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INTEGRATED SEMIGROUP OF UNBOUNDED LINEAR OPERATORS - CAUCHY PROBLEM
Part II

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Abstract

We study relations of an infinitesimal generator A and its restrictions $\tilde{A}^{\omega}, \omega > 0$ which correspond to a family of unbounded operators $(S(t))_{t\geq 0}$ and a family of Banach spaces $(\mathcal{D}_{\omega}, ||\cdot||_{\omega})$. Results are applied to a Cauchy problem $\frac{d}{dt}u(t) = Au(t) + f(t), u(0) = u_0$ with appropriate f and u_0.

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1. Introduction

Since the appearance of Arendt's papers ([1],[2]), integrated semigroups of bounded linear aperators have been studied by many authors in the last decade ([4],[5],[8],[12],[13],[14],[17],[18]). On the other hand in [7] Hughes's studied families of unbounded operators forming C_0-semigroups on appropriate domains. Hughes's paper inspired us to introduce and study integrated semigroups of unbounded linear operators.

This paper is a continuation of [9], where we have given the structural properties of Banach spaces $(E_\omega, \|\cdot\|_\omega)$ and infinitesimal generators $A^\omega, \omega > 0$ which are related to a family of unbounded linear operators $(S(t))_{t\geq 0}$ on a Banach space E satisfying the composition law for an integrated semigroup on a subdomain $D \subset E$.

First, by introducing an equivalent norm in each $(E_\omega, \|\cdot\|_\omega), \omega > 0$, we characterize the infinitesimal generator A (equal to A^ω on E_ω) of $(S(t))_{t\geq 0}$. Then we apply the theory to a Cauchy problem $\frac{d}{dt}u(t) = Au(t) + f(t)$, $u(0) = u_0$.

We note that the infinitesimal generator A is not closed. But the integrated semigroup of unbounded linear operator $(S(t))_{t\geq 0}$ is analyzed through the families $(S^\omega(t))_{t\geq 0}$, $\omega > 0$ of exponentially bounded integrated semigroups whose infinitesimal generators $\tilde{A}^\omega$ are closed on the family of subspaces $(\mathcal{D}_\omega)_{\omega>0}$ with the stronger norm $\|\cdot\|_\omega$($\tilde{A}^\omega$ is the restriction of A). Moreover, in this case, differentiability and integrability are conserved. Thus we can find solutions to inhomogeneous Caushy problems with $\tilde{A}^\omega$ instead of A. With the aid of these solutions we find a solution to the posed Cauchy problem.

2. Notation

Let $(S(t))_{t\geq 0}$ be a family of unbounded linear operators in a Banach space $(E, \|\cdot\|)$. Denote by $D(S(t))$ a domain of $S(t)$ and set

$$D = \{x \in \bigcap_{s,t\geq 0} D(S(s)S(t)) \mid S(0)x = 0,\ S(t)x \text{ is strongly continuous for } t \geq 0,$$

$$(1) \qquad S(s)S(t)x = \int_0^s (S(r+t) - S(r))x\,dr = S(t)S(s)x \text{ for } s,t \geq 0\}$$

If $D \neq \{0\}$, the $(S(t))_{t\geq 0}$ is said to be *an integrated semigroup of unbounded operators* in E.

Differentiation spaces $C^n, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are: $C^0 = D$, $C^n = \{x \in$

$D; S(t)x$ is $n-$ times strongly continuously differentiable function of $t \geq 0\}$. A set

$$\mathcal{N} = \{x \in D; S(t)x = 0, t \geq 0\}$$

is called a *degeneration space.* A semigroup $(S(t))_{t\geq 0}$ is called *non-degenerate* if $\mathcal{N} = \{0\}$ and *degenerate* otherwise.

We will assume that $(S(t))_{t\geq 0}$ is a non-degenerate integrated semigroup of unbounded linear operators.

Definition 1. *For* $\omega \in \mathbb{R}^+ = (0, \infty)$, *and* $x \in \bigcap_{t\geq 0} D(S(t))$, *let* $E_\omega := \{x \in D; \|x\|_\omega < \infty\}$, *where*

$$\|x\|_\omega := \sup_{t\geq 0} e^{-\omega t}\|S(t)x\|.$$

Let $\overline{E}_\omega$ *denote the closure of the set* E_ω *under the norm* $\|\cdot\|$ *and* $S(t)|\overline{E}_\omega$ *is the part of* $S(t)$ *in* $\overline{E}_\omega$ *i.e.*

$$D(S(t)|\overline{E}_\omega) = \{x \in \overline{E}_\omega; x \in D(S(t)) \text{ and } S(t)x \in \overline{E}_\omega\}.$$

Remark 1. We have

$$\|S(t)x\|_\omega \leq \frac{2e^{\omega t}}{\omega}\|x\|_\omega \text{ and } e^{-\omega(t+s)}\|S(s)S(t)x\| \leq \frac{2}{\omega}\|x\|_\omega.$$

In this paper, we assume

(2) For every $\omega > 0$ there exists $C_\omega > 0$ such that $\|x\|_\omega \geq C_\omega\|x\|, x \in D$. *Also, we assume that* $S(t)|\overline{E}_\omega$ *is closed in* $\overline{E}_\omega$ *for* $t \geq 0$, *and* $\omega > 0$, which implies E_ω is a Banach space ([9], Theorem 1).

