Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 2 (1995), 93-101

Review of Research
Faculty of Science
Mathematics Series

REGULAR NULL-ADDITIVE MONOTONE SET FUNCTIONS

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Abstract

A singleton characterization of an atom of regular null-additive monotone set function is given. Autocontinuous from above monotone set function which is inner regular and exhaustive on the family of compact sets is always regular.

AMS Mathematics Subject Classification (1991): 28B10
Key words and phrases: null -additive, autocontinuous from above, regular set function.

1. Introduction

Fuzzy measures were introduced by Sugeno [4]. A fuzzy measure is a non-negative extended real - valued set function μ defined on σ - algebra $\mathcal{F}$ and with the properties:

(FM_1) $\qquad \mu(\emptyset) = 0,$

(FM_2) $\qquad E \subset F \quad \Rightarrow \quad \mu(E) \leq \mu(F),$

(FM_3) $\qquad E_1 \subset E_2 \subset \ldots \quad , E_n \in \mathcal{F} \quad \Rightarrow \quad \mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n),$

(FM_4) $E_1 \supset E_2 \supset ...$, $E_n \in \mathcal{F}$ and there exists n_0 such that $\mu(E_{n_0}) < \infty \Rightarrow \mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

Wang [7] and Suzuki [5],[6] have investigated fuzzy measures with special properties as null - additivity and autocontinuity. In this paper we shall investigate the regularity of set functions with properties (FM_1) and (FM_2) defined on σ - algebra $\mathcal{B}$ of Borel sets of a locally compact set. Using our previous methods from papers [1],[2] and [3] on $\perp$ - decomposable measures with respect to a t -conorm $\perp$ we shall prove a theorem on singleton characterization of an atom of regular null - additive set function and that inner regularity together with autocontinuity and exhaustivity on compact sets imply regularity of set function.

2. Null - additive set functions

Throughout this paper, let X be a locally compact Hausdorff topological space and let $\mathcal{K}$ be the lattice of all compact subsets of X. Borel σ - algebra $\mathcal{B}$ is the smallest σ - algebra containing $\mathcal{K}$. We shall denote by $\mathcal{O}$ the class of all open sets belonging to $\mathcal{B}$.
All the considered set functions in this paper are supposed to be with values in $[0,\infty]$, monotone and equal to zero on the empty set.
We have by Wang [7]

Definition 2.1.

A set function μ,

$$\mu : \mathcal{B} \to [0,\infty],$$

is called null - additive, if we have

$$\mu(E \cup F) = \mu(E)$$

whenever $E \in \mathcal{B}$, $F \in \mathcal{B}$, $E \cap F = \emptyset$ and $\mu(F) = 0$.

Example 1. *$\perp$ - decomposable measure $m : \mathcal{B} \to [0,1]$ with respect to a t - conorm $\perp$ is always null - additive.*

Example 2. *Let $\mu(E) \neq 0$ whenever $E \in \mathcal{B}, E \neq \emptyset$. Then μ is null - additive.*

Example 3. *Let* $X = \{x, y\}$ *and define* μ *in the following way :* $\mu(X) = 1$ *and* $\mu(E) = 0$ *for* $E \neq X$. *Then* μ *is not null - additive.*

Definition 2.2. *A set function* μ *is called* σ *- finite if there exists a sequence* $\{X_n\}$ *such that*

$$X_1 \subset X_2 \subset ..., \qquad \cup_{n=1}^{\infty} X_n \in \mathcal{B} \qquad and \qquad \mu(X_n) < \infty \quad (n \in N).$$

Throughout this paper the set function μ always will be σ - finite.

Definition 2.3. *A set* $A \in \mathcal{B}$ *is an atom of* μ *iff* $\mu(A) > 0$ *and either* $\mu(B) = 0$ *or* $\mu(B) = \mu(A)$ *and* $\mu(A \setminus B) = 0$ *for* $B \subset A, B \in \mathcal{B}$.

