

# ON PROJECTING OF $\mathbb{P}^N$ ONTO $\mathbb{P}^{N-M}$ AND THE VARIETY OF PROJECTION CENTERS

L. Medvedeva

Russia

*Email address:* scout@uniyar.ac.ru

M. Popovich

Yugoslavia

## Abstract

The present paper is the final one in the series of papers [1] – [6] devoted to a central projection of the projective space  $\mathbb{P}^n$  on its subspace  $\mathbb{P}^k$ . In these papers, as well as in the present one, we consider a projection with the following property: the reference points  $A_i$  ( $i = \overline{1, n+2}$ ) have images  $A'_i$  which are projectively equivalent to so-called model, i.e., an ordered set of generally disposed points  $T_i$  in  $\mathbb{P}^k$ .

It is known [1], [2] that for any  $k \in \mathbb{N}$  ( $0 < k < n$ ) there exists a projection of such a kind which can be defined in different ways: a set of projection centers is a  $(n-k)$ -dimensional variety which consists of  $(n-k-1)$ -dimensional planes. If  $k = n - 1$ , this set is a normal rational curve except finite number of points [3]. For  $k = n - 2$  and  $k = n - 3$  a description of the variety of projection centers can be found in [4]-[6]. Below we generalize this description for any  $k < n$ .

If  $k = n - m$  ( $m \geq 1$ ), then a projecting can be done by means of  $m$ -dimensional planes which belong to a sheaf of planes with  $(m - 1)$ -dimensional vertex.

**Theorem 1.** *If  $n \leq 2m$  ( $m \geq 2$ ), then the set of projection centers is a subvariety  $\sigma_k^{m+1}$  in the Grassmanian manifold  $G(m, n + 1)$ . It can be parametrized by points of  $k$ -dimensional plane if  $n = m + k$  ( $0 < k \leq m - 1$ ), and it can be parametrized by points of  $m$ -dimensional plane if  $n = 2m$ .*

**Proof.** Let  $\mathbb{P}^{n-m} = \langle A_1, A_2, \dots, A_{n-m}, A_{n-m+1} \rangle$ . Then for the projection  $f : \mathbb{P}^n \rightarrow \mathbb{P}^{n-m}$  we have  $f(A_j) = A_j$  for  $j = \overline{1, n - m + 1}$ . If we choose the image  $A'_{n-m+2} \in \mathbb{P}^{n-m}$  of the point  $A_{n-m+2}$ , we will uniquely determine the images of all points  $A_j$  ( $j = \overline{n - m + 2, n + 2}$ ) because the image  $A'_i$  and the model  $T_i$  are projectively equivalent. The projection center, that is,  $(m - 1)$ -dimensional plane, has to intersect the lines  $\langle A_j, A'_j \rangle$  ( $j =$

$\overline{n - m + 2, n + 2}$ ) and has not to intersect  $\mathbb{P}^{n-m}$ . It is known [7] that the variety of  $(m - 1)$ -dimensional planes, which intersect  $m + 1$  lines in  $\mathbb{P}^n$ , can be represented in the Grassmanian manifold  $G(m, n + 1)$  by means of the intersection of Schubert cycles  $\sigma_{n-m}$ , and the multiplicity of intersection is  $m + 1$ ; so this intersection is  $\sigma_{n-m}^{m+1}$ . The latter is not empty if  $(n - m)(m + 1) \leq m(n - m + 1)$ , or  $n \leq 2m$ .

If  $n = 2m$  then  $\sigma_{n-m}^{m+1} = \sigma_m^{m+1} = \sigma_{m+1, m+1, \dots, m+1}$ , where the index  $m + 1$  is repeated  $m$  times. It means that for the projection  $f : \mathbb{P}^{2m} \rightarrow \mathbb{P}^m$  each point  $A'_{n-m+2} = A'_{m+2} \in \mathbb{P}^m$  determines a unique projection center, consequently, in such a case the variety of projection centers can be parametrized by points of an  $m$ -dimensional plane.

