}\delta_{(k-A)a}\delta y^{0a}+\tbinom{k-A+1}{1}\delta_{(k-A+1)a}\delta y^{1a}+\ldots\\
&\quad \ldots+\tbinom{k-1}{A-1}\delta_{(k-1)a}\delta y^{(A-1)a}+\tbinom{k}{A}\delta_{ka}\delta y^{Aa})\\
&= \tbinom{k-A}{0}(k-A+1)\delta_{(k-A+1)a}\delta y^{0a}+\\
&\quad +\tbinom{k-A+1}{1}(k-A+2)\delta_{(k-A+2)a}\delta y^{1a}+\ldots+\tbinom{k-1}{A-1}k\delta_{ka}\delta y^{(A-1)a}\\
&= (k-A+1)[\tbinom{k-A+1}{0}\delta_{(k-A+1)a}\delta y^{0a}\\
&\quad +\tbinom{k-A+2}{1}\delta_{(k-A+2)a}\delta y^{1a}+\ldots+\tbinom{k}{k-A}\delta_{ka}\delta y^{(A-1)a}]\\
&= (k-A+1)\Gamma_{A-1},
\end{aligned}$$

because

$$k\tbinom{k-1}{A-1}=\tfrac{k!}{(A-1)!\,(k-A)!}=(k-A+1)\tbinom{k}{A-1},\quad \ldots,$$
$$(k-A+1)\tbinom{k-A}{0}=(k-A+1)\tbinom{k-A+1}{0}.$$

On the other hand

$$\begin{aligned}
\Gamma_AJ &= \tbinom{k-A+1}{1}\delta_{(k-A+1)a}\delta y^{0a}+\ldots\\
&\quad \ldots+\tbinom{k-1}{A-1}\delta_{(k-1)a}(A-1)\delta y^{(A-2)a}+\tbinom{k}{A}\delta_{ka}A\delta y^{(A-1)a}\\
&= (k-A+1)[\tbinom{k-A+1}{0}\delta_{(k-A+1)a}\delta y^{0a}+\ldots\\
&\quad \ldots+\tbinom{k-1}{A-2}\delta_{(k-1)a}\delta y^{(A-2)a}+\tbinom{k}{A-1}\delta_{ka}\delta y^{(A-1)a}],
\end{aligned}$$

because

$$A\binom{k}{A} = (k-A+1)\binom{k}{A-1}, (A-1)\binom{k-1}{A-1} = (k-A+1)\binom{k-1}{A-2}, \dots \qquad \square$$

Theorem 3.5. *The k-tangent structure $\bar{J}$ acts on the Liouville vector fields $\bar{\Gamma}_A$ in the following way:*

(3.9)

$$\bar{J}\bar{\Gamma}_A = \bar{\Gamma}_A\bar{J} = (k-(A-1))\bar{\Gamma}_{A-1}, \quad A = 1, 2, \dots, k, \qquad \bar{J}\bar{\Gamma}_0 = \bar{\Gamma}_0\bar{J} = 0,$$

i.e.

$$\begin{aligned} &\bar{J}\bar{\Gamma}_0 = \bar{\Gamma}_0\bar{J} = 0, \quad \bar{J}\bar{\Gamma}_1 = \bar{\Gamma}_1\bar{J} = k\bar{\Gamma}_0, \qquad \bar{J}\bar{\Gamma}_2 = \bar{\Gamma}_2\bar{J} = (k-1)\bar{\Gamma}_1, \\ &\dots, \quad \bar{J}\bar{\Gamma}_{k-1} = \bar{\Gamma}_{k-1}\bar{J} = 2\bar{\Gamma}_{k-2} \quad \bar{J}\bar{\Gamma}_k = \bar{\Gamma}_k\bar{J} = \bar{\Gamma}_{k-1}. \end{aligned}$$

Proof.

$$\begin{aligned} \bar{\Gamma}_A\bar{J} &= [\tbinom{k-A}{0}\delta y^{0a}\delta_{(k-A)a} + \tbinom{k-A+1}{1}\delta y^{1a}\delta_{(k-A+1)a} + \cdots + \tbinom{k}{A}\delta y^{Aa}\delta_{ka}] \\ &\quad [\delta y^{0a} \otimes \delta_{1a} + 2\delta y^{1a} \otimes \delta_{2a} + \cdots + k\delta y^{(k-1)a} \otimes \delta_{ka}] \\ &= \tbinom{k-A}{0}\delta y^{0a}(k-A+1)\delta_{(k-A+1)a} + \tbinom{k-A+1}{1}\delta y^{1a}(k-A+2)\delta_{(k-A+2)a} + \cdots \\ &\quad \cdots + \tbinom{k-1}{A-1}\delta y^{(A-1)a}k\delta_{ka} \\ &= (k-A+1)[\tbinom{k-A+1}{0}\delta y^{0a}\delta_{(k-A+1)a} + \tbinom{k-A+2}{1}\delta y^{1a}\delta_{(k-A+2)a} + \cdots \\ &\quad \cdots + \tbinom{k}{A-1}\delta y^{(A-1)a}\delta_{ka}] \\ &= (k-A+1)\Gamma_{A-1}. \qquad \square \end{aligned}$$

4. The k-sprays and k-antisprays

Definition 4.1. A k-spray on E is a vector field $\bar{S} \in T(E)$, with the property

$$\bar{S}\bar{J} = \bar{J}\bar{S} = \bar{\Gamma}_{k-1},$$

where $\bar{\Gamma}_{k-1}$ is given in (3.4))

Theorem 4.1. *The vector field given by*

$$\bar{S} = \bar{\Gamma}_k + \alpha\delta y^{0a}\delta_{ka}, \qquad \alpha \in \mathbb{R}, \tag{4.1}$$

is a k-spray on $T(E)$.

Proof.

$$\bar{J}\bar{S} = \bar{J}\bar{\Gamma}_k + \alpha(\bar{J}\delta y^{0a})\delta_{ka} = \bar{J}\bar{\Gamma}_k = \bar{\Gamma}_{k-1},$$

$$\bar{S}\bar{J} = \bar{\Gamma}_k\bar{J} + \alpha\delta y^{0a}(\delta_{ka}\bar{J}) = \bar{\Gamma}_k\bar{J} = \bar{\Gamma}_{k-1},$$

because of $\bar{J}\delta y^{0a} = 0$ and $\delta_{ka}\bar{J} = 0$, [5]. □

Let $\tilde{c}$ be an one-parameter real curve in $E = Osc^kM$, $t \mapsto \tilde{c}(t) \in E$:

$$\tilde{c}(t) \; : \; (y^{0a}(t), y^{1a}(t), y^{2a}(t), \dots, y^{ka}(t)).$$

The position vector $r(t)$ of an arbitrary point on $\tilde{c}(t)$ is given by

$$r(t) = y^{0a}\partial_{0a} + y^{1a}\partial_{1a} + \cdots + y^{(k-1)a}\partial_{(k-1)a} + y^{ka}\partial_{ka},$$

so, the tangent vector to the curve, by use of (3.3), in [2] is expressed as

$$\begin{aligned} dr &= dy^{0a}\partial_{0a} + dy^{1a}\partial_{1a} + \cdots + dy^{ka}\partial_{ka} \\ &= \delta y^{0a}\delta_{0a} + \delta y^{1a}\delta_{1a} + \cdots + \delta y^{ka}\delta_{ki} \\ &= \bar{\Gamma}_k. \end{aligned} \tag{4.2}$$

