THE SPACE OF FUNCTIONS WITH A LIMIT AT EACH POINT

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Abstract

We consider the set \( \Lambda(X, Y) \) of functions \( f: X \to Y \) which have a limit at each point of \( X \). If \( X \) is a compact space, \( \Lambda(X, Y) \) is a Banach algebra. The representation of bounded linear functionals on \( \Lambda(X, Y) \) is given.


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1. Introduction

In all further considerations, \((X, \mathcal{O}_X)\) will denote a topological space without isolated points, while \((Y, \mathcal{O}_Y)\) will be a Hausdorff space. These conditions will ensure the uniqueness of the limit \( \lim_x f \), where \( f: X \to Y \) and \( x \in X \). We consider the set

\[
\Lambda(X, Y) = \{ f \in Y^X : \forall x \in X \exists y_x \in Y \ y_x = \lim_x f \}
\]

which is larger than \( C(X, Y) \). This is shown by the following well-known examples.
Example 1.1. Let \( Q = \{ q_n : n \in \mathbb{N} \} \) be the space of rationals and let \( f : Q \to R \) be defined by \( f(q_n) = 1/n, \ n \in \mathbb{N} \). Then, for each \( q \in Q \) we have \( \lim_{x \to q} f(x) = 0 \neq f(q) \). So, \( f \) is discontinuous at each point although \( f \in \Lambda(Q, R) \).

Example 1.2. The function \( f : R \to R \) given by:

\[
f(x) = \begin{cases} 
1/n & \text{if } x = q_n, \ n \in \mathbb{N}; \\
0 & \text{if } x \in R \setminus Q
\end{cases}
\]

is continuous at each point of \( R \setminus Q \) and it is discontinuous at each rational point. Moreover, \( \lim_{x \to \infty} f(x) = 0 \) for all \( x \in R \), thus \( f \in \Lambda(R, R) \).

Theorem 1.1. Suppose \((X, O_X)\) is a \( T_1 \)-space, \((Y, O_Y)\) is a regular space, \( D \subset X \), \( D' = X \) and \( f : D \to Y \) where

\[
\forall x \in X \ \exists y_x \in Y \ y_x = \lim_{t \to x}^{(0)} f(t).
\]

Then, the function \( F : X \to Y \) defined by \( F(x) = y_x \), for all \( x \in X \), is continuous.

Proof. a) By the assumption for each \( x \in X \) there is \( y_x \in Y \) such that

\[
(1) \quad \forall W \in \mathcal{U}(y_x) \ \exists V \in \mathcal{U}(x) \ f(V \setminus \{x\} \cap D) \subset W.
\]

Let us prove the continuity of \( f \) at arbitrary point \( x \in X \). Let \( U \in \mathcal{U}(y_x) \). Since \( Y \) is a regular space there is \( W \in \mathcal{U}(y_x) \) such that \( W \subset U \), and by (1), there exists \( V \in \mathcal{U}(x) \) satisfying

\[
(2) \quad d \in V \setminus \{x\} \cap D \Rightarrow f(d) \in W.
\]

Suppose that \( z \in V \) and \( f(z) = y_z \notin W \). Then \( y_z \in V \setminus W \in \mathcal{U}(y_x) \) and (1) gives \( G \in \mathcal{U}(x) \) such that

\[
(3) \quad d \in G \setminus \{z\} \cap D \Rightarrow f(d) \in V \setminus W.
\]

\( z = x \) would imply \( y_z = y_x \in W \cap V \setminus W = \emptyset \), therefore \( z \neq x \) and \( z \in G \cap V \setminus \{z\} \in \mathcal{U}(x) \). Since \( D' = X \) there is a \( d \in D \cap G \cap V \setminus \{z, x\} \) and from (2) and (3) it follows that \( f(d) \in W \cap V \setminus W = \emptyset \) which is impossible. Thus \( F(V) \subset W \subset U \) and \( F \) is continuous at the point \( e \). \( \square \)

As a special case of the previous theorem, for \( D = X \), we have
Corollary 1.1. If \((X, \mathcal{O}_X)\) is a \(T_1\)-space, \((Y, \mathcal{O}_Y)\) is a regular space and \(f \in \Lambda(X, Y)\), then the function \(F : X \to Y\) given by \(F(x) = y_x\) is continuous.

Example 1.3. If \(f\) is the function from the Example 1.1, then \(F(x) = 0 \neq f(x)\), for all \(x \in X\).

