TRIANGULAR SOLUTIONS OF BOOLEAN EQUATIONS

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Abstract
We give an algorithm which determines the formulas of general reproductive solutions of a given Boolean equation in triangular form. The algorithm also makes simpler formulas of these solutions. This algorithm also does a simplification of the formulas of these solutions.

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Our basic terminology, related to Boolean equations, follows Rudeanu's book [6]. For everything about Boolean equations, not given here, see also [6].

Let $B = (B, \cup, \cap, ^\prime, 0, 1)$ be a Boolean algebra, $n$ be a natural number and $p = 2^n - 1$. Further, let $\{A_0, A_1, \ldots, A_p\} = \{0, 1\}^n$. If $A \in \{0, 1\}^n$ then the $j$-th coordinate of $A$ will be denoted by $(A)_j$, i.e. $A = ((A)_1, \ldots, (A)_n)$.

We shall also use the notation:

$$T_k = (t_1, \ldots, t_k) \ (k = 1, \ldots, n).$$

Especially, if $k = n$ we shall use the notation:

$$T = (t_1, \ldots, t_n) \quad \text{and} \quad X = (x_1, \ldots, x_n).$$
Definition 1. Let $f, \Phi_1, \ldots, \Phi_n : B^n \to B$ be Boolean functions and $\Phi = (\Phi_1, \ldots, \Phi_n)$. The formula

$$X = \Phi(T)$$

or, in a scalar form

$$x_j = \Phi_j(t_1, \ldots, t_n) \quad (j = 1, \ldots, n)$$

defines a general solution of the consistent Boolean equation $f(X) = 0$ if and only if

$$(\forall X \in B^n)(\exists T \in B^n)(f(X) = 0 \leftrightarrow (\exists T \in B^n)X = \Phi(T)).$$

Definition 2. Let $x \in B$. Then

$$x^1 = x, \quad x^0 = x'.$$

If $X = (x_1, \ldots, x_n) \in B^n$ and $A = (a_1, \ldots, a_n) \in \{0, 1\}^n$ then

$$X^A = x_1^{a_1} \cdots x_n^{a_n}.$$

Theorem 1. \cite{[6]} The function $f : B^n \to B$ is Boolean if and only if it can be written in the canonical disjunctive form

$$f(X) = \bigcup_{k=0}^n f(A_k)X^A_k.$$

Theorem 2. \cite{[6]} Let $f : B^n \to B$ be a Boolean function. The equation $f(X) = 0$ is consistent if and only if

$$\prod_{k=0}^n f(A_k) = 0.$$

Theorem 3. \cite{[6]} Let $f : B^n \to B$ be a Boolean function. If $f(X) = 0$ is consistent, then the formula

$$X = \bigcup_{k=0}^p \left( f(A_{1k})A_{1k} \cup f(A_{2k})f'(A_{1k})A_{2k} \cup \left( f(A_{1k})f'(A_{1k})f'(A_{2k})A_{3k} \cup \cdots \cup f(A_{pk})f(A_{1k})f(A_{2k}) \cdots f'(A_{(p-1)k})A_{1k} \right) \right)$$

(1)
defines a general solution of $f(X) = 0$, under conditions

$(i_{a,0}, i_{a,1}, \ldots, i_{a,p})$ are permutations of \(\{0,1,\ldots,p\}\)

and

$(i_{b,0}, i_{b,1}, \ldots, i_{b,p})$ is a permutation of \(\{0,1,\ldots,p\}\).

Bearing in mind the method of successive eliminations, it is known that
every consistent Boolean equation $f(x_1, \ldots, x_n) = 0$ has a triangular general
reproductive solution of the form

$$
z_1 = g_1(t_1) \\
z_2 = g_2(t_1, t_2) \\
\vdots \\
z_n = g_n(t_1, t_2, \ldots, t_n).
$$

We shall prove that the vector $A_{i_{a,b}}$ in Theorem 3 can be chosen such that
the solution (1) is triangular.

