

## ON EXPONENTIAL AUTOREGRESSIVE TIME SERIES MODELS

Momčilo Novković<sup>1</sup>

**Abstract.** This paper presents some recent time series models (the so-called NAREX(1) models) for exponential variables with first order autoregressive structures. They are analogs of the standard AR(1) model and of the EAR(1), NEAR(1), TEAR(1) and AREX(1) models, introduced by Lawrance, Lewis, Gaver, Mališić and others. Some of their models can be obtained from NAREX models as special cases.

The distribution of the innovation sequences (a probability mixture) and the autoregressive structure of NAREX processes are discussed as well.

*AMS Mathematics Subject Classification (1991):* 62M10, 60G25

*Key words and phrases:* positive - valued time series, autoregressive model, stationary, exponential marginal distribution, NAREX model and probability mixture.

### 1. Introduction

It is a well-known fact that the standard first-order autoregressive model (AR(1) model) for a stationary sequence of random variables  $\{X_t, t \in T = \{0, \pm 1, \pm 2, \dots\}\}$  is defined by the difference equation

$$X_t = \varepsilon_t + \beta X_{t-1}, \quad t \in T$$

where  $\beta$  is a parameter ( $\beta \in (0, 1)$ ) and  $\{\varepsilon_t, t \in T\}$  is a sequence of independent and identically distributed (i.i.d.) random variables.

In the Gaussian AR(1) model, the innovation sequence  $\{\varepsilon_t, t \in T\}$  is a sequence of Gaussian random variables. However, many naturally occurring time series are clearly non-Gaussian and for this reason several different models for the generation of non-Gaussian time series have been recently constructed. For example, in many positive-valued time series we have an exponential marginal distribution. This provides strong motivation for introducing some new time series models using exponential variables.

Some earlier papers on exponential time series models can be summarized as follows.

---

<sup>1</sup>University of Novi Sad, Faculty of Technical Sciences, Institute of Mathematics, Novi Sad, Trg Dositeja Obradovića 6, Yugoslavia

Gaver and Lewis (1980) considered an AR model with an exponential  $\varepsilon(\lambda)$  marginal distribution in the following form:

$$X_t = \begin{cases} \rho\varepsilon_t & \text{w.p. } \rho \\ \rho X_{t-1} + \xi_t & \text{w.p. } (1 - \rho) \end{cases}$$

where  $\rho$  is a parameter ( $0 \leq \rho \leq 1$ ) and  $\xi_t, t \geq 0$  are i.i.d. exponential random variables  $\varepsilon(\lambda)$  with the parameter  $\lambda > 0$ . This is an AR(1) model.

Lawrance and Lewis (1981) discussed some new AR models using exponential variables - TEAR(1):

$$X_t = \begin{cases} (1 - \rho)\varepsilon_t & \text{w.p. } 1 - \rho \\ \alpha(1 - \rho)\varepsilon_t + X_{t-1} & \text{w.p. } \rho \end{cases}$$

and NEAR(1)

$$X_t = \begin{cases} \varepsilon_t + \beta X_{t-1} & \text{w.p. } \rho \\ \varepsilon_t & \text{w.p. } 1 - \rho \end{cases}$$

Finally, Mališić (1987) considered some new AR models with an exponential  $\varepsilon(\lambda)$  marginal distribution - AREX(1) model (and some of its applications)

$$X_t = \begin{cases} \varepsilon_t & \text{w.p. } p_0 \\ \alpha X_{t-1} & \text{w.p. } p_1 \\ \beta X_{t-1} + \varepsilon_t & \text{w.p. } q_1 \end{cases}$$

where  $0 \leq p_0, p_1, q_1 \leq 1, p_0 + p_1 + q_1 = 1, 0 < \alpha, \beta < 1$ .

In this paper we present a new form of time series models where marginal distributions are in fact exponential distributions. The forms of these models (the so-called NAREX models) are quite distinct from the previous forms of time series models using exponential variables.