Theorem 1.2. Let \((Y, d)\) be a metric space and \(f \in \Lambda(X, Y)\). Then,

a) For all \(r > 0\), the set \(\Delta_r = \{z \in X \mid d(y_z, f(z)) > r\}\) has no accumulation points in \(X\).

b) \(\Delta_r\) is a closed, discrete subspace of \(X\) and \(|\Delta(f)| \leq e(X)\), where \(\Delta(f) = \{x \in X : f\) is discontinuous at \(x\}\) and \(e(X) = \sup\{|D| : D \subseteq X \text{ is closed and discrete}\}\).

c) If \(X\) is a separable metrizable space, then \(|\Delta(f)| \leq \omega\).

d) For all \(r > 0\), \(\Delta_r\) is a nowhere dense set.

e) If the space \(X\) is metrizable with a complete metric, then the mapping \(f\) is continuous on a set of the second category.

f) If \((X, O)\) is a compact space, then \(\Delta_r\) is a finite set (for each \(r > 0\)) and \(f\) is a bounded function.

Proof. a) Suppose that for some \(r > 0\) we have \(\Delta_r \neq \emptyset\), i.e.

\[
\exists x \in X \quad \forall U \in \mathcal{U}(x) \quad \exists y \in U \setminus \{x\} \cap \Delta_r.
\]

Then, there is a net \(< x_U | U \in \mathcal{U}(x) \to x >\). By the continuity of \(f\) from Corollary 1.1 we have \(< y_{x_U} | U \in \mathcal{U}(x) \to y_x\), hence there is \(W \in \mathcal{U}(x)\) such that

\[
\forall G \in \mathcal{U}(x) | G \subseteq W \Rightarrow d(y_{x_U}, y_x) < r/2.
\]

Since \(\lim_{U} f = y_x\), there is \(V \in \mathcal{U}(x)\) satisfying

\[
\forall t \in V \setminus \{x\} \quad f(t), y_x < r/2.
\]

If \(O \in \mathcal{U}(x)\) and \(O \subseteq V \cap W\), then \(x_0 \in W \cap V \setminus \{x\}\) and by (5) and (6)

\[
d(y_{x_0}, f(x)) \leq d(y_{x_0}, y_x) + d(y_x, f(x_0)) < r.
\]

A contradiction to \(x_0 \in \Delta_r\).
b) From (f), for each \( x \in \Delta \), there is \( U \in \mathcal{U}(x) \) such that \( U \cap \Delta = \{ x \} \), so the discreteness is verified. Since \( \bar{\Delta} = \Delta \cup (\mathcal{U}_z \setminus \Delta_z) \), \( \Delta_1 \) is closed. Finally, for all \( n \in \mathbb{N} \) we have \( | \Delta_1 / \mathcal{N} | \leq e(X) \), hence \( | \Delta(f) | = 1 | \bigcup_{n \in \mathbb{N}} \Delta_1 / \mathcal{N} | \leq \omega_1(X) = e(X) \).

c) In a separable metric space we have \( e(X) = d(X) = \omega \).

d) Suppose that \( x \in \text{int}\Delta_1 = \text{int}\Delta_1 \). Choose \( U \in \mathcal{U}(x) \) such that \( U \cap \Delta_1 = \{ x \} \). Then \( U \cap \text{int}\Delta_1 = \{ x \} \), which is impossible because \( X \) has no isolated points.

e) Follows from (d), \( \Delta(f) = \bigcup_{n \in \mathbb{N}} \Delta_1 / \mathcal{N} \), and the Baire Category Theorem.

f) \( \Delta_1 \) is finite because of (a). Let \( \Delta_1 = \{ x_1, \ldots, x_k \} \) and let \( F \) be the function from Corollary 1.1. Then

\[
\forall z \in X \setminus \Delta_1 \quad d(f(z), F(z)) \leq 1
\]

and for all \( x, y \in X \setminus \Delta_1 \) we have

\[
d(f(x), f(y)) \leq d(f(x), F(x)) + d(F(x), F(y)) + d(F(y), f(y)) \leq 2 + \rho(F(X)).
\]

Now, \( F(X) \) is a compact set in \( Y \), thus it must be bounded. So, \( \rho(f(X \setminus \Delta_1)) \leq 2 + \rho(F(X)) < \infty \) and since \( f(\Delta_1) \) is a bounded set we have \( \rho(F(X)) < \infty \). \( \square \)

2. \( \Lambda(X, R) \) as a Banach algebra

The space of all bounded real-valued functions \( B(X, R) \) with the norm

\[
\| f \| = \sup_{x \in X} | f(x) |
\]

is a commutative Banach algebra with the unit.

