A CLASSIFICATION OF 3-TYPE CURVES IN MINKOWSKI 3-SPACE \( E^3_1 \), I

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Abstract

In this paper we give a complete classification of all planar curves in Minkowski 3-space \( E^3_1 \), which are of type 3. A corresponding classification of all non-planar curves of type 3 in the same space will be given in the part II of this paper.

1. Introduction

The notion of submanifolds of finite type was introduced by B. Y. Chen in [2]. A submanifold \( M \) in the Euclidean space \( E^n_0 \) is said to be of finite type if each component of its position vector field \( x \) can be written as a finite sum of eigenfunctions of the Laplacian \( \Delta \) of \( M \). This means that

\[
x = x_0 + \sum_{i=1}^{k} \Delta x_i = \lambda_i x_i, \quad (1.1)
\]

where \( \lambda_0 < \lambda_1 < \ldots < \lambda_k \) are mutually different eigenvalues of \( \Delta \). When \( M \) is compact, the component \( x_0 \) in (1.1) is constant vector. However, when \( M \) is non-compact, the component \( x_0 \) is not necessary a constant vector.

In particular, a submanifold \( M \) is said to be of \( k \)-type if all eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are different from zero. If one of the \( \lambda_i \)'s is equal to zero \( (i = 1, 2, \ldots, k) \), \( M \) is said to be of null \( k \)-type.

Finite type curves in Euclidean space \( E^n_0 \) were studied intensively in [2], [3] and [4]. The classification of all 2-type curves in \( E^n_0 \) is given in [6].
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Abstract

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1. Introduction

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$$x = x_0 + \sum_{i=1}^k \Delta x_i = \lambda_i x_i,$$

(1.1)

where $\theta = \lambda_0 < \lambda_1 < \ldots < \lambda_k$ are mutually different eigenvalues of $\Delta$. When $M$ is compact, the component $x_0$ in (1.1) is constant vector. However, when $M$ is non-compact, the component $x_0$ is not necessary a constant vector. In particular, a submanifold $M$ is said to be of $k$-type if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different from zero. If one of the $\lambda_i$’s is equal to zero ($i = 1, 2, \ldots, k$), $M$ is said to be of type $k$-type.

Finite type curves in Euclidean space $E^n$ were studied intensively in [2], [3] and [4]. The classification of all 2-type curves in $E^n$ is given in [6].
2. Preliminaries

Minkowski 3 space $E^3_1$ is a manifold $R^3$ equipped with a metric tensor $g$ of index 1. Define the metric tensor $g$ with

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

(2.1)

Let $\alpha$ be a curve in $E^3_1$ parameterized by a pseudo-arclength parameter $s$. Then the Laplace operator $\Delta$ of $\alpha$ is given by

$$\Delta = \pm \frac{d^2}{ds^2}.$$  

(2.2)

Its eigenfunctions are $s, \cos(as), \sin(as), \cosh(as)$ and $\sinh(as)$. Following the definition of Chen, every finite type curve $\alpha$ in $E^3_1$ can be written as

$$\alpha(s) = a_0 + b_0 s + \sum_{i=1}^{k_1} (a_i \cos(p_i s) + b_i \sin(p_i s)) + \sum_{i=1}^{k_2} (c_i \cosh(q_i s) + d_i \sinh(q_i s)),$$

(2.3)

where $a_0, b_0, a_i, b_i, c_i, d_i \in R$ are constants, $i = 1, \ldots, k_1$, $j = 1, \ldots, k_2$ and $0 < p_1 < \ldots < p_{k_1}$, $0 < q_1 < \ldots < q_{k_2}$ are mutually different eigenvalues of $\Delta$. In particular, a curve $\alpha$ in $E^3_1$ is said to be of $k$-type if there are $k$ mutually different eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\Delta$ and they are all different from zero. If one of the $\lambda_i$'s ($i = 1, \ldots, k$), is equal to zero $\alpha$ is said to be of null $k$-type.