Remark 1. *By* σ *- finiteness of* μ, *every atom has a finite measure.*

Definition 2.4. *A set function* μ *is regular if for each set* $A \in \mathcal{B}$ *and each* $\epsilon > 0$ *there exist* $K \in \mathcal{K}$ *and* $V \in \mathcal{O}$ *such that* $K \subset A \subset V$ *and*

$$\mu(V \setminus K) < \epsilon.$$

Remark 2. *A set function* μ *is regular iff*

$$inf\{\mu(V \setminus K) : K \subset A \subset V, K \in \mathcal{K}, V \in \mathcal{O}\} = 0$$

for each set $A \in \mathcal{B}$.

Proposition 1. *Regular monotone set function* μ *is* $\mathcal{O}$ *- exhaustive, i.e.,*

$$\lim_{n \to \infty} \mu(O_n) = 0$$

for each sequence $\{O_n\}$ *of open sets from* $\mathcal{O}$ *which are pairwise disjoint.*

Proof. or a sequence $\{O_n\}$ of open sets we have

$$\cup_{n=1}^{\infty} O_n \in \mathcal{O}.$$

Therefore by the regularity ofμ we have that for any $\epsilon > 0$ there exists a compact set K such that

$$K \subset \cup_{n=1}^{\infty} O_n \qquad and \qquad \mu(\cup_{n=1}^{\infty} O_n \setminus K) < \epsilon.$$

Since $\{O_n\}$ is an open cover of K and therefore exists $n_0 \in N$ such that $K \subset \cup_{n=1}^{n_0} O_n$. The monotonicity of μ implies

$$\mu(O_k) \leq \mu(O_k \cup (\cup_{n=1}^{n_0} O_n \setminus K)) \leq \mu(\cup_{n=1}^{\infty} O_n \setminus K)$$

for $k \geq n_0 + 1$.

We shall give a singleton characterization of atoms of regular null - additive set function.

Theorem 1. *Let μ be a regular null -additive set function. If $A \in \mathcal{B}$ is an atom of μ, then there exists a point $a \in A$ such that*

$$\mu(A) = \mu(\{a\}).$$

Proof. Let $A \in \mathcal{B}$ be an atom of μ. If we denote by $\mathcal{K}_1$ the family of all compact sets $K \subset A$ such that

$$\mu(A \setminus K) = 0,$$

then we shall prove that any K from $\mathcal{K}_1$ is an atom of μ. Since for any $B \subset K, B \in \mathcal{B}$ we have $K \setminus B \subset A \setminus B$, we obtain by the monotonicity of μ that either $\mu(B) = 0$ or $\mu(K \setminus B) = 0$ and so $\mu(K) = \mu((K \setminus B) \cup B) = \mu(B)$. For both cases the null - additivity of μ implies

$$\mu(K) = \mu((A \setminus K) \cup K) = \mu(A) > 0.$$

Let We have for$K_1, K_2 \in \mathcal{K}_1$ that

$$\mu((A \setminus K_1) \cup (A \setminus K_2)) = \mu(A \setminus (K_1 \cap K_2))$$

and $\mu(A \setminus K_2) = 0$. Hence by the null - additivity of μ

$$\mu(A \setminus (K_1 \cap K_2)) = \mu(A \setminus K_1) = 0,$$

i.e., $K_1 \cap K_2 \in \mathcal{K}_1$. We shall prove that

$$K_0 = \bigcap_{K \in \mathcal{K}_1} K$$

is a non - empty compact set . If we would suppose contrary, i.e., $K_0 = \emptyset$, then it would exist some finite subcollection of $\{K\}_{K \in \mathcal{K}_1}$ with the empty

intersection.This is impossible, since this finite subcollection would belong to $\mathcal{K}_1$, but it is an atom as an element of $\mathcal{K}_1$, which is non - empty.
We shall prove that $K_0 \in \mathcal{K}_1$. For $K \in \mathcal{K}_1$ it has to be $\mu(K \setminus K_0) = 0$. If we suppose that this is not true, then for $B \subset K \setminus K_0$ either $\mu(B) = 0$ or $\mu((K \setminus K_0) \setminus B) = 0$, which imply

