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(m+1)-DIMENSIONAL SPACELIKE PARALLEL p_i-EQUIDISTANT RULED SURFACES IN THE MINKOWSKI SPACE R_1^n

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Abstract. In this paper, spacelike parallel p_i-equidistant ruled surfaces in 3-dimensional Minkowski space R_1^3,[1] are generalized to n-dimensional Minkowski space R_1^n. Then some characteristic results related with algebraic invariants of shape operator of the (m+1)-dimensional spacelike parallel p_i-equidistant ruled surfaces are given in the Minkowski space R_1^n.

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1. Introduction

We shall assume throughout that all manifolds, maps, vector fields, etc... are differentiable of class C^∞. First of all, we give some properties of a general submanifold M in R_1^n, [2] . Suppose that $\overline{D}$ is the Levi-Civita connection of R_1^n, while D is the Levi-Civita connection of M. If X and Y are vector fields of M and if V is the second fundamental tensor of M, then we find by decomposing $\overline{D}_X Y$ into a tangent and normal component

(1.1) $\overline{D}_X Y = D_X Y + V(X,Y)$.

The equation (I.1) is called *Gauss Equation*.

If ξ is a normal vector field on M, we find the *Weingarten Equation* by decomposing $\overline{D}_X\ \xi$ in a tangent and a normal component as

(1.2) $\overline{D}_X \xi = -A_\xi(X) + D_X^\perp \xi$.

A_ξ determines at each point a self-adjoint linear map and $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. We use the same notation A_ξ for the linear map and the matrix of the linear map.

If the metric tensor of R_1^n is denoted by $<,>$, we have

(1.3) $< V(X,Y), \xi >=< Y, A_\xi(X) >$.

Let M be an m-dimensional semi-Riemannian manifold in R_1^n and A_ξ be a linear map. If $\zeta \in \chi^\perp(M)$ is a normal unit vector at the point $P \in M$, then

(1.4) $G(P;\xi) = \det A_\xi$

is called the *Lipschitz-Killing curvature* of M at P in the direction ξ.

If $\xi_1, \xi_2, ..., \xi_{n-m}$ constitute an orthonormal base field of the normal bundle $\chi^\perp(M)$, then the *mean curvature* H is given by

(1.5) $H = \sum\limits_{j=1}^{n-m} \frac{\mathrm{tr}_{A_{\xi_j}}}{\dim_M} \xi_j$.

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For every $X_i \in \chi(M)$, $1 \leq i \leq 4$ the 4^{th} order covariant tensor field defined by R as

(1.6) $R(X_1, X_2, X_3, X_4) = < X_1, R(X_3, X_4)X_2 >$

is called the *Riemannian curvature tensor field* and its value at a point $P \in M$ is called *Riemannian curvature* of M at P.

Let $\prod$ be a tangent plane of M at P. For all $X_p, Y_p \in \prod$, the real valued function K defined by

(1.7) $K(X_P, Y_P) = \frac{<R(X_P,Y_P)X_P,Y_P>}{<X_P,X_P><Y_P,Y_P>-<X_P,Y_P>^2}$

is called the *sectional curvature function.* $K(X_P, Y_P)$ is called the *sectional curvature* of M at P.

Let R be the Riemannian curvature tensor of M. The *Ricci curvature tensor field* S of M is by

(1.8) $S(X, Y) = \sum_{i=1}^{m} \varepsilon_i < R(e_i, X)Y, e_i >$,

where $\{e_1, e_2, ..., e_m\}$ is a system of orthonormal base of $T_M(P)$ and the value of $S(X, Y)$ at $P \in M$ is called *the Ricci curvature*, where

$$\varepsilon_i = < e_i, e_i >= \begin{cases} -1, \text{ if } e_i \text{ timelike} \\ 1, \text{ if } e_i \text{ spacelike} \end{cases}$$

The scalar curvature r_{sk} of M is given by

(1.9) $r_{sk} = \sum_{i \neq j} K(e_i, e_j) = 2 \sum_{i<j} K(e_i, e_j)$.

