PARTICULAR F-STRUCTURE ON VECTOR BUNDLE AND COMPATIBLE D-CONNECTIONS

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Abstract

The structures determined by a tensor field of (1, 1) type of constant rank, with the property $f^3 + f = 0$, was studied by many authors. R. Miron and Gh. Atanasiu determined the set of all connections compatible with $f$-structures, the integrability of $f$-structures and studied the case of $(f, g)$-structures in [1].

In the present paper we consider $f$-structures on the total space $E$ of the vector bundle $(E, \pi, M)$ and we shall find the non-linear connections $N$ on $E$ so that the tensor field $f$ has a particular form. In this manner, $f$-structures of type I and type II are studied and in these cases the compatible $d$-connections from general case [2] are determined.

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Let $\xi = (E, \pi, M)$ be a vector bundle over $n$-dimensional manifold $M$, $\chi(E)$ the $F(E)$-modul of the vector fields, $\tau^r_0(E)$ the $F(E)$-modul of the tensor field of $(r, q)$ type and $\tau^p_0(E)$ the algebra of the Finsler tensor fields of $(r, q)$ type.

Let us consider $f$-structures on $E$, $\tau$ tensor fields $f \in \tau^1(E)$ of constant rank $r$, with the property $f^3 + f = 0$.

In the following we shall suppose that there exists an $f$-structure on $E$. We shall consider a non-linear connection $N$ on $E$, then for $\xi \in E$, $TE = N_{\xi} \oplus E_{\xi}$, where dim $N_{\xi} = n$ and dim $E_{\xi} = n$, so it results that total space $TE$ is decomposed $TE = HE \oplus VE$.

Proposition 1 If $f$ is $f$-structure on $E$ then there is a unique decomposition of $f$ in the following $d$-tensor fields:

$$f = f^3 + f^4 + f^4, \text{ where } f \in \tau^1(E), \quad f \in \tau^1(E),$$

(1)
\[ 3f \in \tau^p \| \nu_\alpha (E) \text{ and } 4f \in \tau^p \| \nu_\beta (E), \text{ i.e.} \]
\[ \begin{align*}
3f(\omega, X) &= f(\omega^\beta, X^\beta), 3f(\omega, X) = f(\omega^\alpha, X^\alpha) \\
2f(\omega, X) &= f(\omega^\alpha, X^\alpha), 2f(\omega, X) = f(\omega^\beta, X^\beta)
\end{align*} \quad (2)
\]
\[ \forall \omega \in \chi(\nu) \text{ and } \nu \omega \in \chi(\nu^\alpha). \]
\[ \left\{ \begin{align*}
3f(\nu^\beta) &= 3f(\nu) + 2f(\nu) \\
2f(\nu^\beta) &= 2f(\nu) + 3f(\nu)
\end{align*} \right\}, \quad \forall \nu = X^\alpha + X^\beta. \quad (3)
\]

Let \( \nabla \) be a linear connection on \( E \).

**Definition 1** A linear connection \( \nabla \) on \( E \) is a \( d \)-connection compatible with structure \( f \) on \( E \) if:

a) \( \nabla \) preserves parallelism of the horizontal \( H \) and vertical \( H^\nu \) distribution.

b) \( \nabla_X f - f \nabla_X = 0, \forall X \in \chi(\nu) \).

**Theorem 1** If \( \nabla^\nu \) is a fixed linear \( d \)-connection on \( E \), then the following connection

\[ \nabla_X = \nabla_X^\nu - \frac{1}{2} \left[ \nabla^\nu_X \circ (f + 2f \circ \nabla^\nu_X \circ f - 2f \circ f \circ f + 2f \circ f) \right] \]

\[ = \nabla_X^\nu - \frac{1}{2} \left[ \nabla^\nu_X \circ (f \circ f' + f \circ f' \circ f) - 3 \left( f \circ f' \right) \circ f \circ f' \circ f \circ f' \right] \]

\[ = \nabla_X^\nu - \frac{1}{2} \left[ \nabla^\nu_X \circ (f \circ f' \circ f) - 3 \left( f \circ f' \right) \circ f \circ f' \circ f \circ f' \right] \]

is a \( d \)-connection compatible with the structure \( f \).