If  $n < 2m$  ( $n - m < m$ ), then for any point  $A'_{m+1}$  from  $\mathbb{P}^{n-m}$  we can construct the variety of projection centers  $\sigma_{n-m}^{m+1} = \sigma_k^{m+1}$ . Thus, in such a case the set of projection centers is a  $(n - m)$ -dimensional variety which consists of varieties  $\sigma_{n-m}^{m+1}$ ; it can be parametrized by points of  $(n - m)$ -dimensional plane.

**Theorem 2.** *If  $n = 2m + 1$ , then the variety of projection centers can be parametrized by points of rational hypersurface  $F_m^{m+1} \in \mathbb{P}^{n-m}$  whose degree is  $m + 1$ . If  $n = 2m + k$  ( $k > 1$ ), then the variety of projection centers can be parametrized by points of the variety which is described as follows: it is the intersection of  $k$  hypercone in  $\mathbb{P}^{n-m}$ ; their directrices are the surfaces  $F_m^{m+1}$ ; their vertices are  $(k - 2)$ -dimensional planes which are disposed in a special way in  $\mathbb{P}^{n-m}$ .*

**Proof.** If  $n = 2m + 1$  then we have a projection of  $\mathbb{P}^{2m+1}$  onto  $\mathbb{P}^{m+1}$ . Let  $\mathbb{P}^{m+1}$  be defined as a plane

$$\langle A_1, A_2, \dots, A_{m+2} \rangle: x_{m+3} = x_{m+4} = \dots = x_{2m+2} = 0.$$

We can consider the reference points  $T_i$  ( $i = \overline{1, m + 3}$ ) of  $\mathbb{P}^{m+1}$ ; denote the coordinates of the remainder points of the model, i.e.,  $T_j$  ( $j = \overline{m + 4, 2m + 3}$ ), as  $(\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{j, m+2})$ . Now we can easily obtain the projection  $f$  satisfying the properties

$$f(A_i) = A_i, \quad i = \overline{1, m + 2};$$

$$f(A_{m+3}) = A'_{m+3}, \quad A'_{m+3}(a_1 : a_2 : \dots : a_{m+2} : 0 : \dots : 0);$$

$$f(A_j) = A'_j, \quad j = \overline{m + 4, 2m + 3}, \quad A'_j(a_1 \alpha_{j1} : \dots : a_{m+2} \alpha_{j, m+2} : 0 : \dots : 0).$$

It is defined uniquely because the image  $A'_i$  and the model  $T_i$  ( $i = \overline{1, 2m + 3}$ )

are projectively equivalent. The matrix of the projection has a form

$$A_f = \begin{pmatrix} \rho & 0 & \dots & 0 & \rho_{m+3}a_1 & \rho_{m+4}a_1\alpha_{m+4\ 1} & \dots & \rho_{2m+2}a_1\alpha_{2m+2\ 1} \\ 0 & \rho & \dots & 0 & \rho_{m+3}a_2 & \rho_{m+4}a_2\alpha_{m+4\ 2} & \dots & \rho_{2m+2}a_2\alpha_{2m+2\ 2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \rho & \rho_{m+3}a_{m+2} & \rho_{m+4}a_{m+2}\alpha_{m+4\ m+2} & \dots & \rho_{2m+2}a_{m+2}\alpha_{2m+2\ m+2} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (1)$$

The condition  $f(A_{2m+3}) = A'_{2m+3}$  leads us to a system

$$\begin{aligned} \rho_{2m+3}a_i\alpha_{2m+3\ i} &= \rho + \rho_{m+3}a_i + \rho_{m+4}a_i\alpha_{m+4\ i} + \\ &\dots + \rho_{2m+2}a_i\alpha_{2m+2\ i}, \quad i = \overline{1, m+2}. \end{aligned} \quad (2)$$