Similarly, there is an 1-form field on the curve $\tilde{c}(t)$:

$$\begin{aligned} \delta r &= \partial_{0a}dy^{0a} + \partial_{1a}dy^{1a} + \cdots + \partial_{ka}dy^{ka} \\ &= \delta_{0a}\delta y^{0a} + \delta_{1a}\delta y^{1a} + \cdots + \delta_{ki}\delta y^{ka} \\ &= \Gamma_k. \end{aligned} \tag{4.3}$$

Definition 4.2. The curve $\tilde{c}$ is an integral curve of the k-spray $\bar{S}$ if and only if $\bar{S}$ is a tangent vector field of $\tilde{c}$.

Theorem 4.2. *The curve $\tilde{c}$ is the integral curve of the k-spray $\bar{S} = \bar{\Gamma}_k$.*

Definition 4.3. A k-antispray on E is an 1-form field $S \in T^*(E)$, with the property

$$JS = SJ = \Gamma_{k-1}, \tag{4.4}$$

where (see (3.9))

$$\Gamma_{k-1} = \delta_{1a}\delta y^{0a} + 2\delta_{2a}\delta y^{1a} + 3\delta_{3a}\delta y^{2a} + \ldots + k\delta_{ka}\delta y^{(k-1)a}.$$

Theorem 4.3. *The 1-form field*

$$S = \Gamma_k + \alpha\delta_{ka}\delta y^{0a} \tag{4.5}$$

is a k-antispray on $T^*(E)$.

Proof.

$$JS = J\Gamma_k + \alpha(J\delta_{ka})\delta y^{0a} = J\Gamma_k = \Gamma_{k-1},$$

$$SJ = \Gamma_k J + \alpha\delta_{ka}(\delta y^{0a}J) = \Gamma_k J = \Gamma_{k-1}.$$

Properties of the J structure make the second terms vanish. □

Definition 4.4. The curve $\tilde{c}$ is an integral curve of the k-antispray S if and only if S is parallel with δr.

Theorem 4.4. *The curve $\tilde{c}$ is an integral curve of the k-antispray $S = \Gamma_k$.*

Proof. The result follows from Theorem 4.3 for case $\alpha = 0$. □

Theorem 4.5. *The Liouville vector and 1-form fields $\bar{\Gamma}_A$ and Γ_A, the k-sprays and antisprays $\bar{S}$ and S, and tangent structures $\bar{J}$ and J, are connected in the following way:*

$$\begin{array}{ll} \bar{J}\bar{S} = \bar{S}\bar{J} = \bar{\Gamma}_{k-1} & JS = SJ = \Gamma_{k-1} \\ \bar{J}^2\bar{S} = \bar{S}\bar{J}^2 = 2!\bar{\Gamma}_{k-2} & J^2S = SJ^2 = 2!\Gamma_{k-2} \\ \vdots & \vdots \\ \bar{J}^k\bar{S} = \bar{S}\bar{J}^k = k!\bar{\Gamma}_0 & J^kS = SJ^k = k!\Gamma_0 \\ \bar{J}^{k+1}\bar{S} = \bar{S}\bar{J}^{k+1} = 0, & J^{k+1}S = SJ^{k+1} = 0. \end{array}$$

5. The subspaces in Osc^kM. The specially adapted bases.

Parametrization on the basic manifold M, such that in some local chart (U, φ) there exist two types of parameters, has the following form

$$y^{0a} = y^{0a}(u^{01}, \dots, u^{0m}, v^{0(m+1)}, \dots, v^{0n}) = y^{0a}(u^{0\alpha}, v^{0\widehat{\alpha}}), \tag{5.1}$$

$a, b, c, \dots = 1, 2, \dots, n,\ \alpha, \beta, \gamma, \delta, \dots = 1, 2, \dots, m,\ \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \dots = m+1, \dots, n.$ The new coordinates of the same point with respect to another chart (U', φ'), are $y^{0a'} = y^{0a'}(u^{01}, \dots, u^{0m'}, v^{0(m+1)'}, \dots, v^{n'}) = y^{0a'}(u^{0\alpha'}, v^{0\widehat{\alpha}'})$, with the additional condition

$$u^{0\alpha'} = u^{0\alpha'}(u^{01}, \dots, u^{0m}), \quad v^{0\widehat{\alpha}'} = v^{0\widehat{\alpha}'}(v^{0(m+1)}, \dots, v^{0n}).$$

Here, the following notations will be used:

$$\begin{aligned} &\partial_\alpha = \partial_{0\alpha} = \frac{\partial}{\partial u^{0\alpha}}, \quad B^{\alpha'}_{\alpha} = \partial_{0\alpha} u^{0\alpha'}, \quad B^{a}_{\alpha} = \partial_{0\alpha} y^{0a}, \\ &\partial_{\widehat{\alpha}} = \partial_{0\widehat{\alpha}} = \frac{\partial}{\partial v^{0\widehat{\alpha}}}, \quad B^{\widehat{\alpha}'}_{\widehat{\alpha}} = \partial_{0\widehat{\alpha}} v^{0\widehat{\alpha}'}, \quad B^{a}_{\widehat{\alpha}} = \partial_{0\widehat{\alpha}} y^{0a}. \end{aligned} \tag{5.2}$$

If the transformation $y^{0a} = y^{0a}(u^{0\alpha}, v^{0\widehat{\alpha}})$ is regular, then there exists an inverse transformation: $u^{0\alpha} = u^{0\alpha}(y^{0a}), \quad v^{0\widehat{\alpha}} = v^{0\widehat{\alpha}}(y^{0a})$.