## 2. NAREX(1) models

Let the stationary sequence of random variables  $\{X_t, t \in T\}$  be defined by the equation

$$(2.1) \quad X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t & ; p_0 \\ \beta X_{t-1} + \varepsilon_t & ; p_1 \\ \gamma X_{t-1} & ; p_2, \end{cases}$$

where  $0 \leq p_0, p_1, p_2 \leq 1, p_0 + p_1 + p_2 = 1, 0 < \alpha, \beta, \gamma < 1$  and  $\varepsilon_n$  are some i.i.d. random variables. Let us also, suppose that  $\{X_t\}$  and  $\{\varepsilon_t\}$  are "semi-independent", i.e. that  $X_t$  and  $\varepsilon_m$  are independent if  $t < m$ .

Our first objective is to obtain the distribution of the i.i.d. sequence  $\{\varepsilon_t\}$  which will ensure that the sequence  $\{X_n\}$  in (2.1) has an exponential marginal distribution  $\varepsilon(\lambda)$ .

Let the Laplace-Stieltjes transformation of the  $X$  and  $\varepsilon$  variables be denoted by

$$(2.2) \quad \phi_X(s) = E(e^{-sX}), \quad \phi_\varepsilon(s) = E(e^{-s\varepsilon})$$

We know that the following applies

$$(2.3) \quad \phi_X(s) = \frac{\lambda}{\lambda + s}$$

if  $X$  has an  $\varepsilon(\lambda)$  distribution. If we assume stationarity and "semi-independence", then (2.1), (2.2) and (2.3) give

$$\phi_X(s) = p_0\phi_X(\alpha s)\phi_\varepsilon(s) + p_1\phi_X(\beta s)\phi_\varepsilon(s) + p_2\phi_X(\gamma s)$$

$$\phi_\varepsilon(s) = \frac{\phi_X(s) - p_2\phi_X(\gamma s)}{p_0\phi_X(\alpha s) + p_1\phi_X(\beta s)},$$

$$(2.4) \quad \phi_\varepsilon(s) = \frac{\frac{\lambda}{\lambda+s} - p_2\frac{\lambda}{\lambda+\gamma s}}{p_0\frac{\lambda}{\lambda+\alpha s} + p_1\frac{\lambda}{\lambda+\beta s}} = \frac{(\lambda + \alpha s)(\lambda + \beta s)[(p_0 + p_1)\lambda + (\gamma - p_2)s]}{(\lambda + s)(\lambda + \gamma s)[(p_0 + p_1)\lambda + (p_0\beta^2 + p_1\alpha)s]}.$$

Let  $\alpha, \beta, \gamma, p_0, p_1$  and  $p_2$  be chosen in such a way that in (2.4) there are no common fractions. In this case we shall have

$$\phi_\varepsilon(s) = A_0 + A_1\frac{\lambda}{\lambda + s} + A_2\frac{\lambda}{\lambda + \gamma s} + A_3\frac{\lambda}{(p_0 + p_1)\lambda + (p_0\beta + p_1\alpha)s}.$$

Further calculation gives

$$A_0 = \frac{\alpha\beta(\gamma - p_2)}{\gamma(p_0\beta + p_1\alpha)}, \quad A_1 = \frac{(1 - \alpha)(1 - \beta)}{(1 - \beta)p_0 + (1 - \alpha)p_1},$$

$$A_2 = \frac{-p_2(\gamma - \alpha)(\gamma - \beta)}{\gamma[(\gamma - \beta)p_0 + (\gamma - \alpha)p_1]},$$

$$A_3 = \frac{p_0p_1(p_0 + p_1)(\alpha - \beta)^2[\gamma - (p_0\beta + p_1\alpha + p_2)]}{[(1 - \alpha)p_1 + (1 - \beta)p_0][\gamma - \alpha)p_1 + (\gamma - \beta)p_0]}.$$

It is obvious that  $A_1, A_2, A_3$  and  $\frac{A_3}{p_0 + p_1}$  are the probabilities whose sum equals to 1 if and only if

1. if  $\alpha < p_2 + \alpha p_1 + \beta p_0 < \beta$  (or  $\beta < p_2 + \alpha p_1 + \beta p_0 < \alpha$ ) follow  $\gamma \in (p_2 + \alpha p_1 + \beta p_0, \beta)$  (or  $\gamma \in (p_2 + \alpha p_1 + \beta p_0, \alpha)$ ),
2. if  $p_2 < \alpha < \frac{\alpha p_1 + \beta p_0}{p_0 + p_1} < \beta$  (or  $p_2 < \beta < \frac{\alpha p_1 + \beta p_0}{p_0 + p_1} < \alpha$ ) follow  $\gamma \in (p_2, \alpha)$  (or  $\gamma \in (p_2, \beta)$ ).

