THE INJECTIVE HULL AND THE \( \mathfrak{bc} \)-HULL OF A TOPOLOGICAL SPACE

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Abstract

A close connection between the notion of the \( \mathfrak{bc} \)-hull and the notion of the injective hull (cf. the definitions below) of a topological \( T_\sigma \)-space is established.

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Bounded complete domains (shortly, \( \mathfrak{bc} \)-domains) [1] or, which is the same, complete \( A_\sigma \)-spaces [2] form a subcartesian closed full subcategory of the category \( \text{TOP}_\sigma \) of topological \( T_\sigma \)-spaces. This subcategory is important for denotational semantics.

The author [3] introduced the notion of the \( \mathfrak{bc} \)-hull of a topological space \( X \) as follows. A homeomorphic embedding \( \lambda : X \to B \) of a space \( X \) in a \( \mathfrak{bc} \)-domain \( B \) is called the \( \mathfrak{bc} \)-hull of \( X \) if the following conditions are satisfied:

1. **Unversality.** For any continuous mapping \( f : X \to B' \) from \( X \) to a \( \mathfrak{bc} \)-domain \( B' \) there exists a continuous mapping \( f^*: B \to B' \) such that \( f^* \lambda = f \).

2. **Minimality.** If \( f : B \to B \) is a continuous mapping and \( f \lambda = \lambda \), then \( f = id_B \).

In [3], the existence of the \( \mathfrak{bc} \)-hull of an \( \alpha \)-space (cf. the definition below) is established.
With each topological $(T_0)$-space $X$ we associate binary relations $\leq_X$ and $\prec_X$ on $X$ defined as follows. For $\xi_0, \xi_1 \in X$ we set

$\xi_0 \leq_X \xi_1 \iff$ for any open subset $V \subseteq X$ from $\xi_0 \in V$ it follows that $\xi_1 \in V \iff \xi_0 \in \text{cl}(\xi_1)$, where cl denotes the closure;

$\xi_0 \prec_X \xi_1 \iff \xi_1 \in \text{int}(\xi_1 \mid \xi_0 \leq_X \xi_1)$, where int denotes the interior.

The relation $\leq_X$ is a partial order on $X$ and the relation $\prec_X$ is transitive. The relations $\leq_X$ and $\prec_X$ are connected as follows:

$\xi_0 \prec_X \xi_1 \implies \xi_0 \leq_X \xi_1$,

$\xi_0 \leq_X \xi_0' \prec_X \xi_1^* \leq_X \xi_1 \implies \xi_0 \prec_X \xi_1$.

A space $X$ is called an $\alpha$-space if for any open subset $V \subseteq X$ and point $\xi \in V$ there exists a point $\xi' \in V$ such that $\xi' \prec_X \xi$.

The following characterization of $bc$-domains (cf. [2, Proposition 2, 34]) is essential for the further considerations.

**Proposition 1.** For a topological $T_0$-space $X$ the following conditions are equivalent:

1. $X$ is a $bc$-domain,
2. for any topological space $Y$, everywhere dense subspace $Y_0$, and continuous mapping $f_0 : Y_0 \to X$ there exists a continuous mapping $f : Y \to X$ such that $f_0 = f|Y_0$.

**Remark 2.** The set of extensions $\hat{f}$ of $f_0$ in condition (2) has the largest element, i.e., there exists a continuous mapping $f^* : Y \to X$ such that $f^*|Y_0 = f_0$ and for any continuous mapping $f : Y \to X$ such that $f|Y_0 = f_0$ and any $\eta \in Y$ we have $f(\eta) \leq \hat{f}(\eta)$.

Let $X$ be a subspace of $Y$. The space $Y$ is called an essential extension of $X$ if for any continuous mapping $f : Y \to Z$ from the fact that $f|X$ is a homeomorphic embedding of $X$ in $Z$ it follows that $f$ is a homeomorphic embedding of $Y$ in $Z$.

**Proposition 3.** If $Y \supseteq X$ is the $bc$-hull of $X$, where $\lambda$ is the identity mapping $\text{id}_X$, then $Y$ is an essential extension of $X$. 
Lemma 4. The space $X$ is everywhere dense in $Y$, i.e., $\text{cl}X = Y$.