The Jacobian matrix D of (5.1) and its inverse one D^{-1} are

$$D = \frac{D(y^{01}, \dots, y^{0n})}{D(u^{01}, \dots, u^{0m}, v^{0(m+1)}, \dots, v^{0n})} = \left[\ [B^{(a)}_{(\alpha)}]_{n\times m}\ [B^{a}_{\widehat{\alpha}}]_{n\times(n-m)}\ \right], \tag{5.3}$$

$$D^{-1} = \begin{bmatrix} [B^{(\beta)}_{(b)}]_{m\times n} \\ [B^{(\widehat{\beta})}_{(b)}]_{(n-m)\times n} \end{bmatrix}.$$

As $(y^{0a}) = (y^{0a}(u^{0\alpha}, v^{0\widehat{\alpha}}))$ is a point in M, one can ignore 0 in the superscripts, i.e. $y^{0a} = x^a$, $u^{0\alpha} = u^\alpha$, $v^{0\widehat{\alpha}} = v^{\widehat{\alpha}}$. The explicit forms of the above matrices are:

(5.4)

$$[B^{(a)}_{(\alpha)}] = \begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^m} \\ \vdots & & \\ \frac{\partial x^n}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^m} \end{bmatrix}_{n\times m} \qquad [B^{(a)}_{(\widehat{\alpha})}] = \begin{bmatrix} \frac{\partial x^1}{\partial v^{m+1}} & \cdots & \frac{\partial x^1}{\partial v^n} \\ \vdots & & \\ \frac{\partial x^n}{\partial v^{m+1}} & \cdots & \frac{\partial x^n}{\partial v^n} \end{bmatrix}_{n\times(n-m)}$$

$$[B^{(\beta)}_{(b)}] = \begin{bmatrix} \frac{\partial u^1}{\partial x^1} & \cdots & \frac{\partial u^1}{\partial x^n} \\ \vdots & & \\ \frac{\partial u^m}{\partial x^1} & \cdots & \frac{\partial u^m}{\partial x^n} \end{bmatrix}_{m\times n} \qquad [B^{(\widehat{\beta})}_{(b)}] = \begin{bmatrix} \frac{\partial v^{m+1}}{\partial x^1} & \cdots & \frac{\partial v^{m+1}}{\partial x^n} \\ \vdots & & \\ \frac{\partial v^n}{\partial x^1} & \cdots & \frac{\partial v^n}{\partial x^n} \end{bmatrix}_{(n-m)\times n}.$$

The elements of these matrices are related by (the calculation is given in [3])

$$\begin{aligned} &B^\beta_a B^a_\alpha = \delta^\beta_\alpha, \quad B^{\widehat{\beta}}_a B^a_\alpha = 0, \\ &B^\beta_a B^a_{\widehat{\alpha}} = 0, \quad\;\; B^{\widehat{\beta}}_a B^a_{\widehat{\alpha}} = \delta^{\widehat{\beta}}_{\widehat{\alpha}} \\ &B^a_\alpha B^\alpha_b + B^a_{\widehat{\alpha}} B^{\widehat{\alpha}}_b = \delta^a_b. \end{aligned} \tag{5.5}$$

In the basic manifold there exist two families of submanifolds M_1 and M_2, given by the equations:

$$M_1 : y^{0a} = y^{0a}(u^{0\alpha}, C^{0\widehat{\alpha}}) \qquad M_2 : y^{0a} = y^{0a}(C^{0\alpha}, v^{0\widehat{\alpha}}), \tag{5.6}$$

such that the functions appeared in (5.6) are C^∞.

Consideration of the curves with same real parameter t, on M, M_1 and M_2, generates three osculate spaces $E = Osc^kM$, $E_1 = Osc^kM_1$ and $E_2 = Osc^kM_2$, respectively, and E_1 and E_2 are subspaces of E, $dim(E_1) = (k+1)m$, $\quad dim(E_2) = (k+1)(n-m)$.

The subspaces E_1 and E_2 have the same properties as E, only notation is specific. The coordinates in E_1 are denoted by u, and all indices with small Greek letters. The coordinates in E_2 are denoted by v, and all indices with small Greek letters with hat. A point $u \in E_1$ has the coordinates $(u^{0\alpha}, u^{1\alpha}, \ldots, u^{k\alpha})$, where $(u^{A\alpha} = \frac{d^A}{dt^A}u^{0\alpha})$, and a point $v \in E_2$ has the coordinates $(v^{0\widehat{\alpha}}, v^{1\widehat{\alpha}}, \ldots, v^{k\widehat{\alpha}})$, where $(v^{A\widehat{\alpha}} = \frac{d^A}{dt^A}v^{0\widehat{\alpha}})$.

The coordinate transformations on E_1 and E_2 are of the same type as on E,

$$\begin{aligned} &u^{0\alpha'} = u^{0\alpha'}(u^{01}, u^{02}, \ldots, u^{0m}), && v^{0\widehat{\alpha}'} = v^{0\widehat{\alpha}'}(v^{0(m+1)}, \ldots, v^{0n}), \\ &u^{1\alpha'} = \frac{\partial u^{0\alpha'}}{\partial u^{0\alpha}} u^{1\alpha}, && v^{1\widehat{\alpha}'} = \frac{\partial v^{0\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}}, \\ &u^{2\alpha'} = \frac{\partial u^{1\alpha'}}{\partial u^{0\alpha}} u^{1\alpha} + \frac{\partial u^{1\alpha'}}{\partial u^{1\alpha}} u^{2\alpha}, && v^{2\widehat{\alpha}'} = \frac{\partial v^{1\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}} + \frac{\partial v^{1\widehat{\alpha}'}}{\partial v^{1\widehat{\alpha}}} \partial v^{2\widehat{\alpha}}, \\ &\quad\vdots && \quad\vdots \\ &u^{k\alpha'} = \frac{\partial u^{(k-1)\alpha'}}{\partial u^{0\alpha}} u^{1\alpha} + \ldots + \frac{\partial u^{(k-1)\alpha'}}{\partial u^{(k-1)\alpha}} u^{k\alpha}, && v^{k\widehat{\alpha}'} = \frac{\partial v^{(k-1)\widehat{\alpha}'}}{\partial v^{0\widehat{\alpha}}} \partial v^{1\widehat{\alpha}} + \ldots + \frac{\partial v^{(k-1)\widehat{\alpha}'}}{\partial v^{1(k-1)\alpha}} \partial v^{k\widehat{\alpha}}, \end{aligned}$$

The natural bases $\bar{B}_1$ of $T(E_1)$, further $\bar{B}_2$ of $T(E_2)$ are

$$\bar{B}_1 = \{\partial_{0\alpha}, \partial_{1\alpha}, \dots, \partial_{k\alpha}\},\ \bar{B}_2 = \{\partial_{0\widehat{\alpha}}, \partial_{1\widehat{\alpha}}, \dots, \partial_{k\widehat{\alpha}}\},$$

and their dual bases $\bar{B}_1^*$ and $\bar{B}_2^*$ are

$$\bar{B}_1^* = \{du^{0\alpha}, du^{1\alpha}, \dots, du^{k\alpha}\},\ \bar{B}_2^* = \{dv^{0\widehat{\alpha}}, dv^{1\widehat{\alpha}}, \dots, dv^{k\widehat{\alpha}}\}.$$