Let $\{\xi_1, \xi_2, ..., \xi_{n-m}\}$ be an orthonormal base field of $\chi^{\perp}(M)$. Then the *scalar normal curvature* K_N of M is given by

(1.10) $K_N = \sum_{i,j=1}^{n-m} \overline{M}\left(A_{\xi_i} A_{\xi_j} - A_{\xi_j} A_{\xi_i}\right)$,

where $\overline{M}$ is defined as $\overline{M}(A) = \sum_{i,j} (a_{ij})^2$, $A = [a_{ij}]$.

2. The Curvatures Of (m+1)-Dimensional Spacelike Parallel p_i-Equidistant Ruled Surfaces in the Minkowski Space R_1^n

I

Let α and α^* be two unit-speed spacelike curves in R_1^n and let $\{V_1, V_2, ..., V_k\}$ and $\{V_1^*, V_2^*, \ldots, V_k^*\}$, $k \leq n$, be their Frenet frames at the points α(t) and α^*(t*), respectively. Let **M** and **M*** be (m+1)-dimensional generalized spacelike ruled surfaces in R_1^n and $E_m(t)$ and $E_m(t^*)$, $1 \leq m \leq k-2$, be spacelike generating spaces of **M** and **M***, respectively. Then **M** and **M*** can be given by the following parametric form:

(2.1) $M : X(t, u_1, \ldots, u_m) = \alpha(t) + \sum_{i=1}^{m} u_i V_i(t)$,

$rank\, \{X_t, X_{u_1}, \ldots, X_{u_m}\} = m + 1$,

(2.2) $M^* : X^* (t^*, u_1^*, \ldots, u_m^*) = \alpha^*(t^*) + \sum_{i=1}^{m} u_i^* V_i^*(t^*)$,

$rank\left\{X^*_{t^*}, X^*_{u^*_1}, \ldots, X^*_{u^*_m}\right\} = m+1,$
where $\{V_1, V_2, \ldots, V_m\}$ and $\{V_1^*, V_2^*, \ldots, V_m^*\}$ are the orthonormal basis of $E_m(t)$ and $E_m(t^*)$, respectively.

Definition 2.1. *Let* $\boldsymbol{M}$ *and* $\boldsymbol{M}^*$ *be (m+1)-dimensional two spacelike ruled surfaces and* p_i *be the distances between the (k-1)-dimensional osculator planes obtained by the vanishing the* i^{th} *term from*
$Sp\{V_1, V_2, ..., V_i, ..., V_k\}$ *and* $Sp\{V_1^*, V_2^*, ..., V_i^*, ..., V_k^*\}$.
If
1) V_1 *and* V_1^* *are parallel,*
2) *the distances* p_i,$1 \leq i \leq k$, *between the (k-1)-dimensional osculator planes at the corresponding points of* αr *and* α^* *are constant, then the pair of ruled surfaces* $\boldsymbol{M}$ *and* $\boldsymbol{M}^*$ *are called* the (m+1)-dimensional spacelike parallel p_i-equidistant ruled surfaces.

From now on **M** and **M*** will be assumed (m+1)-dimensional spacelike parallel p_i-equidistant ruled surfaces.

The following theorem can be given by means of definition 2.1 without proof:

Theorem 2.1. ***i)*** *The Frenet frames* $\{V_1, V_2, \ldots, V_k\}$ *and* $\{V_1^*, V_2^*, \ldots, V_k^*\}$ *are equivalent at the corresponding points on* α *and* α^*.
ii) *For the curvatures* k_i *and* k_i^* *of* α *and* α^*, *respectively, we have*

$$k_i^* = \frac{dt}{dt^*} k_i, \quad 1 \leq i < k.$$

Theorem 2.2. *The relation between the base curves of* $\boldsymbol{M}$ *and* $\boldsymbol{M}^*$, *is*

$$\alpha^* = \alpha + p_1 V_1 + p_2 V_2 + \cdots + p_m V_m + \varepsilon_{m+1} p_{m+1} V_{m+1} + \varepsilon_{m+2} p_{m+2} V_{m+2} + ... + \varepsilon_k p_k V_k.$$

Proof. Since the vector $\alpha\alpha^*$ can be written as:

$$\alpha\alpha^* = a_1 V_1 + a_2 V_2 + \cdots + a_m V_m + a_{m+1} V_{m+1} + ... + a_k V_k, \quad a_i \in IR, \quad 1 \leq i \leq k,$$

we find

$$\begin{cases} \langle \alpha\alpha^*, V_i \rangle = a_i \,,\ 1 \leq i \leq m \\ \langle \alpha\alpha^*, V_i \rangle = a_i \varepsilon_i \,,\ \varepsilon_i = \langle V_i, V_i \rangle \,,\ m+1 \leq i \leq k \end{cases}.$$