For fixed $\omega > 0$ and $\lambda \in \mathbb{C}, Re\lambda > \omega$,

$$R^\omega(\lambda)x = \lambda \int_0^\infty e^{-\lambda t}S(t)x dt, x \in E_\omega.$$

In general, $R^\omega(\lambda)$ is unbounded in $(E, \|\cdot\|)$, while

$$\|\lambda \int_0^\infty e^{-\lambda t}S(t)x dt\| \leq \frac{2|\lambda|}{\omega C_\omega(Re\lambda - \omega)}\|x\|_\omega.$$

Therefore, the family $(R^\omega(\lambda))_{Re\lambda>\omega}$ is the resolvent of a closed linear operator A^ω in the Banach space $(E_\omega, \|\cdot\|_\omega)$ and $A^\omega = \lambda I - (R^\omega(\lambda))^{-1}$ (I is the identity operator) for $\lambda \in \mathbb{C}, Re\lambda > \omega, D(A^\omega) = Range(R^\omega(\lambda))$.

In general, the operators A^ω are not closed in the norm of E.

Definition 2. *[9] Let $D(A) = \bigcup_{\omega>0} D(A^\omega)$. For $x \in D(A)$ let $\omega > 0$ such that $x \in D(A^\omega)$. There exists $y \in E_\omega$ such that $x = R^\omega(\lambda)y$ for $Re\lambda > \omega$. We define*

$$(3) \qquad Ax = \lambda x - y.$$

We call A the infinitesimal generator of $(S(t))_{t\geq 0}$.

Thus, $Ax = A^\omega x$ for $x \in D(A^\omega)$, and it is easy to prove that this definition does not depend on ω with the property $x \in D(A^\omega)$, i.e. if $x \in D(A^{\overline{\omega}})$, then $Ax = A^{\overline{\omega}}x$.

Theorem 1. *[9] Fix $\omega > 0$ and $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$.*

a) $R^\omega(\lambda)x \in D(S(t))$ and $S(t)R^\omega(\lambda)x = R^\omega(\lambda)S(t)x, t \geq 0, x \in E_\omega$.

b) i) $R^\omega(\lambda)(E_\omega) \subset E_\omega$. Moreover,

$$\frac{\omega(Re\lambda - \omega)}{2|\lambda|}\|R^\omega(\lambda)x\|_\omega \leq \|x\|_\omega, x \in E_\omega.$$

ii) For every $x \in E_\omega, \|x\|_{R^\omega} < \infty$, where

$$\|x\|_{R^\omega} := \sup_{n\in\mathbb{N}_0}\sup_{\lambda>\omega} \frac{(\lambda-\omega)^{n+1}}{n!}\left\|\left(\frac{R^\omega(\lambda)}{\lambda}\right)^{(n)}x\right\|, \lambda > \omega.$$

The norm $\|\cdot\|_{R^\omega}$ is equativelent to the norm $\|\cdot\|_\omega$.

iii) If $\omega_1 \leq \omega_2$ and $Re\lambda > \omega_2$, then $R^{\omega_1}(\lambda)x = R^{\omega_2}(\lambda)x, x \in E_{\omega_1}$. Thus, as operators in $E, R^{\omega_1}(\lambda) \subset R^{\omega_2}(\lambda)$ if $Re\lambda > \omega_2$.

Theorem 2. *[9] The family of operators $(R^\omega(\lambda))_{Re\lambda>\omega}$ on $E_\omega, \omega > 0$ is the resolvent of a closed linear operator A^ω in E_ω (closed in the $\|\cdot\|_\omega$ norm topology).*

3. Infinitesimal generator

We have $D(A^{\omega_1}) \subset D(A^{\omega_2})$ or $D(A^{\omega_2}) \subset D(A^{\omega_1})$, where $\omega_1, \omega_2, \in (0,\infty)$. It is easy to prove that $D(A)$ is a subspace of E and A is a linear operator in E.