$$\mu(K \setminus K_0) = \mu(((K \setminus K_0) \setminus B) \cup B) = \mu(B).$$

For both cases by the supposition $\mu(K \setminus K_0) > 0$ and so $K \setminus K_0$ would be an atom of μ.Since A and K are atoms of μ we have $\mu(A) > 0$ and $\mu(K_0) = 0$ (namely, $K_0 \subset K$ and $\mu(K \setminus K_0) > 0$). Null- additivity of μ implies

$$\mu(A) = \mu((A \setminus K) \cup (K \setminus K_0) \cup K_0) = \mu(K \setminus K_0).$$

By the supposition $\mu(K \setminus K_0) > 0$ and the preceding equality we obtain by the atomness of A

$$\mu(A \setminus (K \setminus K_0)) = 0.$$

Therefore $K \setminus K_0$ have to contain an element of $\mathcal{K}_1$, what is impossible, since K_0 is non - empty. Therefore $\mu(K \setminus K_0) = 0$. Hence by null - additivity of μ

$$\mu(A) = \mu((A \setminus K) \cup (K \setminus K_0) \cup K_0) = \mu(K_0),$$

i.e., $\mu(A \setminus K_0) = 0$ by the atomness of A. This implies $K_0 \in \mathcal{K}_1$.
Finally,we shall prove that K_0 is a singleton. If we suppose the contrary, i.e. that K_0 contains at least two distinct elements a_1 and a_2, then, since X is a locally compact Hausdorff topological space, there exists an open neighbourhood V of a_1 such that clV does not contain a_2. Hence

$$K_0 = (K_0 \setminus V) \cup (K_0 \cap clV).$$

Since one of the sets $K_0 \setminus V$ or $K_0 \cap clV$ have to belong to $\mathcal{K}_1$, but K_0 is the least element of $\mathcal{K}_1$, we obtain a contradiction. Hence there exists $a \in A$ such that

$$\mu(A) = \mu(K_0) = \mu(\{a\}).$$

In a special case we obtain the following result from [2]:

Corollary 2.1. *Let $m : \mathcal{B} \to [0,1]$ be a regular $\perp$ - decomposable measure with respect to an arbitrary but fixed t - conorm $\perp$. If $A \in \mathcal{B}$ is an atom of m, then there exist a point $a \in A$ such that*

$$m(A) = m(\{a\}).$$

Corollary 2.2. *Each continuous from above (i.e. with the property* (FM_4) *regular Borel null - additive set function* μ *,* $\mu : \mathcal{B}(\mathcal{R}) \to [0, \infty]$*, which has the property*

$$\mu((a,b)) = g(b-a)$$

on each finite interval (a,b) for some continuous at 0 function g with $g(0) = 0$*, is non - atomic.*

Proof. If we suppose that the theorem is not true, then there exists an atom $A \in \mathcal{B}(\mathcal{R})$ of μ. therefore by Theorem 1 there exists an element a from A such that $\mu(A) = \mu(\{a\})$. So we would have

$$\mu(\{a\}) = \mu(\cap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n})) = \lim_{n\to\infty} \mu((a - \frac{1}{n}, a + \frac{1}{n})) = \lim_{n\to\infty} g(\frac{2}{n}) = 0,$$

i.e., $\mu(A) = \mu(\{a\}) = 0$, what is a contradiction.