Recall that an arbitrary vector $v$ in $E^3_1$ can have one of three causal characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$, and null if $g(v, v) = 0$ and $v \neq 0$. The norm of a vector $v$ is given by

$$||v|| = \sqrt{|g(v, v)|}.$$  

(2.4)

The unit vectors, orthogonality and orthonormality are defined as in the Euclidean spaces. An arbitrary, unit-speed curve $\alpha(s)$ can locally be spacelike, timelike or null curve if respectively all of its velocity vectors $\dot{\alpha}(s)$ are spacelike, timelike or null vectors. An arbitrary plane $\pi$ in $E^3_1$ can be spacelike plane, if $g_{\pi}$ is positive definite, timelike plane, if $g_{\pi}$ is nondegenerate of index 1, or isotropic (lightlike) plane, if $g_{\pi}$ is degenerate.

Curves of finite type in Minkowski space-time have been investigated in [5] and also in [7], independently. The following classification theorem is obtained in [7].
where \( p^2 - 12ae = 0 \), \( \rho, e, a \in \mathbb{R}_0 \).

(vii) \( \alpha(s) = (e \sinh s + a \sinh 3s, e \cosh s + a \cosh 3s, \rho \sinh s) \),

where \( p^2 - 12ae = 0 \), \( \rho, e, a \in \mathbb{R}_0 \).

All closed 3-type curves in Euclidean 3-space \( \mathbb{E}^3 \) were classified by D. E. Blair in [1]. He obtained the following classification theorem.

**Theorem 2.5.** A closed 3-type curve in \( \mathbb{E}^3 \) is either a curve which lies on a quadric of revolution or a curve whose frequency ratio is \( 1 : 3 : 7 \) and the curve belongs to a 3-parameter family of such curves or the frequency ratio is \( 1 : 3 : 5 \) or the curve belongs to a 5-parameter family of such curves. Some curves with frequency ratio \( 1 : 3 : 5 \) or \( 1 : 3 : 7 \) also lie on quadrics of revolution.

3. A classification of planar 3-type curves in \( \mathbb{E}^3_1 \)

In this part we will classify all planar 3-type curves in Minkowski 3-space \( \mathbb{E}^3_1 \). Main results are contained in Theorems 3.1 and 3.2.

**Theorem 3.1.** A planar 3-type curve, lying in an isotropic plane of \( \mathbb{E}^3_1 \), is a null 3-type spacelike curve.

**Proof.** Let \( \alpha \) be a 3-type curve in \( \mathbb{E}^3_1 \), parametrized by an pseudo-arclength parameter \( s \). Then \( \alpha \) can have one of the following forms:

\[
\begin{align*}
\alpha(s) &= a + b s + c \cos(p s) + d \sin(p s) + e \cosh(t s) + f \sinh(t s), \\
\alpha(s) &= a + b s + c \cos(p s) + d \sin(p s) + e \cos(t s) + f \sin(t s), \\
\alpha(s) &= a + b s + c \cosh(p s) + d \sinh(p s) + e \cosh(t s) + f \sinh(t s), \\
\alpha(s) &= a + b \cos(p s) + c \sin(p s) + d \cos(t s) + e \sin(t s) + \\
&\quad f \cosh(q s) + h \sinh(q s), \\
\alpha(s) &= a + b \cos(p s) + c \sin(p s) + d \cos(t s) + e \sin(t s) + \\
&\quad f \cosh(q s) + h \sinh(q s), \\
\alpha(s) &= a + b \cosh(p s) + c \sinh(p s) + d \cosh(t s) + e \sinh(t s) + \\
&\quad f \cos(q s) + h \sin(q s), \\
\alpha(s) &= a + b \cosh(p s) + c \sinh(p s) + d \cosh(t s) + e \sinh(t s) + \\
&\quad f \cosh(q s) + h \sinh(q s), \\
\alpha(s) &= a + b \cosh(p s) + c \sinh(p s) + d \cosh(t s) + e \sinh(t s) + \\
&\quad f \cosh(q s) + h \sinh(q s), \\
\end{align*}
\]


Theorem 2.1. Every curve of finite type in Minkowski plane $E^2_1$ is of 1-type and hence an open part of an orthogonal hyperbola or an open part of a straight line.