Proof. By [2, Proposition 5, §3], $Y_0 = \text{cl}X \subseteq Y$ is a complete $A_0$-space, i.e., $Y_0$ is a $bc$-domain. The space $Y$ is the $bc$-hull of $X$. By the universality condition, there exists a continuous mapping $f : Y \to Y_0$ such that $f|_X = \text{id}_X$. By the minimality condition, $f = \text{id}_Y$, i.e., $Y = Y_0 = \text{cl}X$. □

Proof of Proposition 3. Let $f : Y \to Z$ be a continuous mapping such that $f|_X$ is a homeomorphic embedding of $X$ in $Z$. Since any $(T_0)$-space is homeomorphically embedded in a $bc$-domain, without loss of generality, we assume that $Z$ is a $bc$-domain. Let $Z_0$ be the closure of $f(X)$ in $Z$. By Lemma 4, we have $f(Y) \subseteq Z_0$. Indeed, if $\eta \in Y$ is an element such that $f(\eta) \in Z \setminus Z_0$, then $\eta \in f^{-1}(Z \setminus Z_0)$. Since $V = f^{-1}(Z \setminus Z_0)$ is a nonempty open subset, $V \cap X = \emptyset$, which is impossible. The space $Z_0$ is a $bc$-domain and $X_0 = f(X)$ is everywhere dense in $Z_0$. Let $g_0 : X_0 \to X \subseteq Y$ be a homeomorphism such that $f_0 = \text{id}_X$, and $g_0(f|_X) = \text{id}_Y$. By Proposition 1, there exists a continuous mapping $g : Z_0 \to Y$ extending $g_0$. The continuous mapping $gf : Y \to Y$ is such that $(gf)|_X = g_0(f|_X) = \text{id}_Y$. Hence $gf = \text{id}_Y$ and $f$ is a homeomorphic embedding of $Y$ in $Z_0 \subseteq Z$. □

Remark 5. Proposition 3 gives the positive answer to Question 1 in [3]. An answer to Question 2 in [3] is also positive. Indeed, by the construction of the $bc$-hull $B(X)$ of an $\alpha$-space $X$, there exists a $bc$-domain $B$ such that $B \subseteq B(X) \subseteq B$ and $X$ is a smooth subspace of $B$. Therefore, $X$ is a smooth subspace of any intermediate space.

As is shown in [4], for any $T_0$-space $X$ there exists "the largest" essential extension $\lambda X$. If $\lambda X$ is an injective space, then $\lambda X$ is called the injective hull of $X$. It is convenient to use the following obvious characterization of the injective hull:

A $T_0$-space $Y$ including $X$ as a subspace is the injective hull of $X$ if and only if $Y$ is injective and is an essential extension of $X$.

The following theorem establishes a close connection between the notion of the injective hull and the notion of the $bc$-hull.

Theorem 6. A topological space $X$ possesses the $bc$-hull if and only if $X$ possesses the injective hull.
Lemma 7. Let $Z$ be an essential extension of $X$, $X \subseteq Z$. If $f$ is a continuous mapping from $Z$ to $Z$ such that $f|_Z = \text{id}_Z$, then $f = \text{id}_Z$.

Proof. Let $Z_0 \supseteq Z$ be the largest essential extension of $Z$. Then $Z_0$ is the largest essential extension of $X$. The space $Z_0$ is also the largest essential extension of $f(Z)$, $Z_0 \subseteq f(Z)$, $Z \subseteq Z_0$. Since $Z$ is an essential extension of $X$ and $f|_X = \text{id}_X$, we conclude that $f$ is a homeomorphism from $X$ to $f(X)$. Since the largest essential extension is unique, there exists a homeomorphism $g$ from $Z_0$ onto $Z_0$ such that $g|_Z = f$ (consequently, $g|_X = \text{id}_X$). If $f \neq \text{id}_Z$, then $g \neq \text{id}_Z$. Thus, it suffices to prove the lemma under the assumption that $Z$ is a maximal essential extension of $X$. As is noticed in [5], for any point $\zeta \in Z$ we have $\zeta = \sup \{X_\xi : \xi \in X, \xi \leq \zeta \}$. Since $f$ is monotone, $f(\zeta) = \sup f(X_\xi) = \sup X_\xi = \zeta$. The mapping $f^{-1}$ is also a homeomorphism from $Z$ such that $f^{-1}|_Z = \text{id}_Z$. Hence $f^{-1}(\zeta) = \zeta$ for any point $\zeta \in Z$. Thus, if $f \neq \text{id}_Z$, then there exists a point $\zeta$ such that $f(\zeta) > \zeta$. We have $\zeta > f^{-1}(\zeta) > \zeta$, which is a contradiction. Thus, $f = \text{id}_Z$.

Corollary 8. Let $X \subseteq Y_0$ and $X \subseteq Y_1$ be essential extensions of $X$. Then there exists at most one continuous mapping $f : Y_0 \to Y_1$ such that $f|_X = \text{id}_X$.

Proof. Let $Y \supseteq X$ be the h-closure of $X$. By Proposition 3, $Y$ is an essential extension of $X$. If $Y$ is an injective space, then $Y$ is the injective hull of $X$. Assume that the space $Y$ is not injective. Consider the extension $Y'$ obtained from $Y$ by adding the new isolated largest element $T$.

Lemma 9. The space $Y'$ is injective. This space is an essential extension of $Y$.