The elements of natural tangent and cotangent bases for E_1 and E_2 are not d tensors, so they can be adapted generally or in a special way. In this paper they will be specially adapted by

$$(5.7)\qquad [M^{(\alpha)}_{(\beta)}] = \begin{bmatrix} \binom{0}{0}\delta^\alpha_\beta & 0 & 0 & \cdots & 0 \\ \binom{1}{0}M^{1\alpha}_{0\beta} & \binom{1}{1}\delta^\alpha_\beta & 0 & \cdots & 0 \\ \binom{2}{0}M^{2\alpha}_{0\beta} & \binom{2}{1}M^{1\alpha}_{0\beta} & \binom{2}{2}\delta^\alpha_\beta & \cdots & 0 \\ \vdots & & & & \\ \binom{k}{0}M^{k\alpha}_{0\beta} & \binom{k}{1}M^{(k-1)\alpha}_{0\beta} & \binom{k}{2}M^{(k-2)\alpha}_{0\beta} & \cdots & \binom{k}{k}\delta^\alpha_\beta \end{bmatrix},$$

$$(5.8)\qquad [N^{(\alpha)}_{(\beta)}] = \begin{bmatrix} \binom{0}{0}\delta^\beta_\alpha & 0 & 0 & \cdots & 0 \\ -\binom{1}{0}N^{1\beta}_{0\alpha} & \binom{1}{1}\delta^\beta_\alpha & 0 & \cdots & 0 \\ -\binom{2}{0}N^{2\beta}_{0\alpha} & -\binom{2}{1}N^{1\beta}_{0\alpha} & \binom{2}{2}\delta^\beta_\alpha & \cdots & 0 \\ \vdots & & & & \\ -\binom{k}{0}N^{k\beta}_{0\alpha} & -\binom{k}{1}N^{(k-1)\beta}_{0\alpha} & -\binom{k}{2}N^{(k-2)\beta}_{0\alpha} & \cdots & \binom{k}{k}\delta^\beta_\alpha \end{bmatrix},$$

and $[M^{(\widehat{\alpha})}_{(\widehat{\beta})}]$, $[N^{(\widehat{\beta})}_{(\widehat{\alpha})}]$ obtained from (5.7) and (5.8) by substitution α, β with $\widehat{\alpha}$, $\widehat{\beta}$.

The adapted bases B_1 and B_1^* of $T(E_1)$ and $T^*(E_1)$, further B_2 and B_2^* of $T(E_2)$ and $T^*(E_2)$ respectively, are

$$B_1 = \{\delta_{0\alpha}, \delta_{1\alpha}, \dots, \delta_{k\alpha}\}, \qquad B_2 = \{\delta_{0\widehat{\alpha}}, \delta_{1\widehat{\alpha}}, \dots, \delta_{k\widehat{\alpha}}\},$$

$$B_1^* = \{\delta u^{0\alpha}, \delta u^{1\alpha}, \dots, \delta u^{k\alpha}\}, \qquad B_2^* = \{\delta v^{0\widehat{\alpha}}, \delta v^{1\widehat{\alpha}}, \dots, \delta v^{k\widehat{\alpha}}\}.$$

For the matrix notation here are used analogous abbreviations as for space E, i.e.

$$[\partial_{(\alpha)}] = [\partial_{0\alpha}\, \partial_{1\alpha} \dots \partial_{k\alpha}], \qquad [\partial_{(\widehat{\alpha})}] = [\partial_{0\widehat{\alpha}}\, \partial_{1\widehat{\alpha}} \dots \partial_{k\widehat{\alpha}}]$$

$$[du^{(\alpha)}] = \begin{bmatrix} du^{0\alpha} \\ du^{1\alpha} \\ \vdots \\ du^{k\alpha} \end{bmatrix}, \qquad [dv^{\widehat{\alpha}}] = \begin{bmatrix} dv^{0\widehat{\alpha}} \\ dv^{1\widehat{\alpha}} \\ \vdots \\ dv^{k\widehat{\alpha}} \end{bmatrix},$$

$$[\delta_{(\alpha)}] = [\delta_{0\alpha}\ \delta_{1\alpha} \dots \delta_{k\alpha}], \qquad [\delta_{(\widehat{\alpha})}] = [\delta_{0\widehat{\alpha}}\ \delta_{1\widehat{\alpha}} \dots \delta_{k\widehat{\alpha}}],$$

$$[\delta u^{(\alpha)}] = \begin{bmatrix} \delta u^{0\alpha} \\ \delta u^{1\alpha} \\ \vdots \\ \delta u^{k\alpha} \end{bmatrix}, \qquad [\delta v^{(\widehat{\alpha})}] = \begin{bmatrix} \delta v^{0\widehat{\alpha}} \\ \delta v^{1\widehat{\alpha}} \\ \vdots \\ \delta v^{k\widehat{\alpha}} \end{bmatrix}.$$

$$[B^{(\alpha')}_{(\alpha)}] = \begin{bmatrix} \partial_{0\alpha} u^{0\alpha'} & 0 & \cdots & 0 \\ \partial_{0\alpha} u^{1\alpha'} & \partial_{1\alpha} u^{1\alpha'} & \cdots & 0 \\ \vdots & & & \\ \partial_{0\alpha} u^{k\alpha'} & \partial_{1\alpha} u^{k\alpha'} & \cdots & \partial_{k\alpha} u^{k\alpha'} \end{bmatrix}$$

and $[B^{(\widehat{\alpha}')}_{(\widehat{\alpha})}]$, obtained from $[B^{(\alpha')}_{(\alpha)}]$ by substituting u, α, α' with v, $\widehat{\alpha}$, $\widehat{\alpha}'$ respectively.

Definition 5.1. The specially adapted bases B_1, B_1^*, B_2, B_2^* are defined by

$$[\delta_{(\alpha)}] = [\partial_{(\beta)}][N^{(\beta)}_{(\alpha)}] \qquad [\delta_{(\widehat{\alpha})}] = [\partial_{(\widehat{\beta})}][N^{(\widehat{\beta})}_{(\widehat{\alpha})}],$$

$$[\delta u^{(\alpha)}] = [M^{(\alpha)}_{(\beta)}][du^{(\beta)}] \qquad [\delta v^{(\widehat{\alpha})}] = [M^{(\widehat{\alpha})}_{(\widehat{\beta})}][dv^{(\widehat{\beta})}].$$

In both subspaces E_1 and E_2, the corresponding bases are mutually dual and their elements are transforming as d-tensors if and only if

$$[M^{(\beta)}_{(\gamma)}][N^{(\gamma)}_{(\alpha)}] = \delta^{\beta}_{\alpha} I_{m\times m}, \qquad [M^{(\widehat{\beta})}_{(\widehat{\gamma})}][N^{(\widehat{\gamma})}_{(\widehat{\alpha})}] = \delta^{\widehat{\beta}}_{\widehat{\alpha}} I_{(n-m)\times(n-m)}; \tag{5.9}$$

$$[M^{(\alpha')}_{(\beta')}][B^{(\beta')}_{(\beta)}] = B^{\alpha'}_{\alpha}[M^{(\alpha)}_{(\beta)}], \qquad [M^{(\widehat{\alpha}')}_{(\widehat{\beta}')}][B^{(\widehat{\beta}')}_{(\widehat{\beta})}] = B^{\widehat{\alpha}'}_{\widehat{\alpha}}[M^{(\widehat{\alpha})}_{(\widehat{\beta})}]; \tag{5.10}$$

$$[B^{(\gamma')}_{(\gamma)}][N^{(\gamma)}_{(\alpha)}] = [N^{(\gamma')}_{(\alpha')}]B^{\alpha'}_{\alpha}, \qquad [B^{(\widehat{\gamma}')}_{(\widehat{\gamma})}][N^{(\widehat{\gamma})}_{(\widehat{\alpha})}] = [N^{(\widehat{\gamma}')}_{(\widehat{\alpha}')}]B^{\widehat{\alpha}'}_{\widehat{\alpha}}. \tag{5.11}$$

Further, we consider the previous conditions.