Curves of Chen-type 2 in $E^2_1$ are investigated in [3]. The following result is obtained there.

Theorem 2.2. A plane 2-type curve, lying in an isotropic plane of $E^2_1$, is a null 2-type spacelike curve.

Theorem 2.3. Up to a rigid motions of $E^2_1$, a non-planar curve $\alpha$ in $E^2_1$ is a null 2-type curve if and only if $\alpha$ is a part of one of the following curves:

(i) $\alpha(s) = (as, b\cos s, b\sin s)$, $a, b \in R_0, |a| \neq |b|$

(ii) $\alpha(s) = (a\cosh s, a\sinh s, b)$, $a, b \in R_0, |a| \neq |b|$

(iii) $\alpha(s) = (a\sinh s, a\cosh s, b)$, $a, b \in R_0, |a| \neq |b|$

Theorem 2.4. Up to a rigid motions of $E^2_1$, a non-planar curve $\alpha$ in $E^2_1$ is a 2-type curve with both eigenvalues different from zero if and only if $\alpha$ is a part of one of the following curves:

(i) $\alpha(s) = (\rho s\sin s, \rho \cos s + a\cos 3s, \rho \sin s + a\sin 3s)$, where $p^2 - 12\rho = 0, a, \rho \in R_0$

(ii) $\alpha(s) = (a\cosh s + 3\lambda \sinh s, -4e^{3\lambda s} - b\cosh s - 3\lambda \sinh s + 4ae^{3\lambda s}, 2de^{3\lambda s})$, where $d^2 - 6(a - b)c = 0, a, b, c, d \in R_0, \lambda \in [-1, 1]$

(iii) $\alpha(s) = (ae^s + b\cosh 3s, ae^s + b\sinh 3s, ce^{-s})$, where $c^2 + 6ab = 0, a, b, c \in R_0$

(iv) $\alpha(s) = (\rho s\sin s + a\cosh 3s, \rho \sin s + a\sinh 3s, \rho \cosh s)$, where $p^2 + 12\rho = 0, a, \rho \in R_0$

(v) $\alpha(s) = (\rho s + a\cosh 3s, \rho \sin s + a\sinh 3s, \rho \cosh s)$, where $p^2 + 12\rho = 0, a, \rho \in R_0$

(vi) $\alpha(s) = (ae^s + b\sinh 3s, ae^s + b\cosh 3s, ce^{-s})$, where $c^2 - 6ab = 0, a, b, c \in R_0$

(vii) $\alpha(s) = (\rho s + a\sinh 3s, \rho \cosh s + a\cosh 3s, \rho \sinh s)$,
where \( p^2 - 12w = 0 \), \( \rho, \varepsilon, a \in R_0 \);

(vii) \( \alpha(s) = (\varepsilon \sinh s + a \sinh 3s, \varepsilon \cosh s + a \cosh 3s, \rho \sinh s) \),

where \( p^2 - 12a = 0 \), \( \rho, a, c \in R_0 \).

All closed 3-type curves in Euclidean 3-space \( E^3 \) were classified by D. E. Blair in [1]. He obtained the following classification theorem.

**Theorem 2.5.** A closed 3 type curve in \( E^3 \) is either a curve which lies on a quadric of revolution or a curve whose frequency ratio is \( 1 : 3 : 7 \) and the curve belongs to a 3 parameter family of such curves or the frequency ratio is \( 1 : 3 : 5 \) or the curve belongs to a 5 parameter family of such curves. Some curves with frequency ratio \( 1 : 3 : 5 \) or \( 1 : 3 : 7 \) also lie on quadrics of revolution.