Also, the distance between the osculator planes is

$$p_i = \begin{cases} |a_i| \,,\ 1 \leq i \leq m \\ |a_i \varepsilon_i| \,,\ m+1 \leq i \leq k \end{cases}$$

and thus

$$\alpha^* = \alpha + p_1 V_1 + p_2 V_2 + \cdots + p_m V_m + \varepsilon_{m+1} p_{m+1} V_{m+1} + \varepsilon_{m+2} p_{m+2} V_{m+2} + ... + \varepsilon_k p_k V_k$$

□

Theorem 2.3. *All the asymptotic and tangential bundles of* $\boldsymbol{M}$ *and* $\boldsymbol{M}^*$ *are equal.*

Proof. Let A(t) and $A(t^*)$ be asymptotic bundles of $\mathbf{M}$ and $\mathbf{M}^*$, respectively, then we have

$$A(t) = Sp\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m'\}$$

and

$$A(t^*) = Sp\left\{V_1^*, V_2^*, \ldots, V_m^*, {V_1^*}', {V_2^*}', \ldots, {V_m^*}'\right\}.$$

Similarly, if T(t) and $T(t^*)$ are the tangential bundles of $\mathbf{M}$ and $\mathbf{M}^*$, respectively, then from the definition of the tangential bundles we also have

$$T(t) = Sp\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m', \alpha'\}$$

and

$$T(t^*) = Sp\left\{V_1^*, V_2^*, \ldots, V_m^*, {V_1^*}', {V_2^*}', \ldots, {V_m^*}', {\alpha^*}'\right\}.$$

From the definition 2.1 and Theorem 2.1

$$A(t) = A(t^*) = T(t) = T(t^*)$$

is obtained. □

II

In this part, we will study the matrices A_{ξ_j} and $A_{\xi_j^*}$, $1 \leq j \leq n - m - 1$, of $\mathbf{M}$ and $\mathbf{M}^*$, respectively. Using equation (2.1) and (2.2), we can write

$$X_t = V_1 + \sum_{i=1}^{m} u_i V_i', \quad X_{u_1} = V_1, \ldots, \quad X_{u_m} = V_m$$

and

$$X_{t^*}^* = V_1^* + \sum_{i=1}^{m} u_i^* {V_i^*}', \quad X_{u_1^*}^* = V_1^*, \ldots, \quad X_{u_m^*}^* = V_m^*.$$

Thus, we obtain the orthonormal bases $\{V_1, \ldots, V_{m+1}\}$ and $\left\{V_1^*, \ldots, V_{m+1}^*\right\}$ of $\mathbf{M}$ and $\mathbf{M}^*$, respectively. If we take the orthonormal bases of the normal bundles $\mathbf{M}^\perp$ and $\mathbf{M}^{*\perp}$ as

$$\{\xi_1, \ldots, \xi_{k-m-1}, \ldots, \xi_{n-m-1}\} \, and \left\{\xi_1^*, \ldots, \xi_{k-m-1}^*, \ldots, \xi_{n-m-1}^*\right\},$$

respectively, then we get the orthonormal bases

$$\{V_1, \ldots, V_{m+1}, \xi_1, \ldots, \xi_{k-m-1}, \ldots, \xi_{n-m-1}\}$$

and

$$\left\{V_1^*, \ldots, V_{m+1}^*, \xi_1^*, \ldots, \xi_{k-m-1}^*, \ldots, \xi_{n-m-1}^*\right\}$$

of R_1^n at $P \in \mathbf{M}$ and at $P^* \in \mathbf{M}^*$, respectively, where $\xi_i = V_{m+1+i}$ and $\xi_i^* = V_{m+1+i}^*$, $1 \leq i \leq k-m-1$. Let the connections of R_1^n, $\mathbf{M}$ and $\mathbf{M}^*$ be $\bar{D}$, D and D^*, respectively. Then we have the following Weingarten equations:

$$(2.3)\begin{cases} \bar{D}_{V_1}\xi_j = \sum\limits_{i=1}^{m+1} a_{1i}^j V_i + \sum\limits_{q=1}^{n-m-1} b_{1q}^j \xi_q, & 1 \leq j \leq n-m-1 \\ \vdots & \\ \bar{D}_{V_{m+1}}\xi_j = \sum\limits_{i=1}^{m+1} a_{(m+1)i}^j V_i + \sum\limits_{q=1}^{n-m-1} b_{(m+1)q}^j \xi_q, & 1 \leq j \leq n-m-1\,. \end{cases}$$