3. Autocontinuity from above

We have by Wang [7]

Definition 3.1. *A set function* μ *is called autocontinuous from above if we have*

$$\mu(A \cup B_n) \to \mu(A)$$

whenever $A \in \mathcal{B},\ B_n \in \mathcal{B},\ A \cap B_n = \emptyset\ (n \in N),\ \mu(B_n) \to 0.$

Proposition 2. *A monotone set function* μ *is autocontinuous from above iff for every* $A \in \mathcal{B}$ *and* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon, A) > 0$ *such that, whenever*

$$A \in \mathcal{B},\ B \in \mathcal{B},\ A \cap B = \emptyset,\ \mu(B) < \delta$$

implies

$$\mu(A) - \epsilon \le \mu(A \cup B) \le \mu(A) + \epsilon.$$

Remark 3. *Obviously, autocontinuity from above of* μ *implies null - additivity of* μ*.*

Proposition 3. *Let μ be an autocontinuous from above set function. If $\{E_n\}$ and $\{F_n\}$ are two decreasing sequences from $\mathcal{B}$ such that $E_1 \cap F_1 = \emptyset$ and*

$$\lim_{n\to\infty} \mu(E_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \mu(F_n) = 0,$$

then

$$\lim_{n\to\infty} \mu(E_n \cup F_n) = 0.$$

We have that inner regularity and autocontinuity from above implies regularity:

Theorem 2. *Let μ be a set function which is autocontinuos from above, $\mathcal{K}$ - exhaustive and satisfies the equality*

$$\mu(A) = sup\{\mu(K) : K \in \mathcal{K}, K \subset A\} \qquad (A \in \mathcal{B}). \tag{1}$$

Then μ is a regular Borel set function.

*Proof.*Suppose that the theorem is not true. Then there exist a set $A \in \mathcal{B}$ and a number $\epsilon > 0$ such that

$$\mu(V \setminus K) > \epsilon \tag{2}$$

for each $K \in \mathcal{K}, V \in \mathcal{O}, K \subset A \subset V$. Let us fix such sets K_0 and V_0. The equality (1) implies that there exist

$$C_1, D_1 \in \mathcal{K},\ C_1 \subset V_0 \setminus A,\ D_1 \subset A \setminus K_0$$

such that

$$\mu(C_1) \geq \frac{1}{2}\mu(V_0 \setminus A) \qquad \text{and} \qquad \mu(D_1) \geq \frac{1}{2}\mu(A \setminus K_0).$$

Let $V_1 = V_0 \setminus C_1$ and $K_1 = K_0 \cup D_1$.It is obvious that $V_1 \in \mathcal{O}$, $K_1 \in \mathcal{K}$ and $K_1 \subset A \subset V_1$.Then (2) implies

$$\mu(V_1 \setminus A_1) > \epsilon.$$

Repeating the preceding procedure after n - steps we obtain sets

$$C_n, D_n \in \mathcal{K},\ C_n \subset V_{n-1} \setminus A,\ D_n \subset A \setminus K_{n-1}$$

such that

$$(3) \qquad \mu(C_n) \geq \frac{1}{2}\mu(V_{n-1} \setminus A), \qquad \mu(D_n) \geq \frac{1}{2}\mu(A \setminus K_{n-1}).$$

We have obtained two sequences $\{C_n\}$ and $\{D_n\}$ of pairwise disjoint sets from $\mathcal{K}$. Now $\mathcal{K}$ - exhaustivity of μ implies

$$\lim_{n\to\infty} \mu(C_n) = 0 \qquad \lim_{n\to\infty} \mu(D_n) = 0.$$

Hence by (3)

$$(4) \qquad \lim_{n\to\infty} \mu(V_n \setminus A) = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \mu(A \setminus K_{n-1}) = 0.$$

In this way we have obtained decreasing sequences $\{V_n \setminus A\}$ and $\{A \setminus K_{n-1}\}$ from $\mathcal{B}$ (namely,$V_n \subset V_{n-1}$ and $K_n \supset K_{n-1}$) such that (4) holds and

$$(V_0 \setminus A) \cap (A \setminus K_0) = \emptyset.$$

Therefore by Proposition 3.

$$\lim_{n\to\infty} \mu(V_n \setminus K_n) = \lim_{n\to\infty} \mu((V_n \ A) \cup (A \setminus K_n)) = 0.$$

Contradiction with (2).

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Received by the editors April 24, 1991