3. A classification of planar 3-type curves in \( E^3 \)

In this part we will classify all planar 3-type curves in Makowski 3-space \( E^3 \). Main results are contained in Theorems 3.1 and 3.2.

**Theorem 3.1.** A planar 3-type curve, lying in an isotropic plane of \( E^3 \), is a null 3 type spacelike curve.

**Proof.** Let \( \alpha \) be a 2-type curve in \( E^3 \), parametrized by a pseudo-arclength parameter \( s \). Then \( \alpha \) can have one of the following forms:

\[
\alpha(s) = a + b s + c \cos(ps) + d \sin(ps) + e \cosh(ts) + f \sinh(ts),
\]

(i) \( \alpha(s) = a + b s + c \cos(ps) + d \sin(ps) + e \cos(ts) + f \sin(ts), \)

(ii) \( \alpha(s) = a + b s + c \cos(ps) + d \sin(ps) + e \cosh(ts) + f \sinh(ts), \)

(iii) \( \alpha(s) = a + b \cos(ps) + c \sin(ps) + d \cos(ts) + e \sin(ts) + f \cosh(ts) + h \sinh(ts), \)

(iv) \( \alpha(s) = a + b \cos(ps) + c \sin(ps) + d \cos(ts) + e \sin(ts) + f \cosh(ts) + h \sinh(ts), \)

(v) \( \alpha(s) = a + b \cos(ps) + c \sin(ps) + d \cos(ts) + e \sin(ts) + f \cosh(ts) + h \sinh(ts), \)

(vi) \( \alpha(s) = a + b \cos(ps) + c \sinh(ps) + d \cosh(ts) + e \sinh(ts) + f \cosh(ts) + h \sinh(ts), \)

(vii) \( \alpha(s) = a + b \cosh(ps) + c \sinh(ps) + d \cosh(ts) + e \sinh(ts) + f \cosh(ts) + h \sinh(ts), \)

(viii) \( \alpha(s) = a + b \cosh(ps) + c \sinh(ps) + d \cosh(ts) + e \sinh(ts) + f \cosh(ts) + h \sinh(ts), \)

where \( a, b, c, d, e, f, h, p \) are constants.
where $a, b, c, d, e, f \in R^3$ and suppose $0 < p < t < q$.

Next, suppose that $\alpha$ lies in an isotropic (lightlike) plane in $E^4_1$, with the equation $x_4 = x_2^2$. Then the vectors $b, c, d, e, f$ are of the form $b = (b_1, b_2, b_3, b_4), c = (c_1, c_2, c_3), d = (d_1, d_2, d_3), e = (e_1, e_2, e_3), f = (f_1, f_2, f_3)$, i.e., they are space-like or null vectors. Besides, we can assume that $a = (0, 0, 0, 3)$, up to a translation and let $g = -dx_1^2 + dx_2^2 + dx_3^2$. In the sequel, we shall consider cases (i)-(vii) separately.

**Case (i).** $\alpha(s) = a + bs + c \cos(ps) + d \sin(ps) + e \cosh(ts) + f \sinh(ts)$.

Since $g(\alpha, \alpha) = \pm 1$, using the linear independence of the functions $\sin x, \cos x, \sinh x, \cosh x$ we get the system of equations:

\[ g(b, b) + \frac{c^2}{2} (g(c, e) + g(d, d)) + \frac{d^2}{2} (g(f, f) - g(e, e)) = \pm 1, \]  
\[ g(d, d) - g(c, c) = 0, \]  
\[ g(f, f) + g(e, e) = 0, \]  
\[ g(b, c) = g(b, d) = g(b, e) = g(b, f) = 0, \]  
\[ g(c, d) = g(c, e) = g(c, f) = 0, \]  
\[ g(d, e) = g(d, f) = 0, \]  
\[ g(e, f) = 0. \]  

