THE NUMBER OF 1-FACTORS IN SOME POLYHEXES

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Abstract. We define a new class of benzenoids (hexagonal systems), in continuation of the previous works [1,2]. Some combinatorial K formulas are derived, where K designates the Koschá structure counts, which generalizes some previously known K formulas. We point out the connections with some combinatorial problems.

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1. General formula

All definitions used in this paper are taken from [3].

Let H be any hexagonal system. If we replace each vertical edge of H (together with both incident vertices) with a vertex, we obtain a square system $S_H$. We say that $S_H$ is associated square system for $H$ (Fig. 1).

Fig. 1

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We can choose any class of parallel edges to be vertical. So, in fact, to each hexagonal system we can associate three square systems, which are non-isomorphic in general case.

This operation preserves peaks and valleys. Moreover, it is easy to see that the number of monotonic path systems is the same for a hexagonal system and its associated square system (Fig. 2).

![Diagram](image)

**Fig. 2**

So, we can determine the number $K$ of a hexagonal system $H$ by determining the number of monotonic path systems of its associated square system $S_H$.

The advantage of this approach is that we can use some known combinatorial results concerning square systems, i.e. concerning path systems in the square grid.

Denote by $S(n, n, 3)$ the hexagonal system consisting of $n$ layers with the corresponding number of hexagons $s_i, s_i + \mu, s_i + 2\mu, \ldots, s_i + (n - 1)\mu$. In fact, such a system can be obtained from a parallelogram $L(n, n(n - 1) + 1)$ by removing $(n - 1)\mu$ rightmost hexagons from the $i$-th layer, $i = 1, 2, \ldots, n - 1$. An $S(5, 5, 3)$ is given in Fig. 3.

The associated square system of $S(5, 5, 3)$ is given in Fig. 4.

In [2] is given a result which can be formulated in the following way:

- The number of monotonic paths from point $A$ to the point $B$ in the square system given in Fig. 4 is

$$\frac{s + 1}{(\mu + 1)n + s + 1} \left(\frac{\mu + 1)n + s + 1}{n}\right)$$

- So, we have the following theorem.
The number of 1-factors in some polyhexes

Theorem 1.

\[ K[S(n, z, \mu)] = \frac{z + 1}{(\mu + 1)n + z + 1} \binom{(\mu + 1)n + z + 1}{n} \]

2. Some special cases

If \( z = \mu \), we denote \( S(n, \mu, \mu) \) by \( R(n, \mu) \). \( R(n, \mu) \) may also be considered as \( S(n + 1, a, \mu) \). In Fig. 5 an \( R(4, 3) \) is given.
From Theorem 1 it follows:

**Corollary 2.**

\[ K(T(n, n + s - 1, n - 1)) = \frac{s + 1}{2n + s + 1} \left( \frac{2n + s + 1}{n} \right). \]
Putting \( m = n + s - 1 \), we have

\[
K(T(n, m, n - 1)) = \frac{m - n + 2}{m + n + 2} \binom{m + n + 2}{n}.
\]

In [3], it is proved that

\[
K(T(n, m, n - 1)) = \binom{m + n}{n} - \binom{m + n}{n - 2}.
\]

Hence follows the identity

\[
\binom{m + n}{n} - \binom{m + n}{n - 2} = \frac{m - n + 2}{m + n + 2} \binom{m + n + 2}{n},
\]

which can also be proved in a different way.

In order to specialize further Theorem 1, we take at the same time \( s = \mu = 1 \). We obtain \( R(n, 1) \) which may be also considered as \( S(n + 1, 0, 1) \). In [1] and [3], this hexagonal system is denoted by \( T(n) \). A \( T^*(5) \) is given in Fig. 7.

**Fig. 7**

From Corollary 2, we obtain:

**Corollary 3.**

\[
K(T^*(n)) = \frac{1}{2n + 3} \binom{2n + 3}{n + 1},
\]

i.e.,

\[
K(T^*(n)) = C_n + 1,
\]

where \( C_i \) is the \( i \)-th Catalan number. The last formula is also derived in [3].
References


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