FIXED POINTS IN TWO METRIC SPACES

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Abstract. We give some fixed point theorems in two complete metric spaces. Then we improve and extend some results due to D. Dalučo, B. Fisher and V. Popa.

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In [4], to give a unified approach for contraction mappings, D. Dalić considered the set $F$ of all continuous functions $g : [0, +\infty)^3 \to [0, +\infty)$ satisfying the following conditions:

(1) $g(1, 1, 1) = h < 1,$
(2) $u, v, w \in [0, +\infty)$ are such that $u \leq g(v, v, u) + u \leq g(v, v, v)$ or $u \leq g(u, u, v) + u \leq g(u, v, v),$ then $u \leq hu,$

and proved the following:

Theorem A. Let $(X, d)$ be a complete metric space. If $S$ and $T$ are two mappings from $X$ into itself, satisfying the following conditions:

\[ d(Sx, Ty) \leq g(d(x, y), d(x, Sx), d(y, Ty)) \]

for all $x, y \in X,$ where $g \in F,$ then $S$ and $T$ have a unique common fixed point in $X.$

Some authors proved many kinds of fixed point theorems for contraction type mappings and expansive mappings by using Dalić’s set ([1]-[3], [7], [8], [10]). On the other hand, in [5] and [6], B. Fisher proved some fixed point theorems in two complete metric spaces as follows:

Theorem B. Let $(X, d)$ and $(Y, e)$ be complete metric spaces, if $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X,$ satisfying the following conditions:

\[ e(Tx, TSy) \leq c \cdot \max\{d(x, Sy), e(y, Tsy), c(y, Tsy)\} \]

\[ d(Sy, STx) \leq c \cdot \max\{e(y, Tsy), d(x, Sy), d(x, STx)\} \]

for all $x, y \in X,$ where $0 \leq c < 1,$ then $ST$ have a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y.$ Further, $Tz = w$ and $Sw = z.$

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Recently, in [9], V. Popa extended and improved the results of B. Fischer and V. Popa's results, we introduce a new class $G$ of all functions $g : [0, +\infty)^3 \rightarrow [0, +\infty)$ satisfying some conditions and prove some fixed point theorems in two complete metric spaces by using our class. Our results also extend and improve the results of B. Fisher [5], [6] and V. Popa [9].

Let $G$ be the set of all continuous functions $g : [0, +\infty)^3 \rightarrow [0, +\infty)$ satisfying the following conditions:

(1) $g(0, 0, 0) = 0.$
(2) If $u, v \in [0, +\infty)$ be such that $u^2 \leq g(u, v, 0)$ or $v^2 \leq g(0, u, 0)$ or $u^2 \leq g(0, 0, u)$, then $u \leq v$ for some $0 \leq c < 1.$

**Example 1.** (1) If we define a function $g : [0, +\infty)^3 \rightarrow [0, +\infty)$ by

$$g(u, v, w) = c \cdot \max\{u, v, w\}$$

for all $u, v, w \in [0, +\infty)$, then $g \in G.$

(2) If we define a function $g : [0, +\infty)^3 \rightarrow [0, +\infty)$ by

$$g(u, v, w) = c \cdot \max(uw, vw, uv)$$

for all $u, v, w \in [0, +\infty)$, where $0 \leq c < 1$, then $g \in G.$

(3) If we define a function $g : [0, +\infty)^3 \rightarrow [0, +\infty)$ by

$$g(u, v, w) = u + w - bw + cuw$$

for all $u, v, w \in [0, +\infty)$, where $0 \leq b, c < 1$, then $g \in G.$

(4) If we define a function $g : [0, +\infty)^3 \rightarrow [0, +\infty)$ by

$$g(u, v, w) = (ua^k + bv + cw)^2$$

for all $u, v, w \in [0, +\infty)$, where $k > 1, 0 < a, b, c < 1$, then $g \in G.$

Now, we give our theorem as follows:

**Theorem 1.** Let $(X, d)$ and $(Y, e)$ be two complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions:

(1) $e(Tx, Ty) \leq d(Sy, Ty) + d(x, Sy) + e(y, Ty) + e(y, Tx) + e(Ty, Ty),$ $d(Sy, Ty)$

