BISEMILATTICE-VALUED FUZZY SETS

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Abstract

Bisemilattice-valued fuzzy set (B-fuzzy set) is defined to be a mapping from a nonempty set to a bisemilattice (an algebraic structure with two binary operations, both commutative, associative and idempotent).

To each B-fuzzy set corresponds two collections of subsets, levels (cuts) of that fuzzy set. These two families are determined by two ordering relations on a bisemilattice. Theorems of decomposition of B-fuzzy sets into levels are formulated and proved in the paper. Starting from two families of subsets of a set X, we give conditions for synthesis of a B-fuzzy set. Some other properties of B-fuzzy sets are also formulated.

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1. Preliminaries

Bisemilattice

A bisemilattice \( B = (H, \wedge, \vee) \) is an algebra of the type \((2, 2)\), where \((Y, \wedge)\) and \((H, \vee)\) are semilattices. Therefore, the operations \(\wedge\) (called "meet") and \(\vee\) ("join") satisfy commutative, associative and idempotency laws. Since a
lattice is a bisemilattice satisfying the absorption laws, bisemilattices are a
generalization of lattices.

A bisemilattice was introduced by J. Plonka in [6] under the name of

There are two orderings corresponding to a bisemilattice \((B, \land, \lor)\) [2]:

\[ x \leq_\lor y \text{ if and only if } x \lor y = y \]

and

\[ x \leq_\land y \text{ if and only if } x \land y = x. \]

\[ x \geq_\lor y \text{ is a notation for } y \leq_\lor x, \text{ and } x \geq_\land y \text{ for } y \leq_\land x. \]

Since a bisemilattice \((B, \land, \lor)\) is uniquely represented by two orderings,
it can also be considered as a relational system, as follows. \((B, \leq_\land, \leq_\lor)\) is a
bisemilattice, if \(B\) is a nonempty set and \(\leq_\land, \leq_\lor\) are ordering relations on
\(B\), such that \((B, \leq_\lor)\) is a join-semilattice (\(V\)-semilattice, i.e., the poset in
which every two-element subset has the least upper bound), and \((B, \leq_\land)\) is
a meet-semilattice (\(\land\)-semilattice: each two-element subset has the greatest
lower bound).

**Remark.** Naturally, a diagram of a bisemilattice consists of two Hasse
diagrams, one for each ordering. We use the following convention: if \(x \leq_\lor y\)
and \(z \leq_\land t\), then \(z\) is below \(y\), and \(z\) below \(t\) in the
**corresponding** diagrams.

**Fuzzy set**

As is known, a fuzzy set (by the original definition of Zadeh) is a mapping
from a nonempty set to the real interval \([0, 1]\). This notion has been generalized
in several steps. A codomain was first taken to be a Boolean algebra
instead of a real interval, later on it was taken to be a lattice, then a par-
tially ordered set and finally a relational system. By taking a special kind
of a relational system, one gets the most general concept in this direction
([8]) (the starting point is the lattice as a relational system). In another
direction, when the lattice is considered to be an algebra of the type \((2,2)\),
a more general approach could be obtained by taking a weaker structure of the
same type, instead of a lattice. This is a subject of the present article.

We advance some common elementary notions and properties for any of
the above mentioned (and already defined) fuzzy sets, concerning the
ordering relation included in the codomain.
Let \( P \) be a foregoing ordered structure (real interval, lattice, poset, etc.) with the order \( \leq \), and \( A : P \rightarrow P \) a \( P \)-fuzzy subset of a nonempty set \( A \) (or, a \( P \)-fuzzy set on \( A \)). Then, for \( p \in P \), a level set (level, \( p \)-cut) of \( A \) is a (crisp) subset \( A_p \) of \( A \), such that \( x \in A_p \) if and only if \( A(x) \geq p \).

The characteristic function of \( A_p \) is denoted by \( \bar{A}_p \) and is called the level function of \( A \). Its codomain is the set \( \{0,1\} \). It is well known that for \( p, q \in P \), \( p \leq q \) implies \( A_p \subseteq A_q \).