**Definition 3.** Let $A = (a_1, \ldots, a_n) \in \{0,1\}^n$. We define a sequence

$$B(A) = B_0(A), B_1(A), \ldots, B_p(A)$$

in the following way:

(I) $B_0(A) = (a_1, \ldots, a_n)$

$B_1(A) = (a_1, \ldots, a_{n-1}, a'_n)$

(II) for every $k \in \{1, \ldots, n-1\}$

$B_{2k}(A) = (a_1, \ldots, a'_{n-k}, D_k(0))$

$B_{2k+1}(A) = (a_1, \ldots, a'_{n-k}, D_k(1))$

$\vdots$

$B_{2k+2^{n-1}-1}(A) = (a_1, \ldots, a'_{n-k}, D_k(2^k - 1))$

where $D_k(s)$ is the $k$-tuple of the binary digits of the number $s \in \{0,1,\ldots,2^k-1\}$
in binary expansion.

**Lemma 1.** Let $A = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) = (E_k, F_{n-k}) \in \{0,1\}^n$. Then

for $r \geq 0$

$$B_{2^{k+r}}(E_k, F_{n-k})$$

does not depend on $F_{n-k}$.
Proof. Bearing in mind Definition 1 we have

\[ E_{2m+k}(e_1, e_{2m+k}) = (e_1, \ldots, e_m, D_{n-m}(e)) \]

(for some \( m \) and some \( i \), where \( m \leq k \) and \( 0 \leq i \leq 2^{n-m} - 1 \))

\[ = E_{2m+s+i}(e_1, \ldots, e_m, D_{n-m}(e)) \quad \Box \]

Theorem 4. Let \( f(X) = 0 \) be a consistent Boolean equation. The formula

\[
X = \bigcup \left\{ f(B_0(A))B_0(A) \cup f(B_1(A))f(B_1(A))B_1(A) \cup \cdots \right\}
\]

(2)

\[
\cup f(B_0(A))f(B_1(A)) \cdots f(B_s(A))B_s(A) \right] T^A
\]

defines a general solution of \( f(X) = 0 \).

Proof. Since

\[ \{B_0(A), B_1(A), \ldots, B_s(A)\} = \{0, 1\}^n \]

and

\[ \{B_0(A) | A \in \{0, 1\}^n\} = \{A | A \in \{0, 1\}^n\} = \{0, 1\}^n \]

the conditions of Theorem 3 are fulfilled.

Remark 1. Since the equation \( f(X) = 0 \) is consistent i.e.

\[ f(B_0(A)) f(B_1(A)) \cdots f(B_s(A)) = 0 \]

we can omit \( f(B_s(A)) \) from \( f(B_0(A)) f(B_1(A)) \cdots f(B_s(A)) \) because of \( ab = 0 \Rightarrow ab = a \).

Remark 2. The formula (1) can be written as

\[
X = \bigcup \left\{ \bigwedge_{i=0}^{s} f(B_0(A))B_i(A) \bigwedge_{j=0}^{i} f(B_j(A)) \right\} T^A
\]

i.e. we can write

\[
x_k = \bigcup \left\{ \bigwedge_{i=0}^{s} f(B_0(A))B_i(A) \bigwedge_{j=0}^{i} f(B_j(A)) \right\} T^A \quad (k = 1, \ldots, n;)
\]

(we assume that \( \bigwedge_{j=0}^{i} f(B_j) = 1 \)).
Lemma 2. Let \( A = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) = (E_k, F_{n-k}) \in \{0,1\}^n \). Then

\[
(4) \quad \bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) = \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))
\]

does not depend on \( F_{n-k} \).