Theorem 3.

a) For $x \in E_\omega$, the resolvent equation

$$(\lambda I - A)y = x, Re\lambda > \omega, \tag{4}$$

has a unique solution belonging to E_ω and $y = R^\omega(\lambda)x$.

b) Let $\omega > 0$. Then for $t \geq 0$ $S(t)D(A^\omega) \subset D(A^\omega)$ and

$$S(t)A^\omega x = A^\omega S(t)x, \;\; x \in D(A^\omega).$$

c) If $x \in D(A)$, then there exists $\omega' > 0$ such that

$$R^\omega(\lambda)Ax = AR^\omega(\lambda)x, \;\; \omega \geq \omega'.$$

Proof. a) Let $x \in E_\omega$. Then, by Theorem 1b), $R^\omega(\lambda)x \in E_\omega$ and, by Theorem 2, $R^\omega(\lambda)x$ is the unique solution of the resolvent equation $(\lambda I - A)y = x$ and therefore is the unique solution of resolvent equation (4) because $A|E_\omega = A^\omega$.

b) Let $x \in D(A^\omega)$. Then, $x = R^\omega(\lambda)y = \lambda \int\limits_0^\infty e^{-\lambda t}S(t)ydt$ for some $y \in E_\omega$ and $Re\lambda > \omega$. Using Theorem la), we have $S(t)R^\omega(\lambda)y = R^\omega(\lambda)S(t)y \in D(A^\omega)$ and

$$\begin{aligned} A^\omega S(t)x = A^\omega S(t)R^\omega(\lambda)y &= A^\omega R^\omega(\lambda)S(t)y \\ &= \lambda S(t)x - R^\omega(\lambda)^{-1}R^\omega(\lambda)S(t)y \\ &= \lambda S(t)x - S(t)y = S(t)(\lambda x - y) = S(t)A^\omega x. \end{aligned}$$

c) For $x \in D(A)$ there exists $\omega' > 0$ such that $x \in D(A^{\omega'})$ and

$$R^{\omega'}(\lambda)A^{\omega'}x = R^{\omega'}(\lambda)\Big(\lambda x - (R^{\omega'}(\lambda))^{-1}x\Big) = \lambda R^{\omega'}(\lambda)x - x,$$

$$A^{\omega'}R^{\omega'}(\lambda)x = \lambda R^{\omega'}(\lambda)x - x.$$

If $\omega \geq \omega'$ then $A^\omega \supset A^{\omega'}$ and $R^\omega(\lambda) \supset R^{\omega'}(\lambda)$. Then,

$$A^\omega R^\omega(\lambda)x = R^\omega(\lambda)A^\omega x. \tag{5}$$

Since $A|D(A^\omega) = A^\omega$, (5) implies

$$AR^\omega(\lambda)x = R^\omega(\lambda)Ax. \qquad \square$$

Theorem 4. [10] *a) For $x \in D(A), S(t)x$ is a differentiable function of $t, t \geq 0$, with respect to $\|\cdot\|$ and*

$$S'(t)x - x = S(t)Ax \tag{6}$$

or equivalently

$$S(t)x - tx = \int_0^t S(s)Axds. \tag{7}$$

b) If $x \in D(A^n)$ and $t \geq 0$, then

$$S^{(n)}(t)x = A^{n-1}x + S(t)A^n x, \qquad n \in \mathbb{N}.$$

c) If $x \in E' = \bigcup_{\omega>0} E_\omega$, then $\int_0^t S(s)xds \in D(A)$ and

$$A \int_0^t S(s)xds = S(t)x - tx.$$

Spaces D_ω

Definition 3. *For $\omega > 0$, let*

$$\mathcal{D}_\omega := \overline{D(A^\omega)}^{\|\cdot\|_\omega}.$$

Clearly, $\mathcal{D}_\omega \subset E_\omega$. Let $A^\omega|\mathcal{D}_\omega$ denote the part of A^ω in $\mathcal{D}_\omega$ with the domain

$$D(A^\omega \mid \mathcal{D}_\omega) = \{x \in \mathcal{D}_\omega;\ x \in \mathcal{D}(A^\omega) \quad \text{and} \quad A^\omega x \in \mathcal{D}_\omega\}.$$

Theorem 5. *a) For all $x \in \mathcal{D}_\omega$.*

$$\lim_{\lambda\to\infty} \|\lambda R^\omega(\lambda)x - x\|_\omega = 0.$$

b) For all integers $n \geq 1$,

$$\overline{D((A^\omega)^n)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega, \quad \overline{D((A^\omega)^n)} = \overline{E}_\omega \text{ and } \overline{D(A)} = \overline{E'},$$

where $E' = \bigcup_{\omega>0} E_\omega$.