2. Bismellatite-Valued Fuzzy Sets

In the sequel we investigate bismellatite-valued fuzzy sets, and we give some of their properties, specially connected with the decomposition and synthesis of a fuzzy set.

A bismellatite valued fuzzy set (\( B \)-fuzzy set) is a mapping \( \bar{A} : X \rightarrow B \) from a nonempty set \( X \) to a bismellatite \( B = (B, \wedge, \vee) \).

For each \( p \in B \), there are two level subsets defined as follows:

\[
A^*_p = \{x \in X | \bar{A}(x) \geq \vee p\}
\]

and

\[
A^*_p = \{x \in X | \bar{A}(x) \geq \wedge p\}.
\]

And the corresponding level functions are:

\[
\bar{A}^*_p(x) = 1 \text{ if and only if } \bar{A}(x) \geq \vee p
\]

and

\[
\bar{A}^*_p(x) = 1 \text{ if and only if } \bar{A}(x) \geq \wedge p.
\]

Thus, for a \( B \)-fuzzy set \( \bar{A} : X \rightarrow B \), there are two families of level subsets:

\[
A^*_B = \{A^*_p | p \in B\}, \text{ and } A^*_B = \{A^*_p | p \in B\}.
\]

Let \( \bar{A} : X \rightarrow B \) be a \( B \)-fuzzy set. Relation \( \sim \) on \( X \), given by: \( x \sim y \) if and only if \( \bar{A}(x) = \bar{A}(y) \) is an equivalence relation. The corresponding partition of the set \( X \) is said to be the partition induced by the \( B \)-fuzzy set \( \bar{A} \). Similarly to the notions given above, this one (partition) is also independent of the fact that \( \bar{A} \) is a \( B \)-fuzzy set; it is defined in the same way for other types of fuzzy sets.

Recall that the smallest element in an ordered set, if such an element exists, is denoted by \( 0 \), and the greatest, if it exists, by \( 1 \).
Theorem 1. Let $X : X \rightarrow B$ be a $B$-fuzzy set. Then, the following is satisfied.

1. If there is a bottom element $0$ in $(B, \wedge)$, then
   
   $$A_0^B = X.$$ 

2. If $p \leq_B q$, then $A^*_p \subseteq A^*_q$, and if $p \leq_B q$, then $A^*_p \subseteq A^*_q$.

3. 
   
   $$\overline{A}(x) = \bigvee_{p \in B} A^*_p(x) = 1;$$
   
   $$\overline{A}(x) = \bigvee_{p \in B} A^*_p(x) = 1$$

(i.e., the supremum at the right side of each equality exists for every $x$ and is equal to $\overline{A}(x)$).

Proof. 

1. Indeed,
   
   $$A_0^B = \{ x \in X \mid \overline{A}(x) \geq_B 0 \} = X.$$ 

2. Let $p \leq_B q$. Then $x \in A^*_p$ if and only if $\overline{A}(x) \geq_B q \geq_B p$, i.e., $x \in A^*_p$. The proof goes analogously for $p \leq_B q$. 

3. Let $\overline{A}(x) = r \in B$. Then $A^*_p(x) = 1$. If for $p \in B$, $A^*_p(x) = 1$, then $r = \overline{A}(x) \geq_B p$. Hence, $r$ is the required supremum, and
   
   $$\overline{A}(x) = \bigvee_{p \in B} A^*_p(x) = 1.$$ 

The proof is the same for the second equality. □

Theorem 2. Let $\overline{A} : X \rightarrow B$ be a $B$-fuzzy set. Then, the following holds.

