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SOME THEOREMS OF FUNCTIONAL ANALYSIS WITH K-CONVERGENCE

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ABSTRACT

In this paper some properties of weakly K-sequences are proved. The main result is that a linear weakly continuous operator from a normed K-space into a normed space is continuous.

1. INTRODUCTION

In 1953. S. Mazur and W. Orlicz have introduced a kind of summable convergence in a topological vector space. Recently, this convergence was rediscovered by members of the Katowice Branch of the Mathematical Institute of the Polish Academy of Sciences. They have developed the theory of K-convergent spaces (K = Katowice). The notion of K-convergence has proven to be quite useful in studying various topics in functional analysis ([2], [3], [4], [6], [7], [8], [9]).

In this paper we study the notion of K-property of the weak convergence on a normed space.

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2. WEAKLY K-SEQUENCES

Let X be a normed vector space. A sequence $\{x_n\}$ from X is a weakly K-sequence if each subsequence of $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that the sequence

$$\left\{ \sum_{i=1}^{n} x_{n_i} \right\}$$

is weakly convergent to an element $x \in X$. We shall say that $\{x_n\}$ is weakly convergent with respect to $(X,G)^*$, where G is the convergence induced by the norm on X and $(X,G)^*$ is the dual of (X,G), i.e. the space of linear functionals which are sequentially continuous with respect to G. X is a weakly K-space if each weakly convergent sequence from X to zero is a weakly K-sequence.

Let us recall the Banach-Saks theorem: If $\{x_n\}$ is a sequence from a Hilbert space H such that it weakly converges to an element x_0 from H, then there exists a subseqence $\{x_{n_i}\}$ of $\{x_n\}$ such that the sequence $\{y_n\}$, where

$$y_n = \frac{1}{n} \sum_{i=1}^{n} x_{n_i},$$

is norm convergent to x_0.

We have for normed vector spaces the following version of the Banach-Saks theorem.

Theorem 1. *Let $\{x_n\}$ be a sequence from a normed vector space X such that the sequence $\{x_n - x_0\}$ is a weakly K-sequence. Then the sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that the sequence $\{y_n\}$, where*

$$y_n = \frac{1}{n} \sum_{i=1}^{n} x_{n_i},$$

is norm convergent to x_0.

Proof. Since the sequence $\{x_n - x_0\}$ is a weakly

K-sequence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that for some $x \in X$

$$f\left(\sum_{i=1}^{n} (x_{n_i} - x_0)\right) \to f(x) \qquad (f \in X^*)$$

for $n \to \infty$. Then the sequence

$$\left\{ \sum_{i=1}^{n} (x_{n_i} - x_0) \right\}$$

is weakly bounded.

Hence, by the uniform boundedness theorem, this sequence is also norm bounded. Then, we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} x_{n_i} - x_0 \right\| = \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{n_i} - x_0) \right\| \to 0$$

for $n \to \infty$, i.e. the sequence $\{y_n\}$ converges in norm to x_0.

Corollary. *Let f be a convex continuous functional on a normed vector space X. If $\{x_n\}$ is a sequence from X such that for some $x_0 \in X$ the sequence $\{x_n - x_0\}$ is a weakly K-sequence, then*

$$\liminf f(x_n) \geq f(x_0).$$

Proof. Let $\{y_n\}$ be a subsequence of $\{x_n\}$ such that $\liminf f(x_n) = \lim_{n\to\infty} f(y_n)$. By Theorem 1 there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that the sequence

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} y_{n_i} \right\}$$

is convergent in norm to x_0. Using the convexity of f and the property of real sequences we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} f(y_{n_i}) = \lim_{k\to\infty} f(y_{n_k}) \geq \lim_{k\to\infty} f\left(\frac{1}{k} \sum_{i=1}^{k} y_{n_i}\right) = f(x_0).$$

This implies the desired inequality.