(2) $d(Tg(x), Ty) \leq d(Tg(x), Ty) + d(x, Ty) + d(Ty, Ty),$ $d(Tg(x), Ty)$

for all $x \in X$ and $y \in Y$, where $g \in G,$ then $ST$ has a unique fixed point $z \in X$ and $TS$ has a unique fixed point $w \in Y.$ Further, $TS = w$ and $ST = z.$
Proof. Let \( x_0 \) be an arbitrary point in \( X \). Define two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) and \( Y \) respectively, as follows:

\[
x_n = (ST)^{n-1}x_0, \quad y_n = T(ST)^{n-1}x_0
\]

for \( n = 1, 2, \ldots \). By (D), we have

\[
d(x_n, x_{n+1}) = d((ST)^{n-1}x_0, (ST)^{n-1}x_0)
\]

\[
= d((ST)^{n-1}x_0, (ST)^{n-1}x_0)
\]

\[
= d(y_n, y_{n+1}) \leq g(c(y_n, y_{n+1}), d(x_n, x_{n+1}), c(y_n, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}))
\]

Thus, by (g'2), we have

\[
d(x_n, x_{n+1}) \leq c(y_n, y_{n+1})
\]

for some \( 0 \leq c_1 < 1 \). Similarly, by (D),

\[
e(x_n, y_{n+1}) \leq e(x_n, y_{n+1})
\]

\[
= e((ST)^{n-1}x_0, (ST)^{n-1}x_0)
\]

\[
= e(y_n, y_{n+1}) \leq g(c(y_n, y_{n+1}), d(x_n, x_{n+1}), c(y_n, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}))
\]

Thus, by (g'2), we have

\[
e(y_n, y_{n+1}) \leq e(y_n, y_{n+1})
\]

for some \( 0 \leq c_2 < 1 \). Therefore, by (F) and (G),

\[
d(x_n, x_{n+1}) \leq \cdots \leq c_n d(x_1, x_n)
\]

which implies that \( \{x_n\} \) is a Cauchy sequence in \((X, d)\) since \( 0 < c_1 < 1 \) and \( \cdots \). Since \((X, d)\) is complete, it converges to a point \( z \) in \( X \). Similarly, the sequence \( \{y_n\} \) is also a Cauchy sequence in \((Y, e)\) with the limit \( w \). By (D),
again, we have

$$e^2(Tz, y_{n+1}) = e^2(Tz, TSy_n)$$

and so, $e(Tz, w) = 0$, i.e., $Tz = w$. On the other hand, by (E) we have

$$d^2(Su_n, x_{n+1}) = d^2(Su_n, (ST)^{n-1} x_n)$$

and so, $d(Su_n, z) = 0$, i.e., $Su_n = z$. Therefore, we have $STz = Sw = z$ and $TSw = Tz = w$, which means that the point $z$ is a fixed point of $ST$ and the point $w$ is a fixed point of $TS$.

To prove the uniqueness of the fixed point $z$, let $z'$ be the second fixed point of $ST$. By (D), we have

$$d^2(z', z') = d^2(STz', STz)$$

which, by (g'-2), implies that

$$d(z', z) = c_{ST}(Tz', Tz)$$

for some $0 < c_3 < 1$. Similarly, by (D), we have

$$e^2(Tz, Tz') = e^2(Tz, TSz)$$
Fixed points in two metric spaces

Thus, by (q-2), it follows that

\[(K)\]

\[c(tx, tx') \leq c(d(x', z))\]

for some 0 \leq c < 1. Therefore, by (J) and (K),

\[d(y, x') \leq c \cdot c(tz, tx') \leq c^2 d(y, x')\]

which implies that \(d(z, z') = 0\), i.e., \(z = z'\), since 0 \leq c \leq 1 and so the uniqueness of the fixed point \(n\) of \(ST\) follows. Similarly, the point \(w\) is also a unique fixed point of \(TS\). On the other hand, if these exist a positive integer \(n\) such that \(d(x_n, x_{n+1}) = 0\) or \(d(y_n, y_{n+1}) = 0\), then the theorem is evident. This completes the proof.

As immediate consequences of Theorem 1, we have the following:

**Corollary 2.** Let \((X, d)\) and \((Y, e)\) be two complete metric spaces. If \(T\) is a mapping from \(X\) into \(Y\) and \(S\) is a mapping from \(Y\) into \(X\) satisfying the following conditions:

\[(L)\]

\[c^2(TX, TSX) \leq c_1 \cdot \max\{d(x, SY)\} + \min\{d(x, SY)\}, e(y, TSX)\]

\[(M)\]

\[d^2(SY, STX) \leq c_2 \cdot \max\{e(y, TZX)\} + \min\{e(y, TZX)\}, d(x, STX)\]

d for all \(x \in X\) and \(y \in Y\), where \(0 < c, c_1 < 1\), then \(ST\) has a unique fixed point in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(TS = w\) and \(ST = z\).