Theorem 5.1. *The specially adapted tangent bases B, B_1, B_2 of $T(E)$, $T(E_1)$ and $T(E_2)$ are related by*

$$\delta_{Aa} = B^{\alpha}_{a}\delta_{A\alpha} + B^{(\widehat{\alpha})}_{a}\delta_{A\widehat{\alpha}}, \quad A = 0, 1, \ldots, k, \tag{5.12}$$

if and only if

$$[B^{(\beta)}_{(b)}][N^{(b)}_{(a)}] = [N^{(\beta)}_{(\alpha)}][B^{(\alpha)}_{(a)}], \quad [B^{(\widehat{\beta})}_{(b)}][N^{(b)}_{(a)}] = [N^{(\widehat{\beta})}_{(\widehat{\alpha})}][B^{(\widehat{\alpha})}_{(a)}]. \tag{5.13}$$

The specially adapted cotangent bases B^*, B_1^* and B_2^* are related by

$$\delta y^{Aa} = B^{a}_{\alpha}\delta u^{A\alpha} + B^{a}_{\widehat{\alpha}}\delta v^{A\widehat{\alpha}}, \quad A = 0, 1, \ldots, k, \tag{5.14}$$

if and only if

$$[M^{(a)}_{(b)}][B^{(b)}_{(\beta)}] = [B^{(a)}_{(\alpha)}][M^{(\alpha)}_{(\beta)}], \quad [M^{(a)}_{(b)}][B^{(b)}_{(\widehat{\beta})}] = [B^{(a)}_{(\widehat{\alpha})}][M^{(\widehat{\alpha})}_{(\widehat{\beta})}], \tag{5.15}$$

Proof. The proof is given in [3]. □

6. Tangent structures, Liouville fields and k-sprays and k-antysprays in the subspaces

The J structure in the specially adapted bases B and B^* of $T(E)$ and $T^*(E)$ can be written in the form:

$$J = \delta_{1a} \otimes \delta y^{0a} + 2\delta_{2a} \otimes \delta y^{1a} + 3\delta_{3a} \otimes \delta y^{2a} + \cdots + k\delta_{ka} \otimes \delta y^{(k-1)a}. \tag{6.1}$$

The substitution of (5.12) and (5.14) into (6.1) results

$$\begin{aligned} J \;=\; & (B_a^\alpha \delta_{1\alpha} + B_a^{\widehat{\alpha}} \delta_{1\widehat{\alpha}}) \otimes (B_\beta^a \delta u^{0\beta} + B_{\widehat{\beta}}^a \delta v^{0\widehat{\beta}}) \\ & + 2(B_a^\alpha \delta_{2\alpha} + B_a^{\widehat{\alpha}} \delta_{2\widehat{\alpha}}) \otimes (B_\beta^a \delta u^{1\beta} + B_{\widehat{\beta}}^a \delta v^{1\widehat{\beta}}) \\ & + 3(B_a^\alpha \delta_{3\alpha} + B_a^{\widehat{\alpha}} \delta_{3\widehat{\alpha}}) \otimes (B_\beta^a \delta u^{2\beta} + B_{\widehat{\beta}}^a \delta v^{2\widehat{\beta}}) + \cdots \\ & + k(B_a^\alpha \delta_{k\alpha} + B_a^{\widehat{\alpha}} \delta_{k\widehat{\alpha}}) \otimes (B_{\widehat{\beta}}^a \delta u^{(k-1)\widehat{\beta}} + B_{\widehat{\beta}}^a \delta v^{(k-1)\widehat{\beta}}). \end{aligned}$$

As $B_a^\alpha B_\beta^a = \delta_\beta^\alpha$, $B_a^{\widehat{\alpha}} B_{\widehat{\beta}}^a = \delta_{\widehat{\beta}}^{\widehat{\alpha}}$, $B_a^\alpha B_{\widehat{\beta}}^a = 0$, $B_a^{\widehat{\alpha}} B_\beta^a = 0$, the previous equation proves:

Theorem 6.1. *The J structure, which in the specially adapted bases B and B^* has the form (6.1), in the bases B_1, B_2, B_1^*, B_2^* can be written in the form*

$$J = J' + J'', \tag{6.2}$$

where

$$J' = \delta_{1\alpha} \otimes \delta u^{0\alpha} + 2\delta_{2\alpha} \otimes \delta u^{1\alpha} + 3\delta_{3\alpha} \otimes \delta u^{2\alpha} + \cdots + k\delta_{k\alpha} \otimes \delta u^{(k-1)\alpha}, \tag{6.3}$$

$$J'' = \delta_{1\widehat{\alpha}} \otimes \delta v^{0\widehat{\alpha}} + 2\delta_{2\widehat{\alpha}} \otimes \delta v^{1\widehat{\alpha}} + 3\delta_{3\widehat{\alpha}} \otimes \delta v^{2\widehat{\alpha}} + \cdots + k\delta_{k\widehat{\alpha}} \otimes \delta v^{(k-1)\widehat{\alpha}}. \tag{6.4}$$

From Theorem 6.1 and duality of the bases B_1 and B_1^* as well as B_2 and B_2^* one gets some important facts:

Theorem 6.2. *The following relations are valid:*

$$\begin{aligned} & J\delta_{0\beta} = J'\delta_{0\beta} = \delta_{1\beta}, && J\delta_{1\beta} = J'\delta_{1\beta} = 2\delta_{2\beta}, \\ & J\delta_{2\beta} = J'\delta_{2\beta} = 3\delta_{3\beta} && ,\ldots, \\ & J\delta_{(k-1)\beta} = J'\delta_{(k-1)\beta} = k\delta_{k\beta}, && J\delta_{k\beta} = 0 \end{aligned} \tag{6.5}$$

$$\begin{aligned} & J\delta_{0\widehat{\beta}} = J''\delta_{0\widehat{\beta}} = \delta_{1\widehat{\beta}}, && J\delta_{1\widehat{\beta}} = J''\delta_{1\widehat{\beta}} = 2\delta_{2\widehat{\beta}}, \\ & J\delta_{2\widehat{\beta}} = J''\delta_{2\widehat{\beta}} = 3\delta_{3\widehat{\beta}} && ,\ldots, \\ & J\delta_{(k-1)\widehat{\beta}} = J''\delta_{(k-1)\widehat{\beta}} = k\delta_{k\widehat{\beta}}, && J\delta_{k\widehat{\beta}} = 0. \end{aligned} \tag{6.6}$$