From the equations (1)-(7) follows that $b = (b_1, b_2, \pm 1), c = (c_1, c_2, 0), d = (d_1, d_2, 0), e = (e_1, e_2, 0), f = (f_1, f_2, 0)$, so the curve $\alpha$ reads:

\[ \alpha(s) = (b_1 s + c_1 \cos(ps) + d_1 \sin(ps) + e_1 \cosh(ts) + f_1 \sinh(ts)), \]  
\[ b_2 s + c_2 \cos(ps) + d_1 \sin(ps) + e_2 \cosh(ts) + f_1 \sinh(ts), \]  

where $b_1, c_1, d_1, e_1, f_1 \in R$, $c_2$ and $d_2$ are not both zero, $e_1$ and $f_1$ are not both zero. Consequently, $\alpha$ is null 5-space lightlike curve.

**Cases (ii) and (iii).** The proof in these cases is analogous to the proof of case (i). So in these cases we get the curves:

\[ \alpha(s) = (b_2 s + c_1 \cos(ps) + d_1 \sin(ps) + e_1 \cosh(ts) + f_1 \sinh(ts)), \]  
\[ b_2 s + c_2 \cos(ps) + d_1 \sin(ps) + e_2 \cosh(ts) + f_1 \sinh(ts), \]  

where $b_2, c_1, d_1, e_1, f_1 \in R$, $c_2$ and $d_2$ are not both zero, $e_1$ and $f_1$ are not both zero. Consequently, $\alpha$ is null 5-space lightlike curve.
where $c_1, c_2, d_1, c_1, f_1 \in R$, $c_1$ and $d_1$ are not both 0, $c_1$ and $f_1$ are not both 0. Consequently, $a$ is a null 3-type spacelike curve.

Case (iv). If $0 < p < t < q$, we differ the next subcases:

$$2p = t - p < p + t < 2t < 2q$$

Therefore, we have the following subcases:

(iv.1) $2p = t - p$; (iv.2) $2p \neq t - p$;

Since $g(a, a) = \pm 1$, using the linear independence of the functions $\sin x, \cos x, \sinh x$ and $\cosh x$, in these subcases we obtain different systems of the equations:

(iv.1). In this case, the corresponding system reads:

$$\begin{align*}
\frac{\partial}{\partial t}(g(b, c) + g(c, e)) + \frac{\partial}{\partial t}(g(d, d) + g(e, e)) + \frac{\partial}{\partial t}(g(h, h) + g(f, f)) &= \pm 1, \\
\frac{\partial}{\partial t}(g(c, e) - g(b, b)) + pt(g(b, c) + g(c, e)) &= 0, \\
g(c, e) - g(d, d) &= 0, \\
g(b, h) + g(f, f) &= 0, \\
-\nu g(b, c) + pt(g(b, c) - g(c, d)) &= 0, \\
g(c, e) - g(d, d) &= 0, \\
g(c, c) + g(b, c) &= 0, \\
g(d, e) &= 0, \\
g(f, h) &= 0, \\
g(h, f) &= g(h, b) = g(e, f) = g(c, e) = g(d, f) = g(d, h) = 0
\end{align*}$$

Now equations (3) and (9) imply $e = (c_1, c_2, 0)$, $d = (d_1, d_2, 0)$, so $g(b, d) = g(b, c) = g(c, e) = 0$. Also equations (4) and (9) imply $f = (f_1, f_2, 0)$, $h = (h_1, h_2, 0)$. (2) and (5) imply $b = (b_1, b_2, 0), c = (c_1, c_2, 0)$. All equations (2)–(9) are then satisfied, but equation (1) reads $0 = \pm 1$, thus we obtain a contradiction.

(iv.2). $2p \neq t - p$. Using a similar method as in the case (iv.1), we obtain a contradiction.