Theorem 2.1. If \( (X, O) \) is a compact, then \( \Lambda(X, R) \) is a Banach subalgebra of \( B(X, R) \).

Proof. By Theorem 1.2(f) we have \( \Lambda(X, R) \subset B(X, R) \). Also, if \( \lim_{x} f \) and \( \lim_{x} g \) exist, then \( \lim_{x} (af + bg) \) and \( \lim_{x} fg \) exist, hence \( \Lambda(X, R) \) is a subalgebra of \( B(X, R) \).
Let us prove that \( A(X, R) \) is a closed subset of \( B(X, R) \). Suppose \( f_n \mid n \in N \rightarrow f \), where \( f_n \in A(X, R) \), \( n \in N \), i.e.

\[
(7) \quad \forall \epsilon > 0 \exists n_0 \in N \forall n \forall \epsilon \quad f(x) - f(x) < \epsilon.
\]

A convergent sequence is bounded, thus there is \( M > 0 \) such that

\[
(8) \quad \forall \epsilon > 0 \exists n \in I \mid f_n(x) < M.
\]

Let \( x_0 \in X \) and \( y_{n_0}^\infty = \lim_{n \to \infty} f_n, n \in N \). Since \( \mid \mid \) is a continuous function, from (8) we have \( \lim_{n \to \infty} \mid f_n \mid = \mid y_{n_0}^\infty \mid \leq M \). So, \( \mid y_{n_0}^\infty \mid n \in N \) is a bounded real sequence.

Suppose that \( \alpha \) and \( \beta \) are the accumulation points of the sequence \( y_{n_0}^\infty \mid n \in N \). By (7), for \( \epsilon = \alpha - \beta \mid / 6 \) there is \( n_0 \in N \) satisfying

\[
(9) \quad \forall n \geq n_0 \forall \epsilon \in X \mid f_n(x) - f(x) \mid < \epsilon.
\]

Choose \( n_1, n_2 \geq n_0 \) such that \( \mid y_{n_1}^\infty - \alpha \mid, \mid y_{n_2}^\infty - \beta \mid < \epsilon \). Since \( y_{n_2}^\infty = \lim_{n \to \infty} f_n, i = 1, 2 \), there is \( U \in \mathcal{U}(x_0) \) such that for each \( x \in U \setminus \{x_0\} \) we have \( f_n(x) - y_{n_2}^\infty \mid < \epsilon \), \( i = 1, 2 \). Choose such \( x \in U \setminus \{x_0\} \) (this is possible because \( x_0 \) is not an isolated point). Now, we have: \( \mid \alpha - \beta \mid \leq \alpha - y_{n_2}^\infty + \mid y_{n_1}^\infty - f_n(x) \mid + \mid f_n(x) - f(x) \mid + \mid f(x) - f_n(x) \mid + \mid f_n(x) - y_{n_2}^\infty \mid + \mid y_{n_1}^\infty - y_{n_2}^\infty \mid - \beta \mid < 6 \epsilon = \alpha - \beta \mid \), a contradiction! The sequence \( y_{n_0}^\infty \mid n \in N \) converges.

Let \( y_{n_0}^\infty \mid n \in N \rightarrow y_{n_0} \). We will prove that \( \lim_{n \to \infty} f = y_{n_0} \), i.e.

\[
(10) \quad \forall \epsilon > 0 \exists U \in \mathcal{U}(x_0) \forall \epsilon \in U \setminus \{x_0\} \mid f(x) - y_{n_0} \mid < \epsilon.
\]

Given \( \epsilon > 0 \), for \( \epsilon = \epsilon' / 3 \), there is \( n_0 \in N \) satisfying (9). Let \( m \in N \) be such that \( m \geq n_0 \) and \( \mid y_{n_0}^\infty - y_{n_0} \mid < \epsilon \). Then, by (9)

\[
(10) \quad \forall \epsilon \in X \mid f_m(x) - f(x) \mid < \epsilon.
\]

Since \( y_{n_0}^\infty = \lim_{n \to \infty} f_n, \) there exists a neighbourhood \( U \in \mathcal{U}(x_0) \) such that

\[
\forall \epsilon \in U \setminus \{x_0\} \mid f_n(x) - y_{n_0}^\infty \mid < \epsilon.
\]