Proof. a) We will prove the statement for $x \in D(A^\omega)$ and then for $x \in \mathcal{D}_\omega$. From $A^\omega x = \lambda x - (R^\omega(\lambda))^{-1}x$, it follows

$$R^\omega(\lambda)A^\omega x = \lambda R^\omega(\lambda)x - x.$$

Note, $A^\omega \subset A$ and this implies $S'(t)x - x = S(t)A^\omega x$. We have

$$\Big\|\lambda\int_0^\infty e^{-\lambda t}S(t)A^\omega x\,dt\Big\|_\omega = \Big\|\lambda\int_0^\infty e^{-\lambda t}(S'(t)x - x)dt\Big\|_\omega$$

$$\leq \sup_{s\geq 0}\lambda e^{-\omega s}\int_0^\infty e^{-\lambda t}\|S(s)S'(t)x - S(s)x\|dt$$

$$\leq \sup_{s\geq 0}\lambda e^{-\omega s}\Big(\int_0^\delta e^{-\lambda t}2\epsilon dt + \int_\delta^\infty e^{-\lambda t}\|S(s+t)x - S(s)x\|dt + \int_\delta^\infty e^{-\lambda t}\|S(t)x\|dt\Big)$$

$$\leq \sup_{s\geq 0}\Big[2(1-e^{-\lambda\delta})\epsilon + \lambda\int_\delta^\infty e^{-(\lambda-\omega)t}e^{-\omega(s+t)}\|S(s+t)x\|dt$$

$$+\lambda\int_\delta^\infty e^{-\lambda t}e^{-\omega s}\|(S(s))x\|dt + \lambda\int_\delta^\infty e^{-(\lambda-\omega)t}e^{-\omega t}\|S(t)x\|dt\Big]$$

$$\leq 2(1-e^{-\lambda\delta})\epsilon + \|x\|_\omega\Big(\frac{2\lambda}{\omega-\lambda}e^{(\omega-\lambda)t}\Big|_\delta^\infty - e^{-\lambda t}\Big|_\delta^\infty\Big)$$

$$= 2(1-e^{-\lambda\delta})\epsilon + \|x\|_\omega\Big(\frac{2\lambda}{\lambda-\omega}e^{(\omega-\lambda)\delta} + e^{-\lambda\delta}\Big).$$

Here we use $\|S(s+t)x - S(s)x\| < \epsilon$ and $\|S(t)x\| < \epsilon$ for $0 \leq t \leq \delta$, because

$$\sup_{s\geq 0} e^{-\omega s}\|S(t+s)x - S(s)x\| \leq \sup_{s\geq 0} e^{-\omega s}\frac{1}{C_\omega}\|S(t+s)x - S(s)x\|_\omega$$

$$\leq \sup_{s\geq 0} e^{-\omega s}\frac{1}{C_\omega}\Big(\frac{2}{\omega}e^{\omega(s+t)}\|x\|_\omega + \frac{2}{\omega}e^{\omega s}\|x\|_\omega\Big) \leq \frac{2}{C_\omega\omega}(e^{\omega\delta}+1)\|x\|_\omega.$$

Let $x \in \mathcal{D}_\omega$ and $\{x_n\} \subset D(A^\omega)$ be such that $x_n \to x$ as $n \to \infty$. Then,

$$\|\lambda R^\omega(\lambda)x - x\|_\omega \leq \|\lambda R^\omega(\lambda)x - \lambda R^\omega(\lambda)x_n\|_\omega + \|\lambda R^\omega(\lambda)x_n - x_n\|_\omega + \|x_n - x\|_\omega$$

$$\leq \Big(\lambda\|R^\omega(\lambda)\|_\omega + 2\Big)\epsilon$$

for sufficiently large λ and n.

b) In order to prove $\overline{D\Big((A^\omega)^n\Big)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega$, $n \in \mathbb{N}$. To do this it sufficies to prove that $\overline{D\Big((A^\omega)^2\Big)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega$.

Let $x \in D(A^\omega)$. Then there exists $y \in E_\omega$ such that $x = R^\omega(\mu)y$ for some $\mu > \omega$. Then a) implies

$$\lim_{\lambda\to\infty} \|\lambda R^\omega(\lambda)R^\omega(\mu)y - R^\omega(\mu)y\|_\omega = 0.$$

From $A^\omega R^\omega(\lambda)R^\omega(\mu)y = R^\omega(\lambda)A^\omega R^\omega(\mu)y$ it follows $R^\omega(\lambda)R^\omega(\mu)y \in D((A^\omega)^2)$ and $\overline{D\Big((A^\omega)^2\Big)}^{\|\cdot\|_\omega} \supset D(A^\omega)$. This implies $\overline{D\Big((A^\omega)^2\Big)}^{\|\cdot\|_\omega} \supset \overline{D(A^\omega)}^{\|\cdot\|_\omega}$. It is obvious $\overline{D\Big((A^\omega)^2\Big)}^{\|\cdot\|_\omega} \subset \overline{D(A^\omega)}^{\|\cdot\|_\omega}$. Thus, $\overline{D\Big((A^\omega)^2\Big)}^{\|\cdot\|_\omega} = \overline{D(A^\omega)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega$.