For given  $a_i$  the system contains  $m+2$  equations with  $m+2$  unknowns  $\rho, \rho_{m+3}, \dots, \rho_{2m+3}$  which can't be equal to zero. It means that the determinant of the system (2) is zero:

$$\begin{vmatrix} 1 & a_1 & a_1\alpha_{m+4\ 1} & \dots & a_1\alpha_{2m+2\ 1} & -a_1\alpha_{2m+3\ 1} \\ 1 & a_2 & a_2\alpha_{m+4\ 2} & \dots & a_2\alpha_{2m+2\ 2} & -a_2\alpha_{2m+3\ 2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_{m+2} & a_{m+2}\alpha_{m+4\ m+2} & \dots & a_{m+2}\alpha_{2m+2\ m+2} & -a_{m+2}\alpha_{2m+3\ m+2} \end{vmatrix} = 0. \quad (3)$$

Decomposing the determinant by the first row, we obtain an equation

$$\begin{aligned} A_1 a_2 a_3 \dots a_{m+2} + A_2 a_1 a_3 \dots a_{m+2} + \\ \dots + A_{m+1} a_1 a_2 \dots a_{m+2} + A_{m+2} a_1 a_2 \dots a_{m+1} = 0. \end{aligned} \quad (4)$$

This equation determine a hypersurface of degree  $m+1$  in  $\mathbb{P}^{m+1}$ ; we denote it as  $F_m^{m+1}$ .

Condition (3) means that the points  $(1 : 1 : \dots : 1)$ ,  $A'_{m+3}$ ,  $A'_{m+4}$ ,  $\dots$ ,  $A'_{2m+3}$ , which belong to  $\mathbb{P}^{m+1}$ , are not disposed in a general way because they lay in an  $m$ -dimensional plane. Besides, condition (3) means that the point  $A'_{m+3}$  can't be taken in an arbitrary way. It has to be taken on the hypersurface  $F_m^{m+1}$ , which is uniquely determined by the model.

It is easy to see that any point  $A'_{m+3} \in F_m^{m+1}$  uniquely determines the projection center of  $f$ . In other words, the variety of projection centers is parametrized by points of the variety  $F_m^{m+1}$ .

Here are some properties of the variety  $F_m^{m+1}$ .

1. The points  $A_i$  ( $i = \overline{1, m+2}$ ) lay of  $F_m^{m+1}$ . They are singular points of multiplicity  $m$ .

2. The variety  $F_m^{m+1}$  passes through the point  $(1 : 1 : \dots : 1 : 0 : \dots : 0)$ , which is a tangent point of the variety  $F_m^{m+1}$  and the  $m$ -dimensional plane  $(K_1, \dots, K_m, I)$ . Here  $K_j$  ( $j = \overline{m+4, 2m+3}$ ) have coordinates  $(\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{j,m-2})$  with respect to reference points  $(A_1, \dots, A_{m+2}, I)$  of  $\mathbb{P}^{m+1}$ .

3. The variety  $F_m^{m+1}$  is a rational one.

In order to prove property 3, let us rewrite equation (4) in the form

$$\begin{vmatrix} A_1 a_2 & -A_3 a_4 & A_4 a_2 & -A_5 a_2 & \dots & (-1)^{m-1} A_{m+1} a_2 & (-1)^m A_2 a_1 \\ 0 & a_3 & a_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_4 & a_5 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{m+1} & a_1 \\ (-1)^{m+1} a_{m+2} & 0 & 0 & 0 & \dots & -\frac{A_{m+2}}{A_1} a_{m+1} & a_{m+2} \end{vmatrix} = 0. \tag{5}$$

We can consider (5) a condition which is necessary and sufficient for the corresponding homogeneous system of linear equations to have a non-zero solution. If we consider the variables  $\lambda_0, \lambda_1, \dots, \lambda_m$  coordinates in  $\mathbb{P}^m$  and solve the system for them, we obtain a birational map of  $F_m^{m+1}$  into  $\mathbb{P}^m$ . If we solve the system for  $a_i$  ( $i = \overline{1, m+2}$ ), we obtain formulas of the inverse map. The existence of both maps implies that  $\mathbb{P}^m$  and  $F_m^{m+1}$  are birationally isomorphic, consequently,  $F_m^{m+1}$  is rational.