Proof. The union (4) can be written as the union of two unions

\[
\bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) = \bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))
\]

\[
\bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) = \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})).
\]

Note that

\[
(5) \quad i < 2^{n-k} \Rightarrow (B_i(E_k, F_{n-k})) = e_k,
\]

because of Definition 3. The first union can be written as

\[
\bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) = e_k \bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k})) \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))
\]

(because of (5))

\[
= e_k \bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))
\]

(because of \( a_1' \cup a_2' \cup \cdots \cup a_{k'}' = a_1' \cup a_2' \cup \cdots \cup a_{k'}' \))

\[
= e_k \bigcup_{i=0}^{2^{n-k}-1} f'(E_k, D_{n-k}(i))
\]

(by Definition 3).
The second union can be written as

\[ \bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) a \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) \]

\[ = \bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) b \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) \]

\[ \cdot \prod_{j=0}^{2^{n-k}-1} f(B_j(E_k, F_{n-k})) \prod_{j=2^{n-k}}^{i-1} f(B_j(E_k, F_{n-k})) \]

\[ \text{we assume that} \prod_{j=2^{n-k}}^{i-1} f(B_j) = 1 \]

\[ = \prod_{j=0}^{2^{n-k}-1} f(E_k, D_{n-k}(j)) \bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) b \]

\[ \cdot \prod_{j=2^{n-k}}^{i-1} f(B_j(E_k, F_{n-k})). \]

Since the latter union contains only the members of the form

\[ B_{2^{n-k}+i}(E_k, F_{n-k}) \]

where \( r \geq 0 \), it does not depend of \( F_{n-k} \), because of Lemma 1.

Therefore (4) does not depend on \( F_{n-k} \).

**Remark 3.** Since

\[ \bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) a \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) \]

does not depend on \( F_{n-k} \), we have

\[ \bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) a \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) \]

\[ = \bigcup_{i=0}^{p} f'(B_i(E_k, G^*_{n-k}))(B_i(E_k, G^*_{n-k})) a \prod_{j=0}^{i-1} f(B_j(E_k, G^*_{n-k})), \]

where \( G^*_{n-k} \) is an arbitrary but fixed element from the set \( \{0, 1\}^{n-k} \).
Theorem 5. Let \( f(X) : B^n \rightarrow B \) be a Boolean function. If the equation \( f(X) = 0 \) is consistent, then the formulas

\[
x_k = \bigcup_{E_k \in \{0,1\}^k} \left[ \bigcup_{i=0}^{p} f'(B_i(E_k, G_{n-k}))(B_i(E_k, G_{n-k})) \prod_{j=0}^{i-1} f(B_j(E_k, G_{n-k})) \right] T_k^i
\]

\((k = 1, \ldots, n)\)

defines a general solution of \( f(X) = 0 \), where \( G_{n-k} \) are arbitrary but fixed elements from the sets \( \{0,1\}^{n-k} \) \((k = 1, \ldots, n)\).

**Comment.** Specifically, we can take \( G_{n-k} = (0, \ldots, 0) \).

**Proof.** In accordance with (3) we have for \( k = 1, \ldots, n \)

\[
x_k = \bigcup_{A} \left[ \bigcup_{i=0}^{p} f'(B_i(A))(B_i(A)) \prod_{j=0}^{i-1} f(B_j(A)) \right] T^A
\]

\[
= \bigcup_{E_k \in \{0,1\}^k} \bigcup_{F_{n-k} \subset \{0,1\}^{n-k}} \left[ \bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k})) \right]
\cdot \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))
\]

\[
\cdot \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) \right] T_k^i T_{n-k}^F
\]

\((T_k = (t_1, \ldots, t_k), \ T_{n-k} = (t_{k+1}, \ldots, t_n))\)

\[
= \bigcup_{E_k \in \{0,1\}^k} \bigcup_{F_{n-k} \subset \{0,1\}^{n-k}} \left[ \bigcup_{i=0}^{p} f'(B_i(E_k, G_{n-k}))(B_i(E_k, G_{n-k})) \right] T_k^i T_{n-k}^F
\]

\[
= \bigcup_{E_k \in \{0,1\}^k} \left[ \bigcup_{i=0}^{p} f'(B_i(E_k, G_{n-k}))(B_i(E_k, G_{n-k})) \right] T_k^i
\]

\((because \ \bigcup_{E_k \in \{0,1\}^k} T_{n-k}^F = 1) \).