So, the matrix A_{ξ_j}, $1 \leq j \leq n-m-1$, can be written as:

$$(2.4)\ A_{\xi_j} = -\begin{bmatrix} a_{11}^j & a_{12}^j & \cdots & a_{1(m+1)}^j \\ \vdots & \vdots & & \vdots \\ a_{(m+1)1}^j & a_{(m+1)2}^j & \cdots & a_{(m+1)(m+1)}^j \end{bmatrix}.$$

Since α is a spacelike curve and $E_m(t)$ is a spacelike subspace, we obtain

$$(2.5)\begin{cases} a_{11}^j = <\bar{D}_{V_1}\xi_j, V_1> & \cdots & a_{(m+1)1}^j = <\bar{D}_{V_{m+1}}\xi_j, V_1> \\ \vdots & & \vdots \\ a_{1m}^j = <\bar{D}_{V_1}\xi_j, V_m> & \cdots & a_{(m+1)m}^j = <\bar{D}_{V_{m+1}}\xi_j, V_m> \\ a_{1(m+1)}^j = \varepsilon_{m+1} <\bar{D}_{V_1}\xi_j, V_{m+1}> & \cdots & a_{(m+1)(m+1)}^j = \varepsilon_{m+1} <\bar{D}_{V_{m+1}}\xi_j, V_{m+1}> \end{cases}$$

where $\varepsilon_{m+1} = \langle V_{m+1}, V_{m+1} \rangle$.

Similarly, for any normal vector field ξ^* on $\mathbf{M}^*$, we can write

$$\bar{D}_{X^*}\xi^* = -A_{\xi^*}(X^*) + D_{X^*}^{*\perp}\xi^*.$$

Then we obtain:

$$(2.6)\begin{cases} \bar{D}_{V_1^*}\xi_j^* = \sum\limits_{i=1}^{m+1} c_{1i}^j V_i^* + \sum\limits_{q=1}^{n-m-1} d_{1q}^j \xi_q^*, & 1 \leq j \leq n-m-1 \\ \vdots & \\ \bar{D}_{V_{m+1}^*}\xi_j^* = \sum\limits_{i=1}^{m+1} c_{(m+1)i}^j V_i^* + \sum\limits_{q=1}^{n-m-1} d_{(m+1)q}^j \xi_q^*, & 1 \leq j \leq n-m-1\,. \end{cases}$$

Thus, we obtain the matrix $A_{\xi_j^*}$,$1 \leq j \leq n-m-1$, as follows:

$$(2.7)\ A_{\xi_j^*} = -\begin{bmatrix} c_{11}^j & c_{12}^j & \cdots & c_{1(m+1)}^j \\ \vdots & \vdots & & \vdots \\ c_{(m+1)1}^j & c_{(m+1)2}^j & \cdots & c_{(m+1)(m+1)}^j \end{bmatrix},\ 1 \leq j \leq n-m-1.$$

Since α^* is a spacelike curve and $E_m(t^*)$ is a spacelike subspace, we get

$$(2.8)\begin{cases} c_{11}^j = <\bar{D}_{V_1^*}\xi_j^*, V_1^*> & \cdots & c_{(m+1)1}^j = <\bar{D}_{V_{m+1}^*}\xi_j^*, V_1^*> \\ \vdots & & \vdots \\ c_{1m}^j = <\bar{D}_{V_1^*}\xi_j^*, V_m^*> & \cdots & c_{(m+1)m}^j = <\bar{D}_{V_{m+1}^*}\xi_j^*, V_m^*> \\ c_{1(m+1)}^j = \varepsilon_{m+1} <\bar{D}_{V_1^*}\xi_j^*, V_{m+1}^*> & \cdots & c_{(m+1)(m+1)}^j = \varepsilon_{m+1} <\bar{D}_{V_{m+1}^*}\xi_j^*, V_{m+1}^*> \end{cases}$$

where $\varepsilon_{m+1} = \langle V_{m+1}^*, V_{m+1}^* \rangle$. Hence, the following theorems can be given:

Theorem 2.4. *If $\mathbf{M}$ is (m+1)-dimensional spacelike ruled surface in R_1^n, then*

$$A_{\xi_1} = A_{V_{m+2}} = \begin{bmatrix} 0 & \cdots & 0 & \varepsilon_{m+1}k_{m+1} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)} \quad \text{and } A_{\xi_j} = 0,$$

$2 \leq j \leq n-m-1$.