According to the previous inequalities, for each \( \epsilon \in U \setminus \{x_0\} \) we have

\[
\mid f(x) - y_{n_0} \mid \leq \mid f(x) - f_m(x) \mid + \mid f_m(x) - y_{n_0}^\infty \mid + \mid y_{n_0}^\infty - y_{n_0} \mid < 3 \epsilon = \epsilon',
\]

\]
and (10) is proved. Thus, for each \( x_0 \in X \) there exists \( \lim_{\alpha} f, \) hence \( f \in \Lambda(X, R). \)

**Remark 2.1.** Clearly, the Banach algebra \( \Lambda(X, R) \) is a Banach subalgebra of \( \Lambda(X, R). \) Moreover, for each \( F \in \Lambda(X, R) \) and \( r > 0 \) there is \( f \in B(F, r) \setminus \Lambda(X, R) \) given by

\[
\begin{align*}
    f(x) &= \begin{cases} 
    F(x) & \text{for } x \neq x_0 \\
    F(x_0) + r/2 & \text{for } x = x_0
    \end{cases}
\end{align*}
\]

where \( x_0 \in X. \) \((B(F, r)\) is an open ball). So, \( \Lambda(X, R) \) is a nowhere dense subspace of \( \Lambda(X, R). \)

For a function \( \varphi : X \to R \) and \( r > 0 \) we define a subset \( S_{\varphi, r} \subset X \) as follows:

\[
S_{\varphi, r} = \{ x \in X \mid |\varphi(x)| > r \}.
\]

Also, we will observe the following vector space of real-valued functions:

\[
Z(X, R) = \{ \varphi \in X \mid S_{\varphi, r} \text{ is finite for all } r > 0 \}.
\]

**Theorem 2.2.** Let \((X, O)\) be a compact Hausdorff space. Then

\[
\Lambda(X, R) = Z(X, R) \oplus \Lambda(X, R).
\]

**Proof.** Let \( \varphi \in Z(X, R), x_0 \in X \) and \( \epsilon > 0. \) Then \( W = (X \setminus S_{\varphi, \epsilon/2}) \cup \{x_0\} \)

is a cofinite set containing \( x_0, \) hence \( W \in \mathcal{U}(x_0) \). Since \( |\varphi(x)| < \epsilon \) for all \( x \in W \setminus \{x_0\}, \) we have \( \lim_{\alpha} \varphi = 0. \) So, we proved \( Z(X, R) \subset \Lambda(X, R). \)

Moreover, \( Z(X, R) \) is a subspace of \( \Lambda(X, R) \) because for \( \varphi, \psi \in Z(X, R) \) and \( \alpha \in R \) there holds:

\[
S_{\varphi + \psi, r} \subset S_{\varphi, r/2} \cup S_{\psi, r/2} \quad \text{and} \quad S_{\alpha \varphi, r} \subset S_{\varphi, \alpha r/\epsilon}.
\]

Suppose \( \varphi \in C(X, R) \cap Z(X, R) \setminus \{0\}. \) Then, there is \( x \in X \) such that \( \varphi(x) > 0 \) (or \( \varphi(x) < 0 \)) and \( \varphi^{-1}(\varphi(x)/2, \infty)) = S_{\varphi, \varphi(x)/2} \) is an open, finite set. Since \( X \) has no isolated points this is impossible. Thus, \( Z(X, R) \cap Z(X, R) = \{0\}. \)

Let \( f \in \Lambda(X, R) \) and let \( F \in C(X, R) \) be the function defined in Corollary 1.1. By Theorem 1.2 (f) the set

\[
\Delta_r = \{ x \in X \mid F(x) - f(x) > r \} = S_{1-r, F_r}
\]

is finite for each \( r > 0. \) Thus, \( \varphi = f - F \in Z(X, R) \) and \( f = F + \psi. \) Each function \( f \in \Lambda(X, R) \) has the (unique) representation \( f = F + \psi, \) where \( F \in C(X, R) \) and \( \varphi \in Z(X, R). \) \( \square \)
3. Bounded linear functionals on \( \Lambda(I, R) \)

By \( I \) we will denote an arbitrary segment \([a,b]\) \(\subset R\). The proof of the following lemma is elementary.