**Definition 4.** If \( f : B^n \rightarrow B \) be a Boolean function and \( A \in \{0,1\}^n \), then the term \( S_k(f, A) \) is defined by the following algorithm:

for \( i = 0 \) to \( p \) do

If \((B_i(A))_k = 0\) then
\[ \text{if } (B_{i+1}(A))_k = 0 \text{ then write } (f(B_i(A))) \]
\[ \text{else write } f(B_i(A)) \]
\[ \text{else if } (\exists m > i)(B_m(A))_k = 1 \text{ then write } f'(B_i(A)) \cup \]
\[ \text{else write } f'(B_i(A)), ... \]
("write" \(f'(B_i(A))\)...) means "write \(f'(B_i(A))\) and close all brackets").

Comment. The term \(S_k(f, (E_k, G^n_{n-k}))\) contains every member
\[ f(A_0), f(A_1), ..., f(A_p) \]
at most once, because of Definition 4 and
\[ \{B_0(A), B_1(A), ..., B_q(A)\} = \{0,1\}^n. \]

**Lemma 3.** If \(f(X): B^n \to B\) be a Boolean function and \(A \in \{0,1\}^n\) then
\[ \bigcup_{i=0}^{p} f'(B_i(A))f(B_i(A))_k \cap \bigcap_{j \neq k} f(B_j(A)) = S_k(f, A). \]

The proof follows from Definition 4, distributive law and
\[ a' \cup zb = a' \cup b. \]

**Theorem 6.** Let \(f(X): B^n \to B\) be a Boolean function. If the equation
\[ f(X) = 0 \]
is consistent then the formulas
\[ z_k = \bigcup_{E_k(A) \in \{0,1\}^n} S_k(f, (E_k, G^n_{n-k}))^{T_{k-1}} \quad (k = 1, ..., s) \]
define a general solution of \(f(X) = 0\), where \(G^n_{n-k}\) are arbitrary but fixed elements from the set \(\{0,1\}^{n-k}\) \((k = 1, ..., n)\).

**Proof.** The proof follows from Theorem 5 and Lemma 3.

**Comment.** In accordance with Definition 4, it can be remarked that the algorithm described in Theorem 6 simplifies the formulas of be solutions given in Theorem 3. Namely, the coefficient \(A_{k,j}\) appears in the term \(S_k(f, A)\) at most once.
Example 2. Determine a triangular general solution of the consistent Boolean equation $f(x_1, x_2) = 0$.

$$x_1 = S(f, (0, G_1))t_1 \cup S(f, (1, G_1))t_1 = S(f, (0, 0))t_1^0 \cup S(f, (1, 0))t_1 = (\text{we take } G_1 = 0)$$

$$= (f(0, 0)f(0, 1)(f'(0, 1) \cup f'(1, 1)))t_1^0$$

$$\cup (f'(0, 0) \cup f'(1, 1))t_1$$

$$x_2 = S(f, (0, 0))t_2^0 \cup S(f, (0, 1))t_2^1$$

$$\cup S(f, (1, 0))t_2^0 \cup S(f, (1, 1))t_2$$

$$= (f(0, 0)f(0, 1) \cup f(1, 0)f'(1, 1)))t_2^0$$

$$\cup (f(0, 1) \cup f(0, 0)f(1, 1)f'(1, 1)))t_2^1$$

$$\cup (f(1, 1) \cup f(0, 0)f'(0, 0))t_2^0$$

$$\cup (f'(1, 1) \cup f(1, 0)f'(0, 1))t_2^1.$$

References


[4] Prešić, S., Une méthode de résolution des équations dont toutes les solution


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