1. If $B_1 \subseteq B_2$, then
   
   $$\bigcap\{ A^*_p \mid p \in B_1 \} = A^*_{\bigvee_{p \in B_1} p}.$$ 

If for $B_1 \subseteq B$ there exists the supremum $\bigvee_{p \in B_1}$, then,

$$\bigcap\{ A^*_p \mid p \in B_1 \} = A^*_{\bigvee_{p \in B_1} p}.$$ 

2. 

   $$\bigcup\{ A^*_p \mid p \in B \} = X \quad \text{and} \quad \bigcup\{ A^*_p \mid p \in E \} = X;$$

3. For all $x \in X$,
   
   $$\bigcap\{ A^*_p \mid x \in A^*_p \} \in \overline{A}_B^* \quad \text{and} \quad \bigcap\{ A^*_p \mid x \in A^*_p \} \in \overline{A}_B^*.$$
Proof. 1. Since the semilattice \((B, \land, \lor)\) is complete, the supremum of \(B_1 \subseteq B\) exists, and for \(x \in X\),
\[ x \in \bigcap \{ A^y_p | p \in B_1 \}, \text{ if and only if } x \in A^y_p \text{ for all } p \in B_1, \text{ if and only if } \bar{A}(x) \geq \lor \{p | p \in B_1\}, \text{ if and only if } \bar{A}^y_{\lor \{p | p \in B_1\}}(x) = 1, \text{ if and only if } x \in \bigvee_{y \in Y, p \in B_1} A^y_p. \]

The proof of the second equality is similar.

2. If \(x \in X\), then \(\bar{A}(x) = p \in B\). Hence, \(x \in A^y_p\) and \(x \in A^z_p\). Thus, \(x \in \bigcup \{A^y_p | p \in B\}\) and \(x \in \bigcup \{A^z_p | p \in B\}\), and the required equalities are satisfied.

3. Let \(x \in X\). Since \(x \in A^y_p\) if and only if \(\bar{A}(x) \geq \lor \{p | p \in B\}\) if and only if \(\bar{A}^y_{\lor \{p | p \in B\}}(x) = 1\) by Theorem 1 (3) and by the first part of this theorem,
\[ \bigcap \{A^y_p | x \in A^y_p\} = \bigcap \{A^y_p | \bar{A}_p(x) = 1\} = A^y_{\bigvee_{y \in Y, p \in B} \{p | p \in B\}}. \]

The proof is the same for the second equality. \(\square\)

An important difference between \(B\)-fuzzy sets and the \(L\)-valued ones is that the collection of level sets of an \(L\)-fuzzy set is always a lattice (under inclusion), but the corresponding collection for a \(B\)-fuzzy set is not a biordered lattice. The reason is that the collections of level subsets \(A^y_p\) and \(A^z_p\) of a \(B\)-fuzzy set \(A\) do not coincide; in general, they even do not have the same cardinality. The following example illustrates this case.

Example 1.

Let \((B, \land, \lor)\) be a bisemilattice in Fig. 1, let \(X = \{x, y, z\}\), and a \(B\)-fuzzy set be given by:
\[ \bar{A} = \begin{pmatrix} x & y & z \\ a & c & d \end{pmatrix}. \]
The corresponding families of level sets are: \( A_x' = \{ x \}; A_y' = \emptyset; A_{xy}' = \{ y \}; A_y^x = \{ y, z \}; A_{xy}^y = \{ x, y \} \).\\n\( A_x'' = \{ x, y, z \}; A_y'' = \emptyset; A_{xy}'' = \{ y \}; A_y^x = \{ y, z \}; A_{xy}^y = \emptyset \).

Hence,\\n\( A_y' = \{ \emptyset, \{ x \}, \{ y \}, \{ y, z \}, \{ x, y \} \} \) and\\n\( A_y'' = \{ \emptyset, \{ x, y, z \}, \{ y \}, \{ y, z \} \} \).

The following theorem gives conditions under which two collection of subsets of a set \( X \) are families of the level subsets of a \( B \)-fuzzy set.

Let \( B \) be a family of subsets of a nonempty set \( X \) union of which is \( X \), such that for every \( x \in X \),

\[ \bigcap \{ p \in B \mid x \in p \} \in B. \]

For every \( x \in X \), let

\[ B(x) := \bigcap \{ p \in B \mid x \in p \}. \]

Further, let \( \pi_B(x) := \{ y \in X \mid B(y) = B(x) \} \). The collection \( \{ \pi_B(x) \mid x \in X \} \) is, by the construction, partition of \( X \), and we shall call it the \textbf{partitions induced by} \( B \). Every partition of \( X \) is induced by some family of the above type: the simplest one is the partition itself, together with the empty set. Obviously, different families can induce the same partition.