Let \( \gamma \) be a local chart on \( E \) and \( (x^i, y^i), i = 1, 2, \ldots, n \) the coordinates of point \( u \) in \( \pi^{-1}(U) \).

If \( N^\nu \) is a fixed non-linear connection on \( E \), then in the adapted frame \( \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\} \), the tensor field \( f \) is written in the form

\[ f = f^\beta \frac{\partial}{\partial y^i} \otimes dx^i + f^\alpha \frac{\partial}{\partial x^j} \otimes dy^j + f^\mu \frac{\partial}{\partial y^i} \otimes dx^i + f^\nu \frac{\partial}{\partial x^j} \otimes dy^j. \quad (5) \]
If $B \in \tau^{(1,0)}(E)$ is a $d$-tensor field and $N^\rho = N^\rho_\sigma = B^\rho_\sigma$ is another non-linear connection on $E$ then the tensor field $f$ has the components $(F^\rho_\alpha, F^\rho_\gamma, F^\rho_\alpha, N^\rho_\gamma)$.

\[
\begin{align*}
F^\rho_\alpha &= f^\rho_\alpha - f^\rho_\beta B^\beta_\alpha \\
F^\rho_\gamma &= f^\rho_\gamma \\
F^\rho_\alpha &= f^\rho_\alpha + f^\rho_\beta B^\beta_\alpha - f^\beta_\gamma B^\rho_\beta \\
F^\rho_\gamma &= f^\rho_\gamma + f^\rho_\beta B^\beta_\gamma .
\end{align*}
\]

(6)

When on the total space $E'$ there exists a non-linear connection so that the tensor $f$ has the form $f = (F^\rho_\gamma, 0, 0, N^\rho_\gamma)$, $f$ will be called $f$-structure of type I, respectively $f$-structure of type II, if $f$ has the form $f = (0, F^\rho_\gamma, F^\rho_\gamma, 0)$.

From (6) it is clear that the $f$-structure of type I exists only in the case $f^\rho_\gamma = 0$ and a non-linear connection $N$ for which $f^\rho_\gamma = 0$ exists.

**Proposition 2** The $f$-structure is of type I if and only if there exists a tensor field $B \in \tau^{(1,0)}(E)$ which satisfies the equation

\[
f^\rho_\beta B^\beta_\gamma - f^\beta_\gamma B^\rho_\beta = f^\rho_\gamma
\]

(7)

Since, rank $f = r < n + m$, the system (7) has not always a solution. In the affirmative case, the $d$-tensor field $f$ has the form:

\[
f = f + d^r f, \quad \text{where} \quad f \in \tau^{(1,0)}(E), \quad d^r f \in \tau^{(0,1)}(E)
\]

and the relation $d^r f = 0$ is equivalent with:

\[
\begin{align*}
\{ f \circ f \circ f \circ f - f = 0 \\
\circ f \circ f \circ f \circ f - f = 0 .
\end{align*}
\]

(9)

**Theorem 2** If a linear $d$-connection (distinguished connection) exists on $E$, then the distinguished $f$-connection are of type I and of one of them is given by

\[
\begin{align*}
\nabla_\alpha = \nabla_\alpha^\nu - \frac{1}{2} [ f \circ \nabla_\alpha^\nu \circ f + f \circ \nabla_\alpha^\nu \circ f - \\
2( f \circ f \circ f \circ f \circ f ) \circ \nabla_\alpha^\nu - 2\nabla_\alpha^\nu ( f \circ f \circ f \circ f \circ f ) = \\
3 \circ f \circ f \circ f \circ f \circ f - 3 \circ f \circ f \circ f \circ f \circ f ]
\end{align*}
\]