**Proof.** Define a function \(g : [0, +\infty) \rightarrow (0, +\infty)\) by

\[g(u, v, w) = u \cdot c \cdot \max\{u, v, w\}\]

for all \((u, v, w) \in [0, +\infty)\), where \(0 < c < 1\). Then, from Example 1 (2) follows that \(g \in \mathcal{G}\) and, by Theorem 1, the corollary follows.

**Corollary 3.** Let \((X, d)\) and \((Y, e)\) be two complete metric spaces. If \(T\) is a mapping from \(X\) into \(Y\) and \(S\) is a mapping from \(Y\) into \(X\) satisfying the following conditions:

\[(N)\]

\[c^2(TX, TSX) \leq d(x, SY) + b_1 d(x, SY) + d(x, TSX)\]

\[(O)\]

\[d^2(SY, STX) \leq a_2 e(y, TZX) + b_2 e(y, TZX) + c_2 d^2(SY, STX)\]

d for all \(x \in X\) and \(y \in Y\), where \(a_1, a_2, b_1, b_2, c_1, c_2 \in [0, +\infty]\) with \((a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1\), then \(ST\) has a unique fixed point \(w\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(TS = w\) and \(ST = z\)
Proof. Define a function \( g : [0, +\infty]^3 \to [0, +\infty) \) by
\[
g(a, v, w) = aw + bw + cw
\]
for all \( a, v, w \in [0, +\infty) \), where \( a, b, c \in [0, +\infty) \). Then, from Example 1 (3), follows that \( g \in \mathcal{G} \) and, by Theorem 1, the corollary follows.

Corollary 4. Let \((X, d)\) and \((Y, e)\) be two complete metric spaces. If \( T \) is a mapping from \( X \) into \( Y \) and \( S \) is a mapping from \( Y \) into \( X \) satisfying the following conditions:
\[
\begin{align*}
(P) & \quad d^2(Tx, TSy) \leq a_2d_2^2(z, Sz) + b_2d_2^2(y, Ty) + c_2d_2^2(y, TSz), \\
(Q) & \quad d^2(Sy, STz) \leq a_2d_2^2(y, Ty) + b_2d_2^2(x, Sz) + c_2d_2^2(x, STz)
\end{align*}
\]
for all \( x \in X \) and \( y \in Y \), where \( 0 \leq a_2, b_2, c_2 < 1 \), then \( ST \) has a unique fixed point \( z \in X \) and \( TS \) has a unique fixed point \( w \in Y \). Further, \( Tx = w \) and \( Sw = z \).

Proof. Define a function \( g : [0, +\infty]^3 \to [0, +\infty) \) by
\[
g(u, v, w) = au^2 + bv^2 + cw^2
\]
for all \( u, v, w \in [0, +\infty) \), where \( 0 < a, b, c < 1 \). Then \( g \in \mathcal{G} \) and, by Theorem 1, the corollary follows.

If \((X, d)\) and \((Y, e)\) are the same metric spaces, then by Theorem 1, we have the following:

Theorem 5. Let \((X, d)\) be a complete metric space. If \( S \) and \( T \) are mappings from \( X \) into itself satisfying the following conditions:
\[
\begin{align*}
(R) & \quad d^2(Tx, TSy) \leq g(d(x, Sz)d(y, Ty), d(z, Sz)d(y, TSz)), \\
(S) & \quad d^2(Sy, STz) \leq g(d(y, Ty)d(x, Sz), d(y, TSz)d(x, STz))
\end{align*}
\]
for all \( x, y \in X \), where \( g \in \mathcal{G} \), then \( ST \) has a unique fixed point \( z \in X \) and \( TS \) has a unique fixed point \( w \in Y \). Further, \( Tz = w \) and \( Sw = z \) and, if \( z = w \), then \( z \) is the unique common fixed point of \( T \) and \( S \).

Corollary 6. Let \((X, d)\) be a complete metric space. If \( S \) and \( T \) are mappings from \( X \) into itself satisfying the following conditions:
\[
\begin{align*}
(T) & \quad d^2(Tx, TSy) \leq c_1 \max\{d(x, Sz)d(y, Ty), d(x, Sz)d(y, TSz), \\
(U) & \quad d^2(Sy, STz) \leq c_2 \max\{d(y, Ty)d(x, Sz), d(y, TSz)d(x, STz), \\
& \quad d(z, Sz)d(x, STz))
\end{align*}
\]
for all \( x, y \in X \), where \( 0 \leq c_1, c_2 < 1 \), then \( ST \) has a unique fixed point \( z \in X \) and \( TS \) has a unique fixed point \( w \in X \). Further, \( Tz = w \) and \( Sw = z \) and, if \( z = w \), then \( z \) is the unique common fixed point of \( S \) and \( T \).
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