Proof. Let $X_0$ be a subspace of $X_1$, $g_0 : X_0 \to Y'$ a continuous mapping, and $X_2$ the closure of $g_0^{-1}(Y)$ in $X_1$. Then $X_1 \setminus X_2$ is an open subset, $g_0|_{X_1 \setminus X_2}(Y')$
is a continuous mapping from $g_0^{-1}(Y)$ to $Y$, and $g_0^{-1}(Y)$ is everywhere dense in $X$. By Proposition 1, there exists a continuous mapping $g_2 : X_2 \to Y$ extending $g_0|_{g_0^{-1}(Y)}$. We define a mapping $g_1$ from $X_1$ to $Y$ by setting $g_1(\xi) = g_2(\xi)$ for $\xi \in X_2$ and $g_1(\xi) = \tau$ for $\xi \notin X_2$. It is easy to verify that $g_1$ is continuous and $g_1|_{X_1} = g_0$. Thus, $Y$ is an injective space.

Since the space $Y$ is not injective, there is no largest element in $Y$. Therefore, there exist inconsistent elements $\eta_0$ and $\eta_1$, i.e., there exists no element $\eta$ in $Y$ such that $\eta_0 \preceq_Y \eta$ and $\eta_1 \preceq_Y \eta$. We show that this implies the existence of open subsets $U_0$ and $U_1$ such that $\eta_0 \in U_0$, $\eta_1 \in U_1$, and $U_0 \cap U_1 = \emptyset$. Indeed, since $Y$ is an $\alpha$-space, we have $\eta_0 = \sup\{\eta_0 | \eta \preceq_Y \eta_0\}$ and $\eta_1 = \sup\{\eta_1 | \eta \preceq_Y \eta_1\}$. If every pair $\eta_0 \preceq_Y \eta_0$, $\eta_1 \preceq_Y \eta_1$ is consistent, then the family $\{\eta_0 \vee \eta_1 | \eta \preceq_Y \eta_0, \eta \preceq_Y \eta_1\}$ is directed. But the existence of $\eta = \sup\{\eta_0 \vee \eta_1 | \eta \preceq_Y \eta_0, \eta \preceq_Y \eta_1\}$ contradicts the fact that $\eta_0$ and $\eta_1$ are inconsistent. Let $\eta_0(\preceq_Y \eta_0)$ and $\eta_1(\preceq_Y \eta_1)$ be inconsistent. Then $U_0 = \text{int}(\eta | \eta_0 \preceq_Y \eta)$ and $U_1 = \text{int}(\eta | \eta_1 \preceq_Y \eta)$ satisfy the required conditions.

Now, we will prove that $Y^T$ is an essential extension of $X$. Let $f : Y^T \to Z$ be a continuous mapping such that $f|_Y$ is a homeomorphic embedding of $Y$ in $Z$. Since $f(Y)$ is homeomorphic to $Y$, there is no largest element in $f(Y)$. Hence $f(\top) \neq f(Y)$. Thus, $f$ is a one-to-one mapping. It suffices to prove that $f(\top)$ is an isolated point of $f(Y^T)$. Since $f(U_0)$ and $f(U_1)$ are open subsets of $f(Y^T)$, there exist open subsets $V_0$ and $V_1$ of $Z$ such that $V_0 \cap f(Y) = f(U_0)$ and $V_1 \cap f(Y) = f(U_1)$. We have $(V_0 \cap V_1) \cap f(Y) = f(U_0) \cap f(U_1) \cap f(\top) = f(U_0 \cap U_1) = \emptyset$. Since $f(\top) \notin V_0 \cap V_1$, we conclude that $f(\top)$ is an isolated point of $f(Y^T)$. The lemma is proved.

Lemma 9 completes the proof of the theorem. □

Thus, a $T_0$-space $X$ possesses the $bc$-hull if and only if $X$ possesses the injective hull; moreover, the injective hull coincides with the $bc$-hull or is obtained from the $bc$-hull by adding the new isolated largest element.

In [5], the following characterization of spaces that possess the injective hull is obtained:

A $T_0$-space $X$ possesses the injective hull if and only if for any open subset $U \subseteq X$ and point $\xi \in U$ there exists a finite set $\xi_0, \ldots, \xi_n$ of points in $X$ and a family of open sets $U_0, \ldots, U_n$ such that $\xi_i \prec \xi$, $\xi_i \in U_i$ for all $i \leq n$, and $\bigcap_{i \leq n} U_i \subseteq U$.
In conclusion, we present a simple example of a space that satisfies the conditions of the above characterization and is not an $\alpha$-space.

Let $\mathcal{S}$ be an infinite set, and let $P(\mathcal{S})$ be the family of all subsets of $\mathcal{S}$ endowed with the Scott topology. Consider the subspace

$$X = \{ S_0 \mid S_0 \subseteq \mathcal{S}, S_0 \text{ is infinite or it contains at most one element}\}$$

of $P(\mathcal{S})$. The injective hull of $X$ is $P(\mathcal{S})$, whereas $X$ is not an $\alpha$-space.

References


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