(10)
In local coordinates, if we denote by $D^\nu = (L^\nu_\mu, L^\nu_\mu, C^\mu_\nu_\rho, C^\mu_\nu_\rho)$ the local components of linear $d$-connection $\nabla^\nu$, then the distinguished $f$-connection of type I is characterized by

$$
\begin{align*}
L^\nu_\mu &= L^\nu_\mu + \frac{1}{2} f^\nu_\mu f^\rho_\mu - \frac{1}{2} f^\nu_\mu f^\rho_\mu + \frac{1}{2} f^\nu_\mu f^\rho_\mu f^\tau_\mu f^\tau_\rho \\
L^\nu_\mu &= L^\nu_\mu + \frac{1}{2} f^\nu_\mu f^\rho_\mu - \frac{1}{2} f^\nu_\mu f^\rho_\mu + \frac{1}{2} f^\nu_\mu f^\rho_\mu f^\tau_\mu f^\tau_\rho \\
C^\mu_\nu_\rho &= C^\mu_\nu_\rho + \frac{1}{2} f^\nu_\mu f^\rho_\mu - \frac{1}{2} f^\nu_\mu f^\rho_\mu + \frac{1}{2} f^\nu_\mu f^\rho_\mu f^\tau_\mu f^\tau_\rho \\
C^\mu_\nu_\rho &= C^\mu_\nu_\rho + \frac{1}{2} f^\nu_\mu f^\rho_\mu - \frac{1}{2} f^\nu_\mu f^\rho_\mu + \frac{1}{2} f^\nu_\mu f^\rho_\mu f^\tau_\mu f^\tau_\rho .
\end{align*}
$$

(11)

Theorem 3 The set of all distinguished $f$-connections of type I is given by

$$
\nabla^*_x = \nabla^*_x + \Omega(W_X),
$$

(12)

where $\nabla^*_x$ is the connection (10), $W_X \in \mathfrak{g}(E)$ is an arbitrary tensor field so that $[\Omega(W_X)Y^\mu]^\nu = 0$, $[\Omega(W_X)Y^\nu]^\mu = 0$ and $\Omega$ is Obata operator [4].

The structures of type II, $f = 2f + 3f$, under some conditions, permit to study the general case of $f$-structures.

For the study of these structures we will consider the following cases:

a) $n = \text{dim}H^E > \text{dim}Y^E = m$

If the rank $f \geq m$, then the system $f^\nu_\mu f^\rho_\mu + f^\rho_\mu = 0, k = 1, 2, \ldots , \infty$ admits infinite solutions. Moreover, if there exist $f^{\nu}_\mu$ for which $f^{\nu}_\rho f^{\rho}_\mu - f^{\nu}_\rho = 0$, then there exists a non-linear connection $N^{\mu}_\nu = N^{\mu}_\nu - f^{\mu}_\rho f^{\rho}_\nu$ and the tensor field $f$ can be written in the form $(0, F^\mu_\nu, F^\mu_\nu, 0)$.

b) $n = m$

If we consider rank $f \geq m = n$, rank $(f^{\mu}_\nu = \text{max} = \text{max})$ and the equations $f^{\nu}_\rho f^{\rho}_\mu - f^{\nu}_\rho = 0, f^{\nu}_\rho f^{\rho}_\mu + f^{\rho}_\nu = 0$ are satisfied, then the non-linear connection $N^{\mu}_\nu = N^{\mu}_\nu - f^{\mu}_\rho f^{\rho}_\nu$, the tensor field $f$ is of the form $(0, F^\mu_\nu, 0)$.

c) $n < m$

Let us suppose $F^\mu_\nu = 0, \text{rank } f \geq m$, then the equation $f^{\nu}_\rho f^{\rho}_\mu - f^{\nu}_\rho = 0$

admits infinite solutions. Let $(B^\mu_\nu)$ be a solution for which $f^{\nu}_\rho f^{\rho}_\mu + f^{\rho}_\nu = 0$, then there exists a non-linear connection $N^\mu_\nu = N^\mu_\nu - f^\mu_\rho f^\rho_\nu$ so that $f = (0, F^\mu_\nu, F^\mu_\nu, 0)$. In this case, Y. Ichiijo had designed a natural example [2].