Theorem 2.5. *If $\boldsymbol{M}^*$ is (m+1)-dimensional spacelike ruled surface in R_1^n, then*

$$A_{\xi_1^*} = A_{V_{m+2}^*} = \begin{bmatrix} 0 & \cdots & 0 & \varepsilon_{m+1}k_{m+1}^* \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)} \quad \text{and } A_{\xi_j^*} = 0,$$

$2 \leq j \leq n-m-1$.

Theorem 2.6. *Let $\boldsymbol{M}$ and $\boldsymbol{M}^*$ be (m+1)-dimensional spacelike parallel p_i-equidistant ruled surfaces in R_1^n. For the matrices of $\boldsymbol{M}$ and $\boldsymbol{M}^*$, we have*

$$A_{\xi_1^*} = \frac{dt}{dt^*} A_{\xi_1} \quad , \quad A_{\xi_j^*} = A_{\xi_j} = 0 \quad , \quad 2 \leq j \leq n-m-1.$$

Theorem 2.7. *The Lipschitz-Killing curvatures of $\boldsymbol{M}$ and $\mathbf{M}^*$ in all normal directions are zero.*

Proof. From the definition of Lipschitz-Killing curvature in the direction of ξ_j , we can write

$G(P,\xi_j) = \det\ A_{\xi_j} = 0$ for all $P \in \mathbf{M}$, $1 \leq j \leq n-m-1$.

Similarly, the Lipschitz-Killing curvature in the direction of ξ_j^* of $\mathbf{M}^*$, we get

$G(P^*,\xi_j^*) = \det\ A_{\xi_j^*} = 0$, $1 \leq j \leq n-m-1$, *for all* $P^* \in \boldsymbol{S}^*$. □

Theorem 2.8. *$\boldsymbol{M}$ and $\boldsymbol{M}^*$ are minimal and the scalar normal curvatures of $\boldsymbol{M}$ and $\boldsymbol{M}^*$ are zero.*

Proof. If H and K_N (H* and K_{N^*}) are the mean curvature vector and the scalar normal curvature of $\mathbf{M}$($\mathbf{M}^*$), then from Theorem 2.4 and Theorem 2.5, we have

$H = H^* = 0$ and $K_N = K_{N^*} = 0$.

Thus, $\mathbf{M}$ and $\mathbf{M}^*$ are the minimal ruled surfaces. □

III

If X and Y are vector fields and V is the second fundamental form of $\mathbf{M}$, then from (1.2) and (1.3) we can write

$< \bar{D}_X Y\, ,\, \xi >=< V(X,Y)\, ,\, \xi >=< A_\xi(X)\, ,\; Y >\quad ,\quad \xi \in \mathbf{M}^\perp$ and

$$V(X,Y) = -\sum_{j=1}^{n-m-1} < Y\, ,\, \bar{D}_X \xi_j > \xi_j .$$

So, for the Frenet vectors V_i and V_j, $1 \le i, j \le m+1$, we obtain

$$V(V_i\, ,\, V_j\,) = -\sum_{s=1}^{n-m-1} < V_j\, ,\, \bar{D}_{V_i} \xi_s > \xi_s, \quad 1 \le i, j \le m+1.$$

Thus, from (2.3) we get

$$V(V_i\, ,\, V_j) = -\sum_{s=1}^{n-m-1} \varepsilon_j a_{ij}^s \xi_s .$$

Using Theorem 2.4, we have

$$(2.9)\quad \begin{cases} V(V_1, V_{m+1}) = -\sum\limits_{s=1}^{n-m-1} \varepsilon_{m+1} a_{1(m+1)}^s \xi_s = \varepsilon_{m+1}^2 k_{m+1} V_{m+2} \\ V(V_i, V_j) = -\sum\limits_{s=1}^{n-m-1} \varepsilon_j a_{ij}^s \xi_s = 0 \quad , \quad 1 \le i, j \le m+1\ . \end{cases}$$