Theorem 6.3. *The following relations are also valid:*

$$\begin{aligned}&\delta u^{0\beta}J=0,\quad \delta u^{1\beta}J=\delta u^{1\beta}J'=\delta u^{0\beta},\quad \delta u^{2\beta}J=\delta u^{2\beta}J'=2\delta u^{1\beta},\\ &\delta u^{3\beta}J=\delta u^{3\beta}J'=3\delta u^{2\beta},\quad \dots,\quad \delta u^{k\beta}J=\delta u^{k\beta}J'=k\delta u^{(k-1)\beta},\end{aligned}\tag{6.7}$$

$$\begin{aligned}&\delta v^{0\widehat{\beta}}J=0,\quad \delta v^{1\widehat{\beta}}J=\delta v^{1\widehat{\beta}}J''=\delta v^{0\widehat{\beta}},\quad \delta v^{2\widehat{\beta}}J=\delta v^{2\widehat{\beta}}J''=2\delta v^{1\widehat{\beta}},\\ &\delta v^{3\widehat{\beta}}J=\delta v^{3\widehat{\beta}}J''=3\delta v^{2\widehat{\beta}},\quad \dots,\quad \delta v^{k\widehat{\beta}}J=\delta v^{k\widehat{\beta}}J''=k\delta v^{(k-1)\widehat{\beta}}.\end{aligned}\tag{6.8}$$

Structures J' and J" are defined on the disjunct spaces $T(E_1)$ and $T(E_2)$, which means $J'J''=J''J'=0$, so Theorem 6.1 gives the decomposition of higher degrees J structure in the form

$$J^2=J'^2+J''^2,\qquad J^3=J'^3+J''^3,\ \dots,\quad J^k=J'^k+J''^k,\qquad J^{k+1}=0.$$

Using the same procedure as for the structure J, one gets analogous results:

Theorem 6.4. *The $\bar{J}$ structure, which in specially adapted bases B and B^* has the form*

$$\bar{J}=\delta y^{0a}\otimes\delta_{1a}+2\delta y^{1a}\otimes\delta_{2a}+3\delta y^{2a}\otimes\delta_{3a}+\cdots+k\delta y^{(k-1)a}\otimes\delta_{ka},$$

in the bases B_1, B_2, B_1^*, B_2^* can be written in the form

$$\bar{J}=\bar{J}'+\bar{J}'',\tag{6.9}$$

where

$$\bar{J}'=\delta u^{0\alpha}\otimes\delta_{1\alpha}+2\delta u^{1\alpha}\otimes\delta_{2\alpha}+3\delta u^{2\alpha}\otimes\delta_{3\alpha}+\cdots+k\delta u^{(k-1)\alpha}\otimes\delta_{k\alpha},\tag{6.10}$$

$$\bar{J}''=\delta v^{0\widehat{\alpha}}\otimes\delta_{1\widehat{\alpha}}+2\delta v^{1\widehat{\alpha}}\otimes\delta_{2\widehat{\alpha}}+3\delta v^{2\widehat{\alpha}}\otimes\delta_{3\widehat{\alpha}}+\cdots+k\delta v^{(k-1)\widehat{\alpha}}\otimes\delta_{k\widehat{\alpha}}.\tag{6.11}$$

Theorem 6.5. *The following relations are valid:*

$$\begin{aligned}&\delta_{0\beta}\bar{J}=\delta_{0\beta}\bar{J}'=\delta_{1\beta}, && \delta_{1\beta}\bar{J}=\delta_{1\beta}\bar{J}'=2\delta_{2\beta},\\ &\delta_{2\beta}\bar{J}=\delta_{2\beta}\bar{J}'=3\delta_{3\beta} && ,\dots,\\ &\delta_{(k-1)\beta}\bar{J}=\delta_{(k-1)\beta}\bar{J}'=k\delta_{k\beta}, && \delta_{k\beta}\bar{J}=\delta_{k\beta}\bar{J}'=0\end{aligned}\tag{6.12}$$

$$\begin{aligned}&\delta_{0\widehat{\beta}}\bar{J}=\delta_{0\widehat{\beta}}\bar{J}''=\delta_{1\widehat{\beta}}, && \delta_{1\widehat{\beta}}\bar{J}=\delta_{1\widehat{\beta}}\bar{J}''=2\delta_{2\widehat{\beta}},\\ &\delta_{2\widehat{\beta}}\bar{J}=\delta_{2\widehat{\beta}}\bar{J}''=3\delta_{3\widehat{\beta}} && ,\dots,\\ &\delta_{(k-1)\widehat{\beta}}\bar{J}=\delta_{(k-1)\widehat{\beta}}\bar{J}''=k\delta_{k\widehat{\beta}}, && \delta_{k\widehat{\beta}}\bar{J}=\delta_{k\widehat{\beta}}\bar{J}''=0.\end{aligned}\tag{6.13}$$

Theorem 6.6. *The following relations are also valid:*

(6.14)
$$\bar{J}\delta u^{0\beta} = \bar{J}'\delta u^{0\beta} = 0, \quad \bar{J}\delta u^{1\beta} = \bar{J}'\delta u^{1\beta} = \delta u^{0\beta}, \quad \bar{J}\delta u^{2\beta} = \bar{J}'\delta u^{2\beta} = 2\delta u^{1\beta},$$
$$\bar{J}\delta u^{3\beta} = \bar{J}'\delta u^{3\beta} = 3\delta u^{2\beta}, \quad \ldots, \quad \bar{J}\delta u^{k\beta} = \bar{J}'\delta u^{k\beta} = k\delta u^{(k-1)\beta},$$

(6.15)
$$\bar{J}\delta v^{0\widehat{\beta}} = \bar{J}''\delta v^{0\widehat{\beta}} = 0, \quad \bar{J}\delta v^{1\widehat{\beta}} = \bar{J}''\delta v^{1\widehat{\beta}} = \delta v^{0\widehat{\beta}}, \quad \bar{J}\delta v^{2\widehat{\beta}} = \bar{J}''\delta v^{2\widehat{\beta}} = 2\delta v^{1\widehat{\beta}},$$
$$\bar{J}\delta v^{3\widehat{\beta}} = \bar{J}''\delta v^{3\widehat{\beta}} = 3\delta v^{2\widehat{\beta}}, \quad \ldots, \quad \bar{J}\delta v^{k\widehat{\beta}} = \bar{J}''\delta v^{k\widehat{\beta}} = k\delta v^{(k-1)\widehat{\beta}}.$$

Using Theorem 3.3 and relations (5.2), (5.12) and (5.14) the Liouville vector and 1-form fields $\bar{\Gamma}_0$, and Γ_0 can be written in the following forms:

$$\bar{\Gamma}_0 = \binom{k}{0}(B^a_\alpha \delta u^{0\alpha} + B^a_{\widehat{\alpha}}\delta v^{0\widehat{\alpha}})(B^\beta_a \delta_{k\beta} + B^{\widehat{\beta}}_a \delta_{k\widehat{\beta}}) = \binom{k}{0}\delta u^{0\alpha}\delta_{k\alpha} + \binom{k}{0}\delta v^{0\widehat{\alpha}}\delta_{k\widehat{\alpha}},$$
$$\Gamma_0 = \binom{k}{0}(B^\beta_a \delta_{k\beta} + B^{\widehat{\beta}}_a \delta_{k\widehat{\beta}})(B^a_\alpha \delta u^{0\alpha} + B^a_{\widehat{\alpha}}\delta v^{0\widehat{\alpha}}) = \binom{k}{0}\delta_{k\alpha}\delta u^{0\alpha} + \binom{k}{0}\delta_{k\widehat{\alpha}}\delta v^{0\widehat{\alpha}}$$

The same calculations can be done for the remaining Liouville fields.