Let us denote by $N^\mu_\nu = F^\mu_\nu$ and $f^\mu_\nu = F^\mu_\nu$, in the adapted frame, in all the above cases.

Theorem 4 If there exists a non-linear connection $N$ for which $f = 2f + 3f$, then there exist distinguished $f$-connections of type II, one of them is given
by:

\[ \nabla_X = \nabla_X^\perp - \frac{1}{2} f^\perp \circ \nabla_X \circ f^\perp - 2 \frac{f \circ f^\perp}{f^2} \circ \nabla_X - 2 \frac{\nabla_X \circ f^\perp}{f^2} \circ f - 3 \frac{f \circ f^\perp}{f^2} \circ \nabla_X - \frac{1}{2} f \circ \nabla_X \circ f - 2 \frac{f \circ f^\perp}{f^2} \circ f \circ \nabla_X - 2 \frac{\nabla_X \circ f}{f^2} \circ f - 3 \frac{f \circ f^\perp}{f^2} \circ f \circ \nabla_X \circ f^\perp \quad (13) \]

where \( \nabla_X^\perp \) is a fixed linear d-connection on \( E \).

\[ \nabla = \nabla_X + \Omega(W_X), \quad (14) \]

where \( \nabla_X \) is the connection (12) \( W_X \in T(\mathcal{F}) \) an arbitrary tensor field and \( \Omega(W_X) \odot \odot 1 = 0, \Omega(W_X) \odot \odot 2 = 0 \).

If we denote by \( DT^\perp = (L^\perp_\nu, L^\perp_\mu, L^\perp_\nu, L^\perp_\nu) \) the local coordinates of the linear d-connection \( \nabla^\perp \) then the distinguished f-connection of type II, (13), is characterized by:

\[ \begin{align*}
L^\perp_\nu &= L^\perp_\nu - \frac{2}{3} f^\perp \odot f^\perp + \frac{2}{3} f^\perp \odot f^\perp + \frac{2}{3} f^\perp \odot f^\perp + \frac{2}{3} f^\perp \odot f^\perp + \frac{2}{3} f^\perp \odot f^\perp \\
L^\perp_\nu &= L^\perp_\nu - \frac{3}{4} f^\perp \odot f^\perp + \frac{3}{4} f^\perp \odot f^\perp + \frac{3}{4} f^\perp \odot f^\perp + \frac{3}{4} f^\perp \odot f^\perp \\
C^\perp_\nu &= C^\perp_\nu - \frac{1}{4} f^\perp \odot f^\perp + \frac{1}{4} f^\perp \odot f^\perp + \frac{1}{4} f^\perp \odot f^\perp + \frac{1}{4} f^\perp \odot f^\perp \\
C^\perp_\nu &= C^\perp_\nu - \frac{3}{4} f^\perp \odot f^\perp + \frac{3}{4} f^\perp \odot f^\perp + \frac{3}{4} f^\perp \odot f^\perp + \frac{3}{4} f^\perp \odot f^\perp
\end{align*} \quad (15) \]

Remark. If we consider natural f-structure \( F = (\odot, \odot^2) \) with \( B_\nu C^\perp_\nu = \delta^\perp_\nu \), and if \( f_\nu = -C_\nu, f^\perp_\nu = B^\perp_\nu, f_\nu = f^\perp_\nu = \delta^\perp_\nu \odot \odot, \) then the linear distinguished f-connection (15), represents the natural distinguished f-connection [5].

References