\textbf{Theorem 3.} (Theorem of synthesis) Let \( X \) be a nonempty set, and \( \Pi \) a partition of \( X \). Further, let \( B_1 \subseteq \mathcal{P}(X) \), \( B_2 \subseteq \mathcal{P}(X) \) be two families of the subsets of \( X \), satisfying the following.

(i) \( |B_1| = |B_2| \).
(ii) The poset \((B_1, \subseteq)\) is a \(\wedge\)-semilattice and \((B_2, \subseteq)\) is a \(\vee\)-semilattice.

(iii) \(\bigcup B_i = \bigcup B_2 = X\).

(iv) For every \(x \in X\)

\[\bigcap \{p \in B_1 \mid x \in p\} \in B_1 \quad \text{and} \quad \bigcap \{p \in B_2 \mid x \in p\} \in B_2.\]

(v) Both \(B_1\) and \(B_2\) induce the partition \(\Pi\).

Let \(\overline{A}^\wedge : X \longrightarrow B_1\) be a join-fuzzy set, defined by:

\[\overline{A}^\wedge(x) = \bigcap \{p \in B_1 \mid x \in p\},\]

as a mapping to a semilattice \((B_1, \leq_v)\), where \(p \leq_v q\) if and only if \(q \subseteq p\).

Similarly, let \(\overline{A}^\vee : X \longrightarrow B_2\) be a meet-fuzzy set, defined by:

\[\overline{A}^\vee(x) = \bigcap \{p \in B_2 \mid x \in p\},\]

as a mapping to a semilattice \((B_2, \leq_s)\), where \(p \leq_s q\) if and only if \(q \subseteq p\).

Then, there is a bisemilattice \(\mathcal{B} = (B_1, \leq_v, \leq_s)\), and a \(B\)-fuzzy set \(\overline{A} : X \longrightarrow B\), such that \(B_1\) and \(B_2\) are collections of level sets of \(\overline{A}\).

**Proof.**

Since \(B_1\) and \(B_2\) are of the same cardinality, and partitions on \(X\) induced by \(\overline{A}_L\) and by \(\overline{A}_R\) are the same (straightforward by (v)), we have that also the sets \(C_1 = B_1 \setminus \overline{A}^\wedge(X)\) and \(C_2 = B_2 \setminus \overline{A}^\vee(X)\) have the same cardinality.

Let \(\varphi\) be an arbitrary bijection from \(C_1\) to \(C_2\).

Now, we define a bijection \(\varphi : B_1 \longrightarrow B_2\) in the following way:

If \(p \in B_1\) is the image of an \(x \in X\) by \(\overline{A}_L\), i.e., if \(\overline{A}^\wedge(x) = p\) for an \(x \in X\), then let

\[\varphi(p) := \overline{A}^\vee(x) \in B_2.\]

If \(p \in B_1\) is not the image of any \(x \in X\), then let

\[\varphi(p) := \varphi(q).\]

We consider the bisemilattice \(\mathcal{B} = (B_1, \leq_v, \leq_s)\), such that \(B = B_1\), \(p \leq_v q\) is the same as in \(B_1\) and \(p \leq_s q\) in \(B\) if and only if \(\varphi(p) \leq_s \varphi(q)\) in \(B_2\).
Now, the required $B$-fuzzy set (i.e., such that its level sets are collections $B_1$ and $B_2$), is $\bar{A} : X \rightarrow B$, defined by:

$$\bar{A}(x) := \bar{A}^\vee(x).$$

By the theorem of synthesis for semilattices [8], we have that $A_p^\vee = p$, for $p \in B_1$. Hence $A_p^\vee = B_1$.