From the cases (iv.1) and (iv.2) we conclude that a curve $a$ of the form (iv) does not exist.
Case (v). After calculation $g(\hat{c}, \hat{a}) = \pm 1$, we get that arguments of the functions $\sin z$ and $\cos x$ are the numbers $2p$, $2t$, $p + q$, $p + q + t$, $q - t$, $q + t$, $q - p$, $q - t$, $t - p$, $t - q - p$. If $0 < p < t < q$, we find that $2q = \max\{2p, 2t, p + t, p + q, t + q, t + q - p, q - t, t - p\}$, so the coefficients of $\sin 2q$ and $\cos 2q$ are equal to 0. Therefore, $g(f, f) = g(h, h) = 0$, $g(f, h) = 0$, so $f = (f_1, f_0)$, $h = (h_1, h_0)$. Then $g(f, f) = g(h, h) = g(c, f) = g(c, h) = g(d, f) = g(d, h) = g(e, f) = g(e, h) = 0$. Some of the numbers $2p$, $2t$, $p + q$, $p + q + t$, $q - p$, $q - t$, $t - p$ can be mutually equal. Let us look at $2t$. There are 4 possibilities for $2t$:

(v.1) $2t = p + q$;  
(v.2) $2t = q - t$;  
(v.3) $2t = q - p$;  
(v.4) $2t = p + q, q - t, q - p$.

We shall discuss these cases separately.

(v.1) $2t = p + q$, $q - t = t - p$. In this case the coefficients of $\sin(2ts) = \sin(\hat{c} + \hat{a})$ and $\cos(2ts) = \cos(\hat{c} + \hat{a})$ must be 0, i.e. $g(d, e) = 0$, $g(c, e) = g(d, d) = 0$. Hence $c = (c_1, c_0, 0)$, $d = (d_1, d_0, 0)$. In order to find vectors $b$ and $c$, look at number 2p. There are the following possibilities:

(v.1.1) $2p = q - p$;  
(v.1.2) $2p = q - t$;  
(v.1.3) $2p = \neq q - p, q - t$.

We shall again distinguish between all these cases.

(v.1.1) $2p = q - p$. Since the coefficients of $\sin(2pa) = \sin(q - p)a$ and $\cos(2pa) = \cos(q - p)a$ must be 0, we have

$$- p^2 g(b, c) + p q (g(b, h) - g(c, f)) = 0,$$

$$\frac{p^2}{2} (g(c, c) - g(h, h)) + (g(b, f) + g(c, h)) = 0,$$

whence $g(c, c) - g(h, h) = 0$, $g(c, c) = 0$, and then $b = (b_1, b_0, 0) = c = (c_1, c_0, 0)$. Next the equation

$$\frac{p^2}{2} (g(b, h) + g(c, c)) + \frac{p^2}{2} (g(d, f) + g(e, c)) + \frac{q^2}{2} (g(f, f) + g(h, h)) = \pm 1 \quad \text{(*)}$$

becomes 0 = $\pm 1$, which is a contradiction.

(v.1.2) and (v.1.3) the equation (*) also implies a contradiction.

(v.2) $2t = q - t$. Then we get $e = (c_1, c_0, 0) = (d_1, d_0, 0)$, because the coefficients of $\cos 2ts = \cos(q - t)s$ and $\sin 2ts = \sin(q - t)s$ must be 0. In order to find vectors $b$ and $c$, we shall look at the number 2p. Then the following subcases occur:
(v.2.1) $2p = t - p$;  
(v.2.2) $2p \neq t - p$;
In both cases (v.2.1) and (v.2.2), the equation (\star) also implies a contradiction.

(v.3) $2t = q - p$. Differing the subcases $2p = t - p$ and $2p \neq t - p$, the equation (\star) implies a contradiction.