3. WEAKLY CONTINUOUS OPERATORS

Let X and Y be two normed spaces. A linear operator $T : X \to Y$ is weakly continuous if for each sequence $\{x_n\}$ from X such that $x_n \to x$ weakly for some $x \in X$, then $T(x_n) \to T(x)$ weakly.

We have the following important theorem.

Theorem 3. *Let X be a normed K-space and let Y be a normed space. If $T : X \to Y$ is a linear weakly continuous operator, then T is continuous.*

Proof. Let $\{x_n\}$ be a sequence from X such that $\|x_n\| \to 0$ for $n \to \infty$. Suppose that the theorem is not true. Then for arbitrary $\varepsilon > 0$ there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that

$$(1) \qquad \|T(y_n)\| > \varepsilon \qquad (n \in \mathbb{N}).$$

We may assume that Y is separable since we can always replace Y by the closed linear subspace generated by the sequence $\{T(y_n)\}$. For each $T(y_n)$ there exists $g_n \in Y^*$ such that $\|g_n\| = 1$ and

$$(2) \qquad g_n(T(y_n)) = \|T(y_n)\|.$$

Applying the Banach-Alaoglu theorem on the sequence $\{g_n(T(y_n))\}$, we obtain that there exists a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $\{g_{n_i}\}$ weak * converges to an element $g \in Y^*$.

Let $x_{ij} = |(g_{n_i} - g)(T(y_j))|$ for $i \neq j$ $(i,j \in \mathbb{N})$ and $x_{ii} = 0$ $(i \in \mathbb{N})$. Then $\lim_{j\to\infty} x_{ij} = 0$ $(i \in \mathbb{N})$ and $\lim_{i\to\infty} x_{ij} = 0$ $(j \in \mathbb{N})$. So we can apply on matrix $[x_{ij}]$ the Antosik Diagonal Theorem from [1]. Then there exists an increasing sequence $\{p_i\}$ of natural numbers such that

(3) $$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_{p_i p_j} = 0.$$

Since X is a K-space there exists a subsequence $\{s_i\}$ of $\{p_i\}$ such that

$$\lim_{i\to\infty} \sum_{j=1}^{n} y_{s_j} = y$$

for some $y \in X$.

Then we have

$$(g_{s_i} - g)(T(\sum_{j=1}^{n} y_{s_j})) \to (g_{s_i} - g)(T(y)) \text{ for } j \to \infty.$$

We have for arbitrary $p \in N$

(4) $$|(g_{s_i}-g)(T(y_{s_i}))| \le \sum_{\substack{j=1\\ j\ne i}}^{i+p} |(g_{s_i} - g)T(y_{s_i})| +$$

$$+ |\sum_{j=1}^{i+p} (g_{s_i} - g)(T(y_{s_j}))| = \sum_{\substack{j=1\\ j\ne i}}^{i+p} |(g_{s_i}-g)T(y_{s_i})| +$$

$$+|(g_{s_i} - g)(T(\sum_{j=1}^{i+p} y_{s_j}))|.$$

Hence, letting $p \to \infty$ we obtain by (3) and (4)

$$|(g_{s_i}-g)(T(y_{s_i}))| \le \sum_{j=1}^{\infty} x_{s_i s_j} + |(g_{s_i} - g)(T(y))|.$$

This implies

$$(g_{s_i}-g)(T(y_{s_i})) \to 0 \quad \text{for} \quad i \to \infty.$$

Hence

$$g_{s_i}(T(y_{s_i})) \to 0 \quad \text{for} \quad i \to \infty,$$

which is in contradiction with (1) (using (2)). So we have proved Theorem 2.

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REZIME

NEKE TEOREME FUNKCIONALNE ANALIZE SA K-KONVERGENCIJOM

U radu su dobijene neke osobine slabo K-konvergentnih nizova. Tako je dokazana verzija Banach-Saksove teoreme.

Dokazana je važna teorema, koja kaže da je svaki slabo neprekidan operator nad normiranim K-prostorom, a sa vrednostima u normiranom prostoru, neprekidan operator.

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