Similarly, if X^* and Y^* are vector fields and V^* is the second fundamental form of $\mathbf{M}^*$, then from equations (1.2) and (1.3) we have

$$< \bar{D}_{X^*} Y^*, \xi^* >=< V^*(X^*, Y^*), \xi^* >=< A_{\xi^*}(X^*), Y^* > \quad , \quad \xi^* \in {M^*}^\perp$$

and

$$V^*(X^*, Y^*) = -\sum_{j=1}^{n-m-1} < Y^*, \bar{D}_{X^*} \xi_j^* > \xi_j^* .$$

For the Frenet vectors V_i^* and V_j^* , $1 \le i, j \le m+1$, we have

$$V^*(V_i^*, V_j^*) = -\sum_{s=1}^{n-m-1} < V_j^*, \bar{D}_{V_i^*} \xi_s^* > \xi_s^* \quad , \quad 1 \le i, j \le m+1$$

and from equation (2.6) we get

$$V^*(V_i^*, V_j^*) = -\sum_{s=1}^{n-m-1} \varepsilon_j c_{ij}^s \xi_s^* \quad , \quad 1 \le i, j \le m+1.$$

Using Theorem 2.5, we obtain

$$(2.10)\quad \begin{cases} V^*(V_1^*, V_{m+1}^*) = \varepsilon_{m+1}^2 k_{m+1}^* V_{m+2}^* \ , \\ V^*(V_i^*, V_j^*) = 0 \ 1 \le i, j \le m+1 \end{cases}$$

and from Theorem 2.1 we get

$$(2.11)\quad \begin{cases} V^*(V_1^*, V_{m+1}^*) = \frac{dt}{dt^*} V(V_1, V_{m+1}) \ , \\ V^*(V_i^*, V_j^*) = V(V_i, V_j) = 0 \ \ 1 \le i, j \le m+1. \end{cases}$$

Thus, the following theorems can be given:

Theorem 2.9. *V_1 and V_{m+1} are conjugate vectors iff V_1^* and V_{m+1}^* are conjugate vectors.*

Theorem 2.10. ***i)*** *For the Riemannian curvature of $\boldsymbol{M}$ in two dimensional direction spanned by V_i and V_j, we have*

$$K(V_1, V_{m+1}) = \varepsilon_{m+1}\varepsilon_{m+2}(k_{m+1})^2 \text{ and } K(V_i, V_j) = 0, \quad 1 \leq i, j \leq m+1,\ i \neq j.$$

ii) *For the Riemannian curvature of $\boldsymbol{M}^*$ in two dimensional direction spanned by V_i^* and V_j^*, we have*

$$K(V_1^*, V_{m+1}^*) = \varepsilon_{m+1}\varepsilon_{m+2}(k_{m+1})^2 \text{ and } K(V_i^*, V_j^*) = 0, \quad 1 \leq i, j \leq m+1, i \neq j.$$

Theorem 2.11. *For the Riemannian curvatures of $\boldsymbol{M}$ and $\boldsymbol{M}^*$*

$$\begin{cases} K(V_1^*, V_{m+1}^*) = \left(\frac{dt}{dt^*}\right)^2 K(V_1, V_{m+1}) \ , \\ K(V_i^*, V_j^*) = K(V_i, V_j) = 0 \quad , \quad 1 \leq i, j \leq m+1\,,\ i \neq j \end{cases}$$

are valid.

Theorem 2.12. *If $S(V_i, V_i)$, and r_{sk} ($S(V_i^*, V_i^*)$ and r_{sk}^*), $1 \leq i \leq m+1$, are the Ricci and scalar curvatures of $\mathbf{M}(\mathbf{M}^*)$, then we have*

$$S(V_i^*, V_i^*) = S(V_i, V_i) = 0\,,\ 1 \leq i \leq m,$$

$$S(V_{m+1}^*, V_{m+1}^*) = \left(\frac{dt}{dt^*}\right)^2 S(V_{m+1}, V_{m+1}),$$

