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SOME REMARKS ON ERRORS IN THE CLASS OF BLOCK-CODES WHICH CORRESPOND TO L-VALUED FUZZY SET

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Abstract

By using the results from ([2],[3]), in which conditions for an arbitrary block-code to correspond to an $L-$valued fuzzy set are given, the results related to errors of such block - codes are given in this paper.

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1. For a nonvoid set S and a complete lattice $(L, 0, 1)$, a function $\bar{A} : S \to L$ is an $L-$ valued set on S, or an $L-$ fuzzy set on S. If $p \in L$, then $\bar{A}_p : S \to \{0, 1\}$ is defined as $\bar{A}_p(x) = 1$ iff $\bar{A}(x) \geq p$. Obviously, $\bar{A}_p = \{x \in S | \bar{A}_p(x) = 1\}$.

$$\bar{A}(x) = \bigvee_{p \in L} p \circ \bar{A}_p(x), \tag{1}$$

where $\bigvee$ is the supremum in L, and the operation " $\circ$ " is defined as:

$$p \circ 0 = 0 \in L, \quad \text{and} \quad p \circ 1 = p, \ (p \in L).$$

The equality (1) gives a decomposition of $\bar{A}$ into a family of characteristic functions $\{\bar{A}_p | p \in L\}$, or, equivalently into a collection of corresponding subsets A_p of S.

Some properties of this family are:

(a) the partially ordered set $(\{A_p | p \in L\} \subseteq)$ is a lattice in which the infimum is the intersection;

(b)

$$\bigcap_{p \in K \subseteq L} A_p = A_{\bigvee_{p \in K} p};$$

(c) $A_0 = S$;

(d) $p \leq q$ implies $A_q \subseteq A_p$.

It was proved in [2] that (a),(b) and (c) are necessary and sufficient conditions under which a family of subsets of S, $\{A_p | p \in P\}$, corresponds to a fuzzy set $\bar{A} : S \to L$ in the sense of (1).

A fuzzy set $\bar{A} : S \to L$ induces a partition on L : if $\sim$ is a relation on L given with

$$p \sim q, \quad \text{and only if} \quad A_p = A_q,$$

then $\sim$ an equivalence relation on L, and for every $p \in L$,

$$\bigvee [p]_\sim \in [p]_\sim$$

$$([p]_\sim \stackrel{def}{=} \{q \in L | p \sim q\}) \; [2].$$

Moreover, $p \to p_m = \bigvee [p]_\sim$ is on operation on L.

Let $S = \{1, 2, ..., n\}$, and let L be an arbitrary finite lattice. We say that to every fuzzy set $A : S \to L$ there corresponds a binary block - code V of the length n the following way.

Every class $[p]_\sim$, $p \in L$, uniquely determines a codeword $v_p = x_1 ... x_n$, such that

$$x_i \stackrel{def}{=} \bar{A}_p(i).$$

Let $p_m = \bigvee [p]_\sim, p \in L$. We shall denote the class $[p]_\sim$, as well as tht corresponding codeword, by v_p.

Since $p \to p_m$ is a closure operation on L, it is possible to define an order relation on the collection od $(\sim)$- classes of L:

$$v_p \leq v_q \quad \text{if and only if} \quad p \leq q.$$

According to this order, $L/\sim$ is a lattice isomorphic with the one of closed elements under the closure.

For the binary block - code $V \subseteq \{0,1\}^n$, we shall also consider the dual of the natural ordering relation:

$x_1 ... x_n \leq y_1 ... y_n$ if and only if $x_1 \geq y_1, ..., x_n \geq y_n$,

on the right side being the ordinary ordering among the numbers.

Let now $V \subseteq \{0,1\}^n$ be a bunarry block - code, $S = \{1, ..., n\}$, and L a lattice. We say that a fuzzy set $\bar{A}_V : S \to L$ corresponds to V, if the block - code corresponding to $\bar{A}_V$ (in the sense of (2)) is V.

It was shown in ([3]) that: for a binary block - code $V \subseteq \{0,1\}^n$ there is a fuzzy set which corresponds to V iff the following conditions are satisfied

(i) $(V, \leq)$ is a lattice,

(ii) V is closed under the conjunction defined componentwise;

(iii) $11...1 \in V$.

As it is known ([1]), the Hamming distance $d(x,y)$ between two codewords $x, y \in \{0,1\}^n$ is the number of coordinates in which x and y differ. The code distance $d(V)$ of a code $V \subseteq \{0,1\}^n$ is the minimum distance between two different codewords in V.