By induction, we have $\overline{D\Big((A^\omega)^n\Big)}^{\|\cdot\|_\omega} = \mathcal{D}_\omega$.

Let us prove $\overline{D\Big((A^\omega)^n\Big)} = \overline{E}_\omega$, for $n \in \mathbb{N}$. Let $x \in E_\omega$. For $\lambda > \omega, \mu > \omega$ and $\lambda \neq \mu$ we obtain

$$\lim_{\lambda\to\infty} \|\lambda R^\omega(\lambda)R^\omega(\mu)x - R^\omega(\mu)x\|_\omega = 0. \tag{8}$$

First resolvent equation implies

$$\lim_{\lambda\to\infty} \|\frac{\lambda}{\lambda-\omega}(R^\omega(\mu)x - R^\omega(\lambda)x) - R^\omega(\mu)x\|_\omega = 0$$

and

$$\lim_{\lambda\to\infty} \|R^\omega(\lambda)x\|_\omega = 0.$$

The norm $\|\cdot\|_\omega$ is stronger that the norm $\|\cdot\|$ (i.e. $\|\cdot\| \leq \frac{1}{C_w}\|\cdot\|_\omega$) and implies

$$\lim_{\lambda\to\infty} \|R^\omega(\lambda)x\| = 0. \tag{9}$$

From $A^\omega R^\omega(\mu)x = \lambda R^\omega(\mu)x - (R^\omega(\lambda))^{-1}R^\omega(\mu)x$ it follows

$$R^\omega(\lambda)A^\omega R^\omega(\mu)x = \lambda R^\omega(\lambda)R^\omega(\mu)x - R^\omega(\mu)x.$$

This implies

$$\lim_{\lambda\to\infty} \|R^\omega(\lambda)A^\omega R^\omega(\mu)x\|_\omega = 0 \text{ and } \lim_{\lambda\to\infty} \|R^\omega(\mu)A^\omega R^\omega(\lambda)x\|_\omega = 0$$

because the operator A^ω comutates with $R^\omega(\lambda)$ on $D(A^\omega)$.

The injective operator $R^\omega(\mu)$ is a bounded linear operator with respect to the norm $\|\cdot\|_\omega$ and

$$\lim_{\lambda\to\infty} \|A^\omega R^\omega(\lambda)x\|_\omega = 0.$$

It implies

(10) $$\lim_{\lambda\to\infty} \|A^\omega R^\omega(\lambda)x\| = 0.$$

From (9) and (10) it follows that for a given $\epsilon > 0$ there exists $\lambda_0 > \omega$ such that

$$\|R^\omega(\lambda)x - A^\omega R^\omega(\lambda)x\| < \epsilon \text{ for } \lambda > \lambda_0.$$

Since $R^\omega(\lambda)x \in D(A^\omega)$, it follows $A^\omega R^\omega(\lambda)x \in \overline{D(A^\omega)}$.

Let $x \in E_\omega$. We have $x = \lambda R^\omega(\lambda)x - A^\omega R^\omega(\lambda)x$. This implies $x \in \overline{D(A^\omega)}$ and $\overline{D(A^\omega)} \supset \overline{E}_\omega$. The converse inclusion is obvious. Thus, $\overline{D(A^\omega)} = \overline{E}_\omega$.

From $D(A^\omega) \subset \mathcal{D}_\omega, \overline{D((A^\omega)^n)}^{\|\cdot\|_\omega} = D_\omega$ and $\overline{D((A^\omega)^n)}^{\|\cdot\|_\omega} \subset \overline{D((A^\omega)^n)}$ it follows $D(A^\omega) \subset \overline{D((A^\omega)^n)}, \overline{D(A^\omega)} \subset \overline{D((A^\omega)^n)}$ and $\overline{E}_\omega \subset \overline{D((A^\omega)^n)}$. This implies

$$\overline{D((A^\omega)^n)} = \overline{E}_\omega.$$

Moreover,

$$\overline{E'} = \overline{\bigcup_{\omega>0} E_\omega} \subset \overline{\bigcup_{\omega>0} \overline{E}_\omega} = \overline{\bigcup_{\omega>0} \overline{D(A^\omega)}} = \overline{\bigcup_{\omega>0} D(A^\omega)} = \overline{D(A)}$$

and

$$\overline{D(A)} = \overline{\bigcup_{\omega>0} D(A^\omega)} \subset \overline{\bigcup_{\omega>0} E_\omega} = \overline{E'}.$$