(v.4) $2t \neq p + q, q = p, q = t$. Differing the subcases $2p = q - p$, $2p = q - t$, $2p = t - p$, $2p \neq q - p$, $q = t, t = p$, the equation (\star) implies a contradiction again.
So we obtain that in the cases (v.1), (v.2), (v.3) and (v.4) a curve $\alpha$ of the form (\star) does not exist.

Cassy (vi) and (vii). By using the same methods as in the cases (iv) and (v) respectively, it is easily seen that the curve $\alpha$ of forms (vi) or (vii) does not exist. This completes the proof of Theorem 3.1.

**Theorem 3.2.** There are no planar 3-type curve lying in an spacelike or in an timelike plane in $E^3_1$.

**Proof.** Firstly, suppose that $\alpha$ is a unit-speed curve lying in an spacelike plane in $E^3_1$, with the equation $x_1 = 0$. As we have seen in the proof of Theorem 3.1, the curve $\alpha$ can have one of seven possible forms (i)-(vii), where the vectors $b = (b_1, b_2, b_3), c = (c_1, c_2, c_3), d = (d_1, d_2, d_3), r = (e_1, e_2, e_3)$ and $f = (f_1, f_2, f_3)$ are all spacelike vectors. We may take $\alpha = (0, 0, 0)$, up to a translation and let the metric tensor $g$ be of the form $g = -dx^2_1 + dx^2_2 + dx^2_3$.
In the sequel, we shall consider cases (i)-(vii) separately.

Case (i). Since $g(n, \alpha) = \pm 1$ and using the linear independence of the functions $\sin x, \cos x, \sinh x$ and $\cosh x$ we get the equation $g(e, c) + g(f, f) = 0$. Because $e$ and $f$ are spacelike vectors, it follows that $e = f = 0$. Consequently, we obtain a contradiction.

Cases (ii) and (iii). We differ the subcases $t = 2p$, $t = 3p$, $t \neq 2p, 3p$ and obtain a contradiction.

Case (iv) We differ the next subcases (iv.1) $t = 3p$ and (iv.2) $t \neq 3p$.
In both of these subcases, we have the equation $g(f, f) + g(h, h) = 0$, so we obtain a contradiction.

Case (v) Since $g(n, \alpha) = \pm 1$ and using the linear independence of
functions $\sin x$ and $\cos x$, we get the equations:

$$g(h, h) - g(f, f) = 0. \quad (1)$$
$$g(f, h) = g(h, f) = g(b, h) = g(c, f) - g(c, h) = 0 \quad (2)$$

It follows that $b$ and $c$ are spacelike vectors which are orthogonal to the spacelike plane $\{f, h\}$. Thus $b = c = 0$ implies a contradiction.

Case (vi). As in the previous case (v), we obtain the same equations (1) and (2), so they imply a contradiction.

Case (vii). Since $g(\alpha, \alpha) = \pm 1$ and since the coefficients of $\sinh(2\alpha)$ and $\cosh(2\alpha)$ must be 0, we have $g(f, f) + g(h, h) = 0$, which gives a contradiction.

Next, suppose that $\alpha$ is a unit-speed 3 type curve lying in an timelike plane in $\mathbb{E}^4_1$, with the equation $x_3 = 0$. Again, $\alpha$ is a curve of one of seven possible forms (i) - (vii), as we have seen in the proof of the Theorem 3.1, where the vectors $b, c, d, e$ and $f$ cannot be spacelike, timelike or null vectors. We may take $e = (0, 0, 0, 0)$, up to a translation and take $g = -dx^2 + dx^2 + dx^2$. We shall consider cases (i) - (vii) separately.

Case (i). We get the same system of the equations as in the case (i) of the Theorem 3.1. If the vectors $c$ and $d$ are different from zero and not null vectors, then there would be two mutually orthogonal spacelike or timelike vectors in a timelike plane, which is impossible. It follows that for $c$ and $d$ holds $g(c, c) = g(d, d) = 0$. Since $c$ and $d$ are orthogonal to the timelike plane $\{c, f\}$ and they belong to it, it follows that $c = d = 0$. Consequently, we obtain a contradiction.