Let $\bar{A} : S \to L$ be a fuzzy set. We say that the number of elements of the set S which are mapped into the same element p of L is a degree of the class v_p (i.e. of the corresponding codeword), and we denote it bu $s(v_p)$.

2. In the following theree propositions, V will be a code corresponding to a fuzzy set $\bar{A} : S \to L$, and $V \neq \{11...1\}$. Recall that

$$\bar{A}(S) = \{p \in L | p = \bar{A}(x) \quad \text{for some} \quad x \in S\}.$$

Proposition 1. *The code V enables correction of t errors iff*

$$\min_{p \in \bar{A}(s) \setminus \{1\}} s(v_p) > 2t.$$

Proof. Let V enablels correction of t errors. The for $v_p, v_q \in V$ we have $d(v_p, v_q) > 2t$.

For $q = 1, d(v_p, v_1) > 2t, i.e. s(v_p) > 2t$. This implies

$$\min_{p \in \bar{A}(s) \setminus \{1\}} s(v_p) > 2t.$$

Conversely.From

$$\min_{p\in \bar{A}(s)\setminus\{1\}} s(v_p) > 2t$$

it follows (similarly as in the proof of P.5. [3]) $d(V) > 2t$, and according to [1], from $d(V) > 2t$ follows that the code V enables correction of t- errors. □

Let the fuzzy set $p_1'p_2'...p_n'$ be obtained from the fuzzy set $p_1p_2...p_np_{i_1}...p_{i_k}$ by the replacement of one latter p_m inthe word $p_1p_2...p_n$ for an arbitrary letter $p_j \in L$ i.e. because of one error of fuzzy set $p_1...p_n$.

Proposition 2. *Let $\bar{A}_V$ be a fuzzy set (which corresponds to the code V) of the form $p_1p_2...p_n$ and let $p_{i_1},...,p_{i_k}$ be those elements from $\{p_1,p_2,...,p_n\}$ such that $s(p_{i_1}),...,s(p_{i_k})$ are odd numbers. Then the fuzzy set $p_1p_2...p_np_{i_1}...p_{i_k}$ enables the detection of one error of $\bar{A}_V$.*

Proof. Let the fuzzy set $p_1'p_2'...p_n'p_{i_1}'...p_{i_k}'$ be obtained from the fuzzy set $p_1p_2...p_np_{i_1}...p_{i_k}$ by the replacement of one latter p_m in the word $p_1p_2...p_np_{i_1}...p_{i_k}$ for an arbitrary letter $p_j \in L$ i.e. because of one error. Let

$$v_{p_m} = x_1...x_{i_k} \quad \text{and} \quad v_{p_j} = y_1...y_{i_k}$$

be the codeword for p_m and p_j.

Then it follows immediately that, for this codeword, it holds that

$$x_1 + ... + x_{i_k} \neq 0 \quad (mod\, 2)$$

or

$$y_1 + ... + y_{i_k} \neq 0 \quad (mod\, 2)$$

by which we detect the error. □

Proposition 3. *Let $\bar{A}$ be a fuzzy set (which corresponds to the code V) of the from $p_1p_2...p_n$ and let $p_{i_1},...,p_{i_k}$ be those element from $\{p_1,p_2,...,p_n\}$ such that $s(p_{i_1}),...,s(p_{i_k})$ are odd numbers. Then the fuzzy set*

$$p_1p_2...p_np_1p_2...p_np_{i_1}p_{i_2}...p_{i_k}$$

enables the corection of one error of the $\bar{A}_V$.

Proof. Suppose that from

$$p_1...p_n p_1...p_n p_{i_1}...p_{i_k}$$

because of one error we obtained the fuzzy set

$$p'_1...p'_n p''_1...p''_n p'_{i_1}...p'_{i_k}$$

and let

$$v_{p_s} = x'_1...x'_n...x''_1...x''_n...x'_{i_1}...x'_{i_k}$$

be the codeword for $p_s \in L$.

By considering the following possible cases:

a) $p'_1...p'_n \neq p''_1...p''_n$;

$$x'_1 + ... + x'_n + x'_{i_1} + ... + x'_{i_k} \neq 0 \ \ (mod 2) \quad \text{for some } p_s \in L:$$

b) $p'_1...p'_n \neq p''_1...p''_n$;

$$x''_1 + ... + x''_n + x'_{i_1} + ... + x'_{i_k} \neq 0 \ (mod 2) \quad \text{for some } p_s \in L,$$

and

c) $p'_1...p'_n = p''_1...p''_n$;

$$x_1 + ... + x'_n + x'_{i_1} + ... + x'_{i_k} \neq 0 \ (mod 2) \quad \text{for some } \ p_s \in L$$

the result follows. □

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