Remark. Since A^ω is closed related to $\|\cdot\|_\omega$ and $\lim_{\lambda\to\infty} \|\lambda R^\omega(\lambda)x - x\|_\omega = 0$ for $x \in D(A^\omega)$, Lemma 12.2.1 in [6] implies

$$\mathcal{D}_\omega = \overline{D(A^\omega)}^{\|\cdot\|_\omega} = \overline{D(A^\omega|\mathcal{D}_\omega)}^{\|\cdot\|_\omega}.$$

Theorem 6. *Let $\omega > 0$ be fixed. Then, for $t \geq 0, S(t)\mathcal{D}_\omega \subset \mathcal{D}_\omega$ and $(S^\omega(t))_{t\geq 0} = (S(t)|\mathcal{D}_\omega)_{t\geq 0}$ is an exponetially bounded integrated semigroup on $(\mathcal{D}_\omega, \|\cdot\|_\omega)$ with the infinitesimal generator which is equal $A^\omega|\mathcal{D}_\omega$.*

Proof. Fix $t \geq 0$. Since, $S(t)$ is a bounded operator on $(E_\omega, \|\cdot\|_\omega)$ and $S(t)D(A^\omega) \subset D(A^\omega)$ (Theorem 3) it implies $S(t)\mathcal{D}_\omega \subset \mathcal{D}_\omega$. Let $x \in \mathcal{D}_\omega$ and $\{x_n\}$ be a sequence in $D(A^\omega)$ such that $\|x_n - x\|_\omega \to \infty$. We have $\|S(t)x_n - S(t)x\|_\omega \to 0, n \to \infty$. Since $S(t)x_n \in \mathcal{D}_\omega$, it follows $S(t)x \in \mathcal{D}_\omega$ and $\|S(t)x\|_\omega \leq \frac{2}{\omega}e^{\omega t}\|x\|_\omega$ implies $\|S(t)\|_\omega \leq \frac{2e^{\omega t}}{\omega}$.

We will prove the strong continuouty of $S(t)$ on $\mathcal{D}_\omega$. Let $x \in D(A^\omega)$ and $t, t_1 > 0$. Theorem 4 implies

$$S(t)x = tx + \int_0^t S(r)Axdr$$

and

$$S(t_1)x = t_1x + \int_0^{t_1} S(r)Axdr.$$

Moreover,

$$\|S(t)x - S(t_1)x\|_\omega = \|tx - t_1x + \int_0^t S(r)Axdr - \int_0^{t_1} S(r)Axdr\|_\omega$$

$$\leq |t - t_1|\ \|x\|_\omega + \frac{2}{\omega}e^{\omega t}\|Ax\|_\omega|t - t_1| < \Big(\|x\|_\omega + \frac{2}{\omega}(e^{\omega t_1} + \epsilon)\|Ax\|_\omega\Big)|t - t_1|,$$

for $|t - t_1|$ sufficiently small. This implies $\|S(t)x - S(t_1)x\|_\omega \to 0$ as $t \to t_1$.

Let $x \in \mathcal{D}_\omega \backslash D(\tilde{A}^\omega)$. Then, for a given $\epsilon > 0$ there exists $x_\epsilon \in D(\tilde{A}^\omega)$ such that $\|x - x_\omega\|_\omega < \epsilon$. Moreover, there exists $\delta > 0$ such that $0 < |t - t_1| < \delta$, implies $e^{\omega t} < e^{\omega t_1} + \epsilon$ and $\|S(t)x_\epsilon - S(t_1)x_\epsilon\|_\omega < \epsilon$. Thus,

$$\|S(t)x - S(t_1)x\|_\omega \leq \|S(t)x - S(t)x_\epsilon\|_\omega + \|S(t)x_\epsilon - S(t_1)x_\epsilon\|_\omega + \|S(t_1)x_\epsilon - S(t_1)x\|_\omega$$

$$< \frac{2}{\omega}e^{\omega t}\|x - x_\epsilon\|_\omega + \epsilon + \frac{2}{\omega}e^{\omega t_1}\|x_\epsilon - x\|_\omega$$

$$< \frac{2}{\omega}(e^{\omega t_1} + \epsilon)\epsilon + \epsilon + \frac{2e^{\omega t_1}}{\omega}\epsilon,$$

which implies $\|S(t)x - S(t_1)x\|_\omega \to 0$ as $t \to t_1$. It follows that $S(t)x$ is strongly continuous on $\mathcal{D}_\omega$ for $t \geq 0$.