Cases (ii) and (iii). We differ subcases $t = 2\alpha$, $t = 3\alpha$, $t \neq 2\alpha, 3\alpha$ and again obtain a contradiction.

Case (iv). We differ subcases (iv.1) $t = 3\alpha$ and (iv.2) $t \neq 3\alpha$.

(iv.1). We get the same system of the equations as in the case (iv.1) of the Theorem 3.1. This system implies that $g(c, c) = g(d, d) = 0$ and since $d$ and $c$ are orthogonal to the timelike plane $\{f, h\}$, we have $c = d = 0$. Thus we obtain a contradiction.

(iv.2). Since $g(\alpha, \alpha) = \pm 1$ and by using the linear independence of the
functions $\sin x, \cos x, \sinh x$ and $\cosh x$, we obtain the equations
\begin{align*}
g(c, c) - g(b, b) &= 0, \quad (1) 
g(h, h) + g(f, f) &= 0, \quad (2) 
g(b, c) = g(f, h) = g(b, f) = g(b, h) = g(c, f) = g(c, h) = 0 \quad (3)
\end{align*}
Then $\{f, h\}$ is a timelike plane and the vectors $b$ and $c$ are lying in it and are orthogonal to it. Thus we have $b = c = 0$, which means that $\alpha$ is not a curve of $3$-type. This is a contradiction.

Case (v). After calculating $g(\alpha, \alpha) = \pm 1$ and using the inequality $0 < p < t < q$, we find that there are 4 subcases:
(v.1) $2t = p + q$; \hspace{1cm} (v.2) $2t = q - t$;
(v.3) $2t = q - p$; \hspace{1cm} (v.4) $2t \neq p + q, q - t, q - p$.

In all of these subcases we obtain the equations:
\begin{align*}
g(h, h) - g(f, f) &= 0, \quad (1) 
g(c, c) - g(d, d) &= 0, \quad (2) 
g(f, h) = g(d, c) = g(d, f) = g(d, h) = g(c, f) = g(c, h) = g(b, f) = g(b, h) = g(c, f) = g(c, h) = 0. \quad (3)
\end{align*}
It follows that $b, c, d, e, f, h$ are collinear null vectors, which implies a contradiction.

Case (vi). We refer the subcases (vi.1) $: = 3p$ and (vi.2) $t \neq 3p$.
(vi.1) Now, we obtain the equations:
\begin{align*}
g(h, h) - g(f, f) &= 0, \quad (1) 
g(c, c) + g(d, d) &= 0, \quad (2) 
g(d, c) - g(d, f) = g(d, h) = g(c, f) = g(c, h) = g(f, h) = 0. \quad (3)
\end{align*}
Then $\{d, e\}$ is a timelike plane and the vectors $f$ and $h$ are orthogonal to $\{d, e\}$. Thus we have $f = h = 0$, which implies a contradiction.

(vi.2) This subcase is analogous to the subcase (vi.1) and again we obtain a contradiction.

Case (vii). As in the case (v), there are 4 subcases:
(vii.1) $2t = p + q$; \hspace{1cm} (vii.2) $2t = q - t$.
A classification of 3 type curves in Minkowski 3-space $E^3_1$, I

(vii.3) $2t = q - p$; (vii.4) $2t \neq p, q, a - t, q - p.$

In all of these subcases we obtain the equations:

\[ g(f, f) + g(h, h) = 0, \]  
\[ g(d, d) + g(e, e) = 0, \]  
\[ g(f, h) = g(d, e) = g(d, f) = g(d, h) = g(e, f) = g(e, h) = 0, \]

which means that \( \{f, h\} \) is timelike plane and the vectors \( d \) and \( e \) are orthogonal to \( \{f, h\} \). Thus \( d = e = 0 \), which is a contradiction.

This completes the proof of this theorem.

References