**Lemma 3.1.** (i) \( \sup(A \cup B) = \max(\sup A, \sup B) \), for all nonempty, bounded 
\( A, B \subset R \).

(ii) If \( F \in C(I, R) \) and \( P \subset I \) is a countable set, then \( \sup_{x \in P} | F(x) | = 
\sup_{x \in \bigcup P} | F(x) | \). \( \square \)

**Theorem 3.1.** The functional \( \Psi : \Lambda(I, R) \to R \) is bounded and linear iff there are the unique bounded, linear functionals \( \Gamma : C(I, R) \to R \) and \( \Phi : Z(I, R) \to R \) such that for all \( F \in C(I, R) \) and all \( \varphi \in Z(I, R) \)

\[ \Psi(F + \varphi) = \Gamma(F) + \Phi(\varphi). \]

**Proof.** (\( \Rightarrow \)) Consider the restrictions \( \bar{\Psi} = \Psi | C(I, R) \) and \( \Phi = \bar{\Psi} | Z(I, R) \).

(\( \Leftarrow \)) The linearity of \( \bar{\Psi} \) is obvious. Let \( F = F + \varphi \in \Lambda(I, R) \). By the previous lemma we have

\[ \|F\| = \sup \{|F(x) + \varphi(x)| : x \in I\} \]

\[ = \sup \{|F(x) + \varphi(x)| : x \in \varphi^{-1}(R \setminus \{0\})\} \cup \{|F(x)| : x \in \varphi^{-1}(0)\} \]

\[ = \max \{ \sup_{\varphi(x) \neq 0} |F(x) + \varphi(x)|, \sup_{\varphi(x) = 0} |F(x)| \} \]

\[ \geq \sup_{\varphi(x) = 0} |F(x)| = \sup_{x \in I} |F(x)| = \|F\| \]

because \( \{x \in I : |\varphi(x)| \neq 0\} = \bigcup_{n \in N} S_{\varphi_n} \) is a countable set. Also, \( \|\varphi\| \leq \|\varphi + F\| + \|F\| = \|\varphi\| + \|F\| \leq 2\|F\| \) and \( \bar{\Psi} \) is bounded because

\[ \|\Psi(f)\| = \|\bar{\Psi}(F + \varphi)\| = |\Gamma(F) + \Phi(\varphi)| \leq \|\Gamma\|\|F\| + \|\Phi\|\|\varphi\| \leq \]

\[ \leq (\|\Gamma\| + 2\|\Phi\|)\|F\|. \] \( \square \)

It is well-known that the set of real-valued functions \( \ell_1(I, R) \) defined by:

\[ \ell_1(I, R) = \{h \in L^1 : (x \in I : h(x) \neq 0) \leq \omega \land \sum_{x \in I} |h(x)| < \infty \} \]
is a vector space with the norm
\[ \|h\| = \sum_{x \in I} |h(x)|, \quad h \in \ell_1(I, R). \]

Also, the following statement is folklore.

**Theorem 3.2.** Let \( h \in \ell_1(I, R) \). Then the functional \( \Phi : Z(I, R) \to R \) given by
\[ \Phi(\varphi) = \sum_{x \in I} \varphi(x)h(x), \quad \varphi \in Z(I, R) \]
is a bounded linear functional on \( Z(I, R) \).

Conversely, if \( \Phi : Z(I, R) \to R \) is a bounded, linear functional, then there is the unique \( h \in \ell_1(I, R) \) satisfying \( \|h\| = \|\Phi\| \) and (**) . □

If \( BV(I, R) \) is the space of all real-valued functions of bounded variation, whose domain is \( I = [a, b] \) and
\[ NV_0(I, R) = \{v \in BV(I, R) : v(a) = 0 \land \forall z \in I \setminus \{a\} \quad v(z - 0) = v(z)\} \]
then we have the following consequence of Theorems 3.1, 3.2 and the well-known Riesz Theorem:

**Theorem 3.3.** Let \( \Psi : \Lambda(I, R) \to R \) be a bounded, linear functional. Then, there are the unique \( v \in NV_0(I, R) \) and \( h \in \ell_1(I, R) \) such that for each \( f = F + \varphi \in \Lambda(I, R) \)
\[ \Psi(F + \varphi) = \int_I Fdv + \sum_{x \in I} \varphi(x)h(x). \] (11)

Conversely, for each \( v \in NV_0(I, R) \) and \( h \in \ell_1(I, R) \) the mapping \( \Psi \), defined by (11), is a bounded, linear functional on \( \Lambda(I, R) \). (\( \int_I Fdv \) is the Riemann-Stieljes integral). □

**References**

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