Since $S(t)\mathcal{D}_\omega \subset \mathcal{D}_\omega$ and $\mathcal{D}_\omega \subset D$, so that

$$S(s)S(t)x = \int_0^s (S(r+t) - S(r))x dr, x \in \mathcal{D}_\omega.$$

Let $\tilde{A}^\omega$ be the generator of the exponentially bounded integrated semigroup $(S^\omega(t))_{t\geq 0}$ on the Banach space $(\mathcal{D}_\omega, \|\cdot\|_\omega)$. Then, $x \in D(\tilde{A}^\omega)$ and $\tilde{A}^\omega x = y$ implies $S'(t)x - x = S(t)y$ where $S(t)x$ is differentiable in the norm $\|\cdot\|_\omega$. Thus $S(t)x$ is differentiable in the norm $\|\cdot\|$ and $S'(t)x - x = S(t)y$. Then we have $x \in D(A)$ and $Ax = y$.

For $x \in \mathcal{D}_\omega$ we have $(A^\omega|\mathcal{D}_\omega)x = Ax$, it follows $\tilde{A}^\omega \subset A^\omega|\mathcal{D}_\omega$. Since $\rho(A^\omega) \cap \rho(A^\omega|\mathcal{D}_\omega) \neq \theta$, Lemma 2.17 [7] implies $\tilde{A}^\omega = A^\omega|\mathcal{D}_\omega$. Thus $(S^\omega(t))_{t\geq 0}$ is an exponentially bounded integrated semigrop on the Banach space $(\mathcal{D}_\omega, \|\cdot\|_\omega)$ with the generator $\tilde{A}^\omega$. □

Note that the infinitesimal generator of this integrated semigroup is densely defined.

4. Inhomogeneous Cauchy problem

Consider a Cauchy problem

$$\text{(11)} \qquad \begin{cases} \frac{d}{dt}u(t) = Au(t) + f(t) \\ u(0) = u_0 \end{cases} \quad t \in [0, b]$$

where $b > 0, f \in C([0,b], \mathcal{D})$, $D = \bigcup_{\omega > 0} \mathcal{D}_\omega$ and A is the infinitesimal generator of an integrated semigroup of unbounded linear operators $(S(t))_{t\geq 0}$. Then, u is a solution of (11) if $u \in C^1([0,b], \mathcal{D})$, $u(t) \in D(A), t \in [0,b]$ and (11) holds.

Assume that for every $b > 0$ there exists $\omega_b > 0$ such that

$$\text{(12)} \qquad f \in \mathcal{D}_{\omega_b}, t \in [0,b], u_0 \in D(A^{\omega_b}).$$

Let $v \in C([0,b],\mathcal{D})$ be given by

$$(13) \qquad v(t) = S(t)u_0 + \int_0^t S(s)f(t-s)ds.$$

Note that (12) implies the existence if the integral in (13) in the norm $\|\cdot\|_\omega$ for every $\omega \geq \omega_b$ because the function $t \to S(t)x$ is strongly continuous in $(\mathcal{D}_\omega, \|\cdot\|_\omega)$.

Lemma 1. *For every* $t \geq 0$ $\int_0^t v(s)ds \in D(A)$ *and*

$$(14) \qquad A\int_0^t v(s)ds = v(t) - tu_0 - \int_0^t (t-r)f(r)dr, t \in [0,b].$$

Proof. Recall, $\tilde{A}^\omega = A^\omega|\mathcal{D}_\omega$. Since this is the generator of an exponentially bounded integrated semigroup on $\mathcal{D}_\omega$, the proof which is to follow is similar to the one given in [2], Lema 5.3.

From (2) it follows $v(t) \in \mathcal{D}_\omega$, for $\omega \geq \omega_b$. This implies $\int_0^t v(s)ds \in \mathcal{D}(\tilde{A}^\omega)(c.f.$ Proposition 3.3 in [2]) and

$$\tilde{A}^\omega \int_0^t v(s)ds = \tilde{A}^\omega \int_0^t S(s)u_0 ds + \tilde{A}^\omega \int_0^t\int_0^s S(r)f(s-r)drds,$$

where the integrals are considered in the norm $\|\cdot\|_\omega$. We have

$$(15) \qquad \tilde{A}^\omega \int_0^t S(s)u_0 ds = S(t)u_0 - tu_0.$$

Fubini's theorem and the change of variables imply

$$(16) \qquad \int_0^t\int_0^s S(r)f(s-r)drds = \int_0^t\int_r^t S(r)f(s-r)dsdr$$

$$= \int_0^t \int_0^{t-r} S(r)f(p)dpdr = \int_0^t \int_0^{t-p} S(r)f(r)drdp.$$

Since $f(s) \in \mathcal{D}_\omega$ for $\omega \geq \omega_b, s \in [0,b]$ and $\tilde{A}^\omega$ is closed with respect to the norm $\|\cdot\|_\omega$, we have

$$\tag{17} \tilde{A}^\omega \int_0^t \int_0^s S(r)f(s-r)drds = \tilde{A}^\omega \int_0^t \int_0^{t-p} S(r)f(p)drdp$$

$$= \int_0^t \tilde{A}^\omega \int_0^{t-p} S(r)f(p)drdp = \int_0^t [S(t-p)f(p) - (t-p)f(p)]dp.$$