Then  $\varepsilon_n$  is the following mixture

$$R(\varepsilon_t) = \begin{cases} 0 & ; & A_0 \\ \xi_t & ; & A_1 \\ \gamma\xi_t & ; & A_2 \\ \frac{\alpha p_1 + \beta p_0}{p_0 + p_1} \xi_t & ; & \frac{A_3}{p_0 + p_1}, \end{cases}$$

with  $\xi_t : \varepsilon(\lambda)$ ,  $\lambda > 0$ .

Taking the expectation of both sides in (2.1) we have

$$E\{X_t\} = p_0[\alpha E\{X_{t-1}\} + E\{\varepsilon_t\}] + p_1[\beta E\{X_{t-1}\} + E\{\varepsilon_t\}] + p_2\gamma E\{X_{t-1}\}$$

or

$$E(\varepsilon_t) = \frac{1 - (\alpha p_0 + \beta p_1 + \gamma p_2)}{p_0 + p_1} \cdot \frac{1}{\lambda}.$$

It follows that the autocovariance function for the NAREX(1) time series model (2.1) is defined by

$$\begin{aligned} \gamma(h) &= E(X_t X_{t-h}) - E(X_t)E(X_{t-h}) = \\ &= (\alpha p_0 + \beta p_1 + \gamma p_2)\gamma(h-1) = \dots = (\alpha p_0 + \beta p_1 + \gamma p_2)^h \gamma(0) \end{aligned}$$

so that the autocorrelation function of  $\{X_t\}$  is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = (\alpha p_0 + \beta p_1 + \gamma p_2)^h, \quad h > 0.$$

### 3. Some special cases

- a) If  $p_0 = p_2 = 0$ , the EAR(1) model will apply
- b) If  $\alpha = p_2 = 0$ , the NEAR(1) model will apply
- c) If  $\alpha = 0$ ,  $\gamma = \alpha$ ,  $p_2 = p_1$  and  $p_1 = q_1$ , the AREX(1) model will apply
- d) FNAREX(1)

Let  $p_2 = 0$ . Then

$$X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t & \text{w.p. } p_0 \\ \beta X_{t-1} + \varepsilon_t & \text{w.p. } p_1 = 1 - p_0 \end{cases}$$

and

$$\phi_\varepsilon(s) = \frac{(\lambda + \alpha s)(\lambda + \beta s)}{(\lambda + s)(\lambda + \delta s)}.$$

We shall have

$$\phi_\varepsilon(s) = \frac{\alpha\beta}{\delta} + \frac{(1-\alpha)(1-\beta)}{1-\delta} \cdot \frac{\lambda}{\lambda+s} + \frac{(\delta-\alpha)(\beta-\delta)}{\delta(1-\delta)} \cdot \frac{\lambda}{\lambda+\delta s}$$

$$R(\varepsilon_t) = \begin{cases} 0 & \text{w.p. } \frac{\alpha\beta}{\delta} \\ \xi_t & \text{w.p. } \frac{(1-\alpha)(1-\beta)}{1-\delta} \\ \delta\xi_t & \text{w.p. } \frac{(\delta-\alpha)(\beta-\delta)}{\delta(1-\delta)} \end{cases}$$

where  $\xi_t; \varepsilon(\lambda), \lambda > 0$ .

## References

- [1] Gaver, D.G., Lewis, P.A.W., First-order autoregressive Gamma sequences and Point processes, *Adv. Appl. Prob.* 12 (1980), 727-745
- [2] Lawrance, A.J., Lewis, P.A.W., A new autoregressive time series model in exponential variables (NEAR(1)), *Adv. Appl. Prob.* 13 (1981), 826-845
- [3] Mališić, J., On exponential autoregressive time series models, P. Bauer et al. (eds.), *Math. Statist. And Prob. Theory*, 8 (1987), 147-153

*Received by the editors March 3, 1998.*