Let us consider theorem 2. As  $n = 2m+k$ ,  $k > 1$ , we have  $n-m = m+k$  and the projection  $f : \mathbb{P}^{2m+k} \rightarrow \mathbb{P}^{m+k}$  with the matrix

$$A_f = \begin{pmatrix} \rho & 0 & \dots & 0 & \rho_{m+k+2} a_1 & \rho_{m+k+3} a_1 \alpha_{m+k+3 \ 1} \\ 0 & \rho & \dots & 0 & \rho_{m+k+2} a_2 & \rho_{m+k+3} a_2 \alpha_{m+k+3 \ 2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \rho & \rho_{m+k+2} a_{m+k+1} & \rho_{m+k+3} a_{m+k+1} \alpha_{m+k+3 \ m+k+1} \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \rho_{2m+k+1} a_1 \alpha_{2m+k+1 \ 1} & \dots \\ \dots & \dots & \dots & \dots & \rho_{2m+k+1} a_2 \alpha_{2m+k+1 \ 2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \rho_{2m+k+1} a_{m+k+1} \alpha_{2m+k+3 \ m+k+1} & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots \end{pmatrix}$$

As  $f(A_{n+2}) = A'_{n+2}$ , we have, as well as in the previous case, a sistem

of linear equations

$$\rho + \rho_{m+k+2}a_i + \rho_{m+k+3}a_i\alpha_{m+k+3}i + \dots + \rho_{2m+k+1}a_i\alpha_{2m+k+1}i - \rho_{2m+k+2}a_i\alpha_{2m+k+2}i = 0, \quad i = \overline{1, m+k+1}.$$

which has  $m+2$  unknowns  $\rho, \rho_{m+k+2}, \dots, \rho_{2m+k+2}$ . This system has non-zero solution if its rank  $r$  satisfies the condition  $r \leq m+1$ . Consequently, all minors of order  $m+2$  vanish:

$$\begin{vmatrix} 1 & a_1 & a_1\alpha_{m+k+3}1 & \dots & a_1\alpha_{2m+k+1}1 \\ 1 & a_2 & a_2\alpha_{m+k+3}2 & \dots & a_2\alpha_{2m+k+1}2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{m+1} & a_{m+1}\alpha_{m+k+3}m+1 & \dots & a_{m+1}\alpha_{2m+k+1}m+1 \\ 1 & a_{m+l} & a_{m+l}\alpha_{m+k+3}m+l & \dots & a_{m+l}\alpha_{2m+k+1}m+l \\ \dots & \dots & \dots & \dots & \dots \\ -a_1\alpha_{2m+k+2}1 & & & & \\ -a_2\alpha_{2m+k+2}2 & & & & \\ \dots & & & & \\ -a_{m+1}\alpha_{2m+k+2}m+1 & & & & \\ -a_{m+l}\alpha_{2m+k+2}m+l & & & & \end{vmatrix} = 0, \quad l = \overline{2, k+1}. \tag{6}$$

Each of  $k$  equation of the system (6) defines a hypercone  $K_l^{m+1}$  in  $\mathbb{P}^{m+k}$  whose degree is  $m+1$ . The variety  $F_{ml}^{m+1}$ , which lays in the  $(m+1)$ -dimensional plane  $a_{m+2} = \dots = \hat{a}_{m+l} = \dots = a_{2m+k+1} = 0$ , is the directrix of hypercones (the symbol  $\hat{a}_{m+l}$  means that there is no equation  $a_{m+l} = 0$  in the system).  $(k-2)$ -dimensional plane in  $\mathbb{P}^{m+k}$ , which is defined by equations  $a_1 = \dots = a_{m+1} = a_{m+l} = a_{m+k+2} = \dots = a_{2m+k+1} = 0$ , is the vertex of the hypercone  $K_l^{m+1}$ .

If we choose a point  $A'_{m+k+2} \in \bigcap_{l=2}^{k+1} F_{ml}^{m+1}$ , we uniquely determine the projection center of  $f$ , therefore, the set of projection centers is parametrized by points of the variety  $\bigcap_{l=2}^{k+1} F_{ml}^{m+1}$ .

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