Now (15) and (17) imply

$$\tag{18} \tilde{A}^\omega \int_0^t v(s)ds = v(t) - tu_0 - \int_0^t (t-r)f(r)dr.$$

The integrability with respect to the norm $\|\cdot\|_\omega$ implies the integrability with respect to the norm $\|\cdot\|$ in E. Since $A = \tilde{A}^\omega$ in $\mathcal{D}_\omega(\omega \geq \omega_b)$ it follows

$$A \int_0^t v(s)ds = v(t) - tu_0 - \int_0^t (t-r)f(r)dr$$

Theorem 7. *a) If there exists a solution of the Cauchy problem (11), then* $C^2([0,b],\mathcal{D})$ *and* $u = v'$.

b) If $v \in C^2([0,b],\mathcal{D})$, *then* $u = v'$ *is the solution to (11).*

Proof. a) Let $t \in [0,b]$ and $w(s) = S(t-s)u(s), s \in [0,t]$. After differentation with respect to $\|\cdot\|$, we obtain

$$w'(s) = -u(s) + S(t-s)f(s).$$

This implies

$$\tag{19} S(t)u_0 = w(0) - w(t) = -\int_0^t w'(s)ds = \int_0^t u(s)ds - \int_0^t S(s)f(t-s)ds$$

Now (13) and (19) imply $v(s) = \int_0^t u(s)ds$.

b) Since $\tilde{A}^\omega$ is closed (with respect to $\|\cdot\|_\omega$) for $\omega \geq \omega_b$, by differentation of (18) we obtain

$$\tilde{A}^\omega v(t) = v'(t) - tu_0 - \int_0^t f(r)dr \tag{20}$$

and

$$\tilde{A}^\omega v'(t) = v''(t) - f(t). \tag{21}$$

Thus, with the respect to $\|\cdot\|$,

$$Av'(t) = v''(t) - f(t)$$

because $A = A^\omega$ in $\mathcal{D}_\omega$ and the norm $\|\cdot\|$ is weaker than $\|\cdot\|_\omega$. This implies that v' is a solution to (11). Also (3) implies $v(0) = 0$ and by (20) $0 = v'(0) - u_0$ i.e. $u(0) = u_0$. □

Proposition 1. *Let $f \in C^2([0,b], \mathcal{D}), u_0 \in D(A), u_1 := Au_0 + f(0) \in D(A)$ and let (12) holds. Then, the problem (11) has a unique solution.*

Proof. Since $u_0 \in D(A)$ it follows

$$S(t)u_0 - tu_0 = \int_0^t S(s)Au_0 ds.$$

Thus

$$v(t) = tu_0 + \int_0^t S(s)Au_0 ds + \int_0^t S(s)f(t-s)ds.$$

So, $v \in C^1([0,b], \mathcal{D})$. By differentation

$$v'(t) = u_0 + S(t)(Au_0 + f(0)) + \int_0^t S(s)f'(t-s)ds.$$

Since $Au_0 + f(0) \in D(A)$ if follows $v \in C^2([0,b], \mathcal{D})$, and $u = v'$ the solution to (11) according to Theorem 7. □

Consider now a Cauchy problem in the space $(\mathcal{D}_\omega, \|\cdot\|_\omega)$. Let

$$(22)\qquad \begin{cases} \frac{d}{dt}u^\omega(t) = \tilde{A}^\omega u(t) + f^\omega(t) \\ & t \in [0,b] \\ u^\omega(0) = u_0^\omega \end{cases}$$

where $\tilde{A}^\omega$ is the infinitesimal generator of the integrated exponentially bounded semigroup $(S^\omega(t))_{t\geq 0}, u_0^\omega \in \mathcal{D}_\omega, f \in C([0,b], \mathcal{D}_\omega)$. Suppose that $u^\omega \in D(\tilde{A}^\omega)$ is the solution to (22). For every $\omega_1 > \omega$ the Cauchy problem (22) in $\mathcal{D}_\omega$ is in fact the Cauchy problem in the space $\mathcal{D}_{\omega_1}$ because $\mathcal{D}_\omega \subset \mathcal{D}_{\omega_1}, \tilde{A}^\omega \subset \tilde{A}^{\omega_1}$, and $\|\cdot\|_\omega \geq \|\cdot\|_{\omega_1}$.

Corollary 4.1. *For a sufficiently large $\omega > 0$ the Cauchy problem (11) with the assumption (12) reduces to the Cauchy problem (22).*

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