Theorem 6.7. *In the bases B_1, B_1^*,B_2, B_2^* the Liouville fields are expressed by:*

$$\begin{array}{ll} \bar{\Gamma}_0 = \bar{\Gamma}'_0 + \bar{\Gamma}''_0 & \Gamma_0 = \Gamma'_0 + \Gamma''_0 \\ \bar{\Gamma}_1 = \bar{\Gamma}'_1 + \bar{\Gamma}''_1 & \Gamma_1 = \Gamma'_1 + \Gamma''_1 \\ \vdots & \vdots \\ \bar{\Gamma}_{k-1} = \bar{\Gamma}'_{k-1} + \bar{\Gamma}''_{k-1} & \Gamma_{k-1} = \Gamma'_{k-1} + \Gamma''_{k-1} \\ \bar{\Gamma}_k = \bar{\Gamma}'_k + \bar{\Gamma}''_k & \Gamma_k = \Gamma'_k + \Gamma''_k \end{array} \tag{6.16}$$

where

(6.17)
$$\begin{aligned} \bar{\Gamma}'_0 &= \binom{k}{0}\delta u^{0\alpha}\delta_{k\alpha}, \\ \bar{\Gamma}'_1 &= \binom{k}{1}\delta u^{1\alpha}\delta_{k\alpha} + \binom{k-1}{0}\delta u^{0\alpha}\delta_{(k-1)\alpha}, \\ \bar{\Gamma}'_2 &= \binom{k}{2}\delta u^{2\alpha}\delta_{k\alpha} + \binom{k-1}{1}\delta u^{1\alpha}\delta_{(k-1)\alpha} + \binom{k-1}{0}\delta u^{0\alpha}\delta_{(k-2)\alpha}, \\ &\vdots \\ \bar{\Gamma}'_{k-1} &= \binom{k}{k-1}\delta u^{(k-1)\alpha}\delta_{k\alpha} + \binom{k-1}{k-2}\delta u^{(k-2)\alpha}\delta_{(k-1)\alpha} + \cdots \\ &\quad \cdots + \binom{2}{1}\delta u^{1\alpha}\delta_{2\alpha} + \binom{1}{0}\delta u^{0\alpha}\delta_{1\alpha} \\ \bar{\Gamma}'_k &= \binom{k}{k}\delta u^{k\alpha}\delta_{k\alpha} + \binom{k-1}{k-1}\delta u^{(k-1)\alpha}\delta_{(k-1)\alpha} + \cdots \\ &\quad \cdots + \binom{2}{2}\delta u^{2\alpha}\delta_{2\alpha} + \binom{1}{1}\delta u^{1\alpha}\delta_{1\alpha} + \binom{0}{0}\delta u^{0\alpha}\delta_{0\alpha}. \end{aligned}$$

(6.18)

$$\begin{aligned}
\Gamma'_0 &= \tbinom{k}{0}\delta_{k\alpha}\delta u^{0\alpha},\\
\Gamma'_1 &= \tbinom{k-1}{0}\delta_{(k-1)\alpha}\delta u^{0\alpha}+\tbinom{k}{1}\delta_{k\alpha}\delta u^{1\alpha},\\
\Gamma'_2 &= \tbinom{k-2}{0}\delta_{(k-2)\alpha}\delta u^{0\alpha}+\tbinom{k-1}{1}\delta_{(k-1)\alpha}\delta u^{1\alpha}+\tbinom{k}{2}\delta_{k\alpha}\delta u^{2\alpha},\\
&\vdots\\
\Gamma'_{k-1} &= \tbinom{1}{0}\delta_{1\alpha}\delta u^{0\alpha}+\tbinom{2}{1}\delta_{2\alpha}\delta u^{1\alpha}+\cdots\\
&\quad\cdots+\tbinom{k-1}{k-2}\delta_{(k-1)\alpha}\delta u^{(k-2)\alpha}+\tbinom{k}{k-1}\delta_{k\alpha}\delta u^{(k-1)\alpha},\\
\Gamma'_k &= \tbinom{0}{0}\delta u^{0\alpha}\delta_{0\alpha}+\tbinom{1}{1}\delta_{1\alpha}\delta u^{1\alpha}+\cdots\\
&\quad\cdots+\tbinom{k-1}{k-1}\delta_{(k-1)\alpha}\delta u^{(k-1)\alpha}+\tbinom{k}{k}\delta_{k\alpha}\delta u^{k\alpha}.
\end{aligned}$$

Substitution of α, u in (6.17) by $\widehat{\alpha}$, v respectively, gives the corresponding equations for $\bar{\Gamma}''_0, \bar{\Gamma}''_1, \ldots, \bar{\Gamma}''_k$; also, the substitution of α, u in (6.18) by $\widehat{\alpha}$, v respectively, gives the corresponding equations for $\Gamma''_1, \Gamma''_2, \ldots, \Gamma''_k$.

From Theorem 3.5, and relations (6.2), (6.10) and (6.17) it follows

Theorem 6.8. *The structure $\bar{J}$ acts on the Liouville vector fields on E_1 and E_2 in the following way*

(6.19)

$$\begin{aligned}
&\bar{J}\bar{\Gamma}'_k = \bar{J}'\bar{\Gamma}'_k = \bar{\Gamma}'_{k-1}, && \bar{J}\bar{\Gamma}''_k = \bar{J}''\bar{\Gamma}''_k = 1\bar{\Gamma}''_{k-1},\\
&\bar{J}\bar{\Gamma}'_{k-1} = \bar{J}'\bar{\Gamma}'_{k-1} = 2\bar{\Gamma}'_{k-2}, && \bar{J}\bar{\Gamma}''_{k-1} = \bar{J}''\bar{\Gamma}''_{k-1} = 2\bar{\Gamma}''_{k-2},\\
&\bar{J}\bar{\Gamma}'_{k-2} = \bar{J}'\bar{\Gamma}'_{k-2} = 3\bar{\Gamma}'_{k-3}, && \bar{J}\bar{\Gamma}''_{k-2} = \bar{J}''\bar{\Gamma}''_{k-2} = 3\bar{\Gamma}''_{k-3},\\
&\quad\vdots && \quad\vdots\\
&\bar{J}\bar{\Gamma}'_1 = \bar{J}'\bar{\Gamma}'_1 = k\bar{\Gamma}'_0 && \bar{J}\bar{\Gamma}''_1 = \bar{J}''\bar{\Gamma}''_1 = k\bar{\Gamma}''_0\\
&\bar{J}\bar{\Gamma}'_0 = \bar{J}'\bar{\Gamma}'_0 = 0. && \bar{J}\bar{\Gamma}''_0 = \bar{J}''\bar{\Gamma}''_0 = 0.
\end{aligned}$$

Using the relations (5.12) and (5.14) and Theorem 6.7, one obtains decomposition of the spray in the subspaces.