$$r_{sk} = 2\varepsilon_{m+2} S(V_{m+1}, V_{m+1})\ ,$$

$$r_{sk}^* = 2\varepsilon_{m+2} S(V_{m+1}^*, V_{m+1}^*),$$

$$r_{sk}^* = \left(\frac{dt}{dt^*}\right)^2 r_{sk} \quad .$$

Proof. For the Ricci curvature in the direction V_i , $1 \leq i \leq m+1$, of $\mathbf{M}$, we can write

$$\begin{aligned} S(V_i, V_i) &= \sum_{j=1}^{m+1} \varepsilon_j \left\langle R(V_j, V_i)V_i, V_j \right\rangle,\ \varepsilon_j = \langle V_j, V_j \rangle \\ &= \sum_{j=1}^{m+1} \varepsilon_j \left\{ < V(V_j, V_i)\,, V(V_j, V_i) > - < V(V_j, V_j)\,, V(V_i, V_i) > \right\}. \end{aligned}$$

Using equation (2.9), we have

(2.12) $\begin{cases} S(V_{m+1}, V_{m+1}) = \varepsilon_{m+2}(k_{m+1})^2, \ \varepsilon_{m+2} = \langle V_{m+2}, V_{m+2} \rangle \\ S(V_i, V_i) = 0 \ , \ 1 \le i \le m \ . \end{cases}$

For the scalar curvature of $\mathbf{M}$, we get

$$r_{sk} = \sum_{i \neq j} K(V_i, V_j) = 2 \sum_{i \langle j} K(V_i, V_j).$$

From Theorem 2.10, we obtain

(2.13) $r_{sk} = 2K(V_1, V_{m+1}) = 2\varepsilon_{m+1}\varepsilon_{m+2}(k_{m+1})^2$.

If we use equation (2.12) we have

(2.14) $r_{sk} = 2\varepsilon_{m+2} S(V_{m+1}, V_{m+1}) \ , \varepsilon_{m+2} = \langle V_{m+2}, V_{m+2} \rangle$.

Similarly, for the Ricci curvature in the direction V_i^* , $1 \le i \le m+1$, of $\mathbf{M}^*$ we get

(2.15) $\begin{cases} S(V_{m+1}^*, V_{m+1}^*) = \varepsilon_{m+1}(k_{m+1}^*)^2 \ , \ \varepsilon_{m+1} = \left\langle V_{m+1}^*, V_{m+1}^* \right\rangle \\ S(V_i^*, V_i^*) = 0 \ , \ 1 \le i \le m \ . \end{cases}$

Also, for the scalar curvature of $\mathbf{M}^*$, we find

(2.16) $r_{sk}^* = 2K(V_1^*, V_{m+1}^*) = 2\varepsilon_{m+2} S(V_{m+1}^*, V_{m+1}^*) \ , \ \varepsilon_{m+2} = \left\langle V_{m+2}^*, V_{m+2}^* \right\rangle$.

From Theorem 2.1 we have

$S(V_{m+1}^*, V_{m+1}^*) = \left(\frac{dt}{dt^*}\right)^2 S(V_{m+1}, V_{m+1})$and $r_{sk}^* = \left(\frac{dt}{dt^*}\right)^2 r_{sk}$. □

Theorem 2.13. *Let* $X = \sum\limits_{i=1}^{m+1} a_i V_i$, $Y = \sum\limits_{i=1}^{m+1} b_i V_i \in \boldsymbol{M}$. $\boldsymbol{M}$ *is totally geodesic iff* $V(V_1, V_{m+1}) = 0$ *or* $a_1 b_{m+1} = 0$.

Proof. Since

$$V(X,Y) = \sum_{i,j=1}^{m+1} a_i b_j V(V_i \ , V_j \ ),$$

using equation (2.9), we get

(2.17) $V(X,Y) = a_1 b_{m+1} V(V_1, V_{m+1})$.

Thus, the definition of totally geodesic completes the proof. □

We can give the following corollary:

Corollary 2.1. *If* $a_1 b_{m+1} \neq 0$ *and* $\boldsymbol{M}$ *is totally geodesic, then* $\boldsymbol{M}^*$ *is totally geodesic and the Riemannian curvatures of* $\boldsymbol{M}$ *and* $\boldsymbol{M}^*$ *in the two dimensional direction spanned by* V_i *and* V_j , $1 \le i, j \le m+1$, $i \neq j$, *are zero.*

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