Further on, $x \in A_p^\vee$ if and only if $\bar{A}(x) \geq_p p$ if and only if $\bar{A}^\vee(x) \geq_p p$ if and only if $\varphi(\bar{A}(x)) \geq_p \varphi(p)$ in $B_2$ if and only if $\varphi(\bar{A}^\vee(x)) \leq_p \varphi(p)$ if and only if $\bar{A}^\vee(x) \leq_p \varphi(p)$.

Since $x \in \bar{A}^{\vee}(x)$, the preceding formula implies that $x \in \varphi(p)$.

Conversely, if $x \in \varphi(p)$, then $\bar{A}^{\vee}(x) \leq_p \varphi(p)$, by the definition of $\bar{A}^{\vee}(x)$.

Hence, $A_p^\vee = \varphi(p)$, for $p \in B$, and since $\varphi$ is a bijection, we have that $A_p^\vee = B_2$. $\square$

The following example illustrates the theorem.

**Example 2.**

Let $X = \{1, 2, 3, 4, 5\}$ and let $\Pi = \{\{1, 2\}, \{3\}, \{4, 5\}\}$. $\Pi$ is a partition on $X$, induced by the following two families of subsets of $X$:

$$B_1 = \{\emptyset, \{3\}, \{1, 2\}, \{4, 5\}\}, \quad B_2 = \{\{3\}, \{1, 2, 3\}, \{4, 5, 6\}, \{1, 2, 3, 4, 5\}\}.$$

It is not difficult to see that the conditions (ii) - (v) of Theorem 3 are satisfied. Further on, we have that:

$$\bar{A}^{\vee} = \left(\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\{1, 2\} & \{1, 2\} & \{3\} & \{4, 5\} \\
\{4, 5\} & \{4, 5\} & \{4, 5\} & \{4, 5\}
\end{array}\right),$$

and

$$\bar{A}^\wedge = \left(\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\{1, 2, 3\} & \{1, 2, 3\} & \{3, 4, 5\} & \{3, 4, 5\} \\
\{3, 4, 5\} & \{3, 4, 5\} & \{3, 4, 5\} & \{3, 4, 5\}
\end{array}\right).$$

By the construction, partitions induced by semilattice-valued fuzzy sets $\bar{A}$ and $\bar{A}^\wedge$ coincide with $\Pi$. The bijection $\varphi : B_1 \rightarrow B_2$ is:

$$\varphi = \left(\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\{1, 2\} & \{1, 2\} & \{3\} & \{4, 5\} \\
\{4, 5\} & \{4, 5\} & \emptyset & \{4, 5\} \\
\{1, 2, 3\} & \{1, 2, 3\} & \{3, 4, 5\} & \{1, 2, 3, 4, 5\}
\end{array}\right).$$
These collections are represented in Figure 2. By the construction described in Theorem 3, we obtain the following fuzzy set:

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & a & b & c & c \end{pmatrix},$$

where we denote \{1, 2\} by \(a\), \{3\} by \(b\), \{4, 5\} by \(c\) and \(\emptyset\) by \(d\).

$$\emptyset \quad \{1, 2\} \quad \{3\} \quad \{4, 5\} \quad \{\{1, 2, 3\}, \{4, 5\}\} = \varphi(\{3\})$$

$$\{1, 2, 3\} = \varphi(\{1, 2\}) \quad \{4, 5\} = \varphi(\{4, 5\})$$

$$\{1, 2, 3, 4, 5\} = \varphi(\emptyset) \quad (B_1, \leq)$$

Figure 2

The obtained fuzzy set is \(\mathcal{A} : X \rightarrow B\), where \(B = \{a, b, c, d\}\), and the biunimodular \(\mathcal{A}\) is given in Figure 3. \(\mathcal{A}\) obviously has \(B_1\) and \(B_2\) as the families of level subsets.

$$\emptyset \quad \{a\} \quad \{b\} \quad \{c\} \quad \{d\} \quad \{a, b, c\} \quad \{a, c\} \quad \{b, c\} \quad \{a, b\} \quad \{a, d\} \quad \{b, d\} \quad \{c, d\} \quad \{a, b, c, d\}$$

$$\{a, b\} \quad \{a, c\} \quad \{b, c\}$$

Figure 3

References