Theorem 6.9. *The vector field $\bar{S}$ can be written in the form:*

(6.20)
$$\bar{S} = \bar{S}' + \bar{S}'',$$

where

(6.21)
$$\begin{aligned}
\bar{S}' &= \delta u^{0\alpha}\delta_{0\alpha} + \delta u^{1\alpha}\delta_{1\alpha} + \cdots + \delta u^{(k-1)\alpha}\delta_{(k-1)\alpha} + \delta u^{k\alpha}\delta_{k\alpha},\\
\bar{S}'' &= \delta v^{0\widehat{\alpha}}\delta_{0\widehat{\alpha}} + \delta v^{1\widehat{\alpha}}\delta_{1\widehat{\alpha}} + \cdots + \delta v^{(k-1)\widehat{\alpha}}\delta_{(k-1)\widehat{\alpha}} + \delta v^{k\widehat{\alpha}}\delta_{k\widehat{\alpha}},
\end{aligned}$$

From Theorems 6.4 and 6.8 it follows

$$\begin{aligned}\bar J'\bar S' &= (\delta u^{0\alpha}\delta_{1\alpha} + 2\delta u^{1\alpha}\delta_{2\alpha} + \cdots + k\delta u^{(k-1)\alpha}\delta_{k\alpha}) = \bar\Gamma'_{k-1},\\ \bar J''\bar S'' &= (\delta v^{0\widehat\alpha}\delta_{1\widehat\alpha} + 2\delta v^{1\widehat\alpha}\delta_{2\widehat\alpha} + \cdots + k\delta v^{(k-1)\widehat\alpha}\delta_{k\widehat\alpha}) = \bar\Gamma''_{k-1}.\\ \bar J\bar S &= (\bar J' + \bar J'')(\bar S' + \bar S'') = \bar J'\bar S' + \bar J''\bar S'' = \bar\Gamma'_{k-1} + \bar\Gamma''_{k-1} = \bar\Gamma_{k-1};\end{aligned}$$

Same calculation for $\bar J^2, \bar J^3, \ldots, \bar J^k$ generates the remainder Liouville vector fields $\bar\Gamma'_{k-2}, \ldots, \bar\Gamma'_0$, and $\bar\Gamma''_{k-2}, \ldots, \bar\Gamma''_0$.

Theorem 6.10. *The spray vector fields $\bar S'$ and $\bar S''$ generate all corresponding Liouville vector fields in the subspaces of $T(E)$.*

$$\begin{aligned}\bar J'\bar S' &= \bar\Gamma'_{k-1}; & \bar J''\bar S'' &= \bar\Gamma''_{k-1},\\ (\bar J')^2\bar S' &= 2!\bar\Gamma'_{k-2}; & (\bar J'')^2\bar S'' &= 2!\bar\Gamma''_{k-2},\\ (\bar J')^3\bar S' &= 3!\bar\Gamma'_{k-3}; & (\bar J'')^3\bar S'' &= 3!\bar\Gamma''_{k-3},\\ &\vdots & &\vdots\\ (\bar J')^k\bar S' &= k!\bar\Gamma'_0; & (\bar J'')^k\bar S'' &= k!\bar\Gamma''_0,\\ (\bar J')^{k+1}\bar S' &= 0; & (\bar J'')^{k+1}\bar S'' &= 0.\end{aligned} \tag{6.22}$$

The last three theorems are valid in a dual tangent space, too. The proves are very similar.

Theorem 6.11. *The tangent structure J acts on the Liouville 1-form fields on E_1 and E_2 in the following way*

$$\begin{aligned}& J\Gamma'_k = J'\Gamma'_k = \Gamma'_{k-1}, && J\Gamma''_k = J''\Gamma''_k = \Gamma''_{k-1},\\ & J\Gamma'_{k-1} = J'\Gamma'_{k-1} = 2\Gamma'_{k-2} && J\Gamma''_{k-1} = J''\Gamma''_{k-1} = 2\Gamma''_{k-2}\\ & J\Gamma'_{k-2} = J'\Gamma'_{k-2} = 3\Gamma'_{k-3}, && J\Gamma''_{k-2} = J''\Gamma''_{k-2} = 3\Gamma''_{k-3},\\ & \quad\vdots && \quad\vdots\\ & J\Gamma'_2 = J'\Gamma'_2 = (k-1)\Gamma'_1, && J\Gamma''_2 = J''\Gamma''_2 = (k-1)\Gamma''_1\\ & J\Gamma'_1 = J'\Gamma'_1 = k\Gamma'_0, && J\Gamma''_1 = J''\Gamma''_1 = k\Gamma''_0,\\ & J\Gamma'_0 = J'\Gamma'_0 = 0; && J\Gamma''_0 = J''\Gamma''_0 = 0.\end{aligned} \tag{6.23}$$

Theorem 6.12. *The antispray 1-form field S can be decomposed in the subspaces:*

$$S = S' + S'', \tag{6.24}$$

where

$$\begin{aligned}S' &= \delta_{0\alpha}\delta u^{0\alpha} + \delta_{1\alpha}\delta u^{1\alpha} + \cdots + \delta_{(k-1)\alpha}\delta u^{(k-1)\alpha} + \delta_{k\alpha}\delta u^{k\alpha},\\ S'' &= \delta_{0\widehat\alpha}\delta v^{0\widehat\alpha} + \delta_{1\widehat\alpha}\delta v^{1\widehat\alpha} + \cdots + \delta_{(k-1)\widehat\alpha}\delta v^{(k-1)\widehat\alpha} + \delta_{k\widehat\alpha}\delta v^{k\widehat\alpha},\end{aligned} \tag{6.25}$$

Theorem 6.13. *The antispray 1-form fields S' and S'' generate all corresponding Liouville 1-form fields in the subspaces of $T^*(E)$.*

$$
\begin{aligned}
J'S' &= \Gamma'_{k-1}, & J''S'' &= \Gamma''_{k-1},\\
(J')^2S' &= 2!\Gamma'_{k-2}, & (J'')^2S'' &= 2!\Gamma''_{k-2},\\
(J')^3S' &= 3!\Gamma'_{k-3}, & (J'')^3S'' &= 3!\Gamma''_{k--3},\\
&\vdots & &\vdots\\
(J')^kS' &= k!\Gamma'_0, & (J'')^kS'' &= k!\Gamma''_0,\\
(J')^{k+1}S' &= 0; & (J'')^{k+1}S'' &= 0.
\end{aligned}
\tag{6.26}
$$

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