

A COMMON FIXED POINT THEOREM OF A FAMILY OF
MAPPINGS IN PROBABILISTIC LOCALLY CONVEX SPACES

Mila Stojaković

Fakultet tehničkih nauka, Institut za primenjene osnovne
discipline, 21000 Novi Sad, ul. Veljka Vlahovića 3, Jugoslavija

In this paper a theorem which gives the necessary and sufficient condition for the existence of a unique fixed point for the mappings S, T and A (defined in Theorem 2) is proved.

First we shall give some definitions and notations which we shall use later.

DEFINITION 1 [1]. Let X be a linear space over the real or complex field K and for every i in the index set I there is a function $F^i : X \rightarrow \Delta^+$, where Δ^+ is the family of distribution functions F such that $F(0) = 0$. We shall denote $F^i(p)$ by F_p^i ($i \in I, p \in X$).

$(X, \{F_p^i\}_{i \in I}, t)$ is called a locally convex probabilistic space (LCP-space) if and only if for each $i \in I$ the following conditions are satisfied:

1. $F_0^i = H$, where $H(\epsilon) = \begin{cases} 0, & \epsilon \leq 0 \\ 1, & \epsilon > 0 \end{cases}$,
2. $F_{\lambda p}^i(\epsilon) = F_p^i\left(\frac{\epsilon}{|\lambda|}\right)$ for every $\lambda \in K, p \in X, \epsilon > 0, (\lambda \neq 0)$
3. $F_{p+q}^i(\epsilon_1 + \epsilon_2) \geq t(F_p^i(\epsilon_1), F_q^i(\epsilon_2))$ for every $p, q \in X,$

$\epsilon_1 > 0, \epsilon_2 > 0$, where the mapping $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a T -norm [2].

The (ε, λ) - topology is introduced by the following definition.

DEFINITION 2 [1]. A net $\{p_d\}_{d \in D}$ converges to \circ if and only if for each $i \in I$, every $\varepsilon > 0$ and every $\lambda \in (0, 1)$ there exists $d_0 \in D$ such that

$$F_{p_d}^i(\varepsilon) \geq 1 - \lambda \quad \text{if } d \geq d_0.$$

In what follows we suppose that

$$F_p^i = H \quad \text{for each } i \in I \text{ if and only if } p = \circ.$$

In a similar way one can introduce the notion of a Cauchy sequence and the notion of completeness.

Some fixed point theorems in probabilistic spaces are proved in [1], [2].

THEOREM 1 [3]. Let S and T be continuous mappings of a complete normed space $(X, \|\cdot\|)$ into itself. Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX \cap TX$ which commutes with S and T and satisfies the inequality

$$\|Ax - Ay\| \leq \alpha \|Sx - Ty\|$$

for every $x, y \in X$, where $0 < \alpha < 1$. S, T and A then have a unique common fixed point.

THEOREM 2. $(X, \{F_{i \in I}^i, t)$ be a sequentially complete probabilistic locally convex space with continuous T -norm t and S and T be continuous mappings of X into X .

The mappings S and T have a unique common fixed point in X if and only if there exists a continuous mapping $A: X \rightarrow SX \cap TX$ which commutes with S and T so that AX is a probabilistic bounded subset of X and satisfies the following conditions:

1. For every $i \in I$ there exists $q(i) > 0$ and $f(i) \in I$ such that

$$F_{Ax-Ay}^i(\varepsilon) \geq F_{Tx-Sy}^{f(i)}\left(\frac{\varepsilon}{q(i)}\right)$$

for every $\varepsilon > 0$ and every $x, y \in X$.

2. For every $i \in I$ there exist numbers $n_i \in \mathbb{N}$ and $Q(i) \in (0, 1)$ such that

$$q(f^n(i)) \leq Q(i) < 1$$

for every $n > n_i$.

3. For every $i \in I$ there exists $g(i) \in I$ such that

$$F_X^{f^n(i)}(\varepsilon) \geq F_X^{g(i)}(\varepsilon)$$

for every $\varepsilon > 0$, every $x \in X$ and every $n \in \mathbb{N}$.

Then there exists one and only one element $x^* \in X$ such that Ax^* is the unique common fixed point for the mappings A, S and T .

P r o o f. First, let us prove the necessity of the conditions 1, 2, and 3. Let $z \in X$ be such an element that

$$z = Az = Sz = Tz.$$

The mapping A is defined by $Ax = z$ for all $x \in X$.

The mapping A commutes with S and T since

$$A(Sx) = Ay = z, \quad S(Ax) = Sz = z$$

and

$$A(Tx) = Av = z, \quad T(Ax) = Tz = z.$$

Condition 1 will be satisfied because

$$F_{Ax-Ay}^i(\varepsilon) = F_{z-z}^i(\varepsilon) = 1 \geq F_{Tx-Sy}^i\left(\frac{\varepsilon}{q(i)}\right)$$

for all $x, y \in X$, i.e. $f(i) = i$ for all $i \in I$ and $q(i)$ is any element from $(0, 1)$.

Since $f(i) = i$, condition 3 is also satisfied, i.e.

$$F_X^{f^n(i)}(\varepsilon) = F_X^i(\varepsilon)$$

for all $x \in X$ and all $\varepsilon > 0$. We have that $g(i) = i$, $i \in I$.

Now, we shall prove that conditions 1, 2 and 3 are sufficient. We form a sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ such that

$$Ax_{2n-2} = Sx_{2n-1}, \quad Ax_{2n-1} = Tx_{2n}$$

for all $n \in \mathbb{N}$. Such a sequence always exists because $AX \subset TX \cap SX$.

First, we prove that the sequence $\{Ax_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a Cauchy sequence. It is easy to see that

$$\begin{aligned} F_{Ax_{2n} - Ax_{2n-1}}^i(\varepsilon) &\geq F_{Tx_{2n} - Sx_{2n-1}}^{f(i)}\left(\frac{\varepsilon}{q(i)}\right) = \\ &= F_{Ax_{2n-1} - Ax_{2n-2}}^{f(i)}\left(\frac{\varepsilon}{q(i)}\right) \geq F_{Tx_{2n-2} - Sx_{2n-1}}^{f^2(i)}\left(\frac{\varepsilon}{q(i)q(f(i))}\right) \geq \\ &\geq F_{Ax_{2n-2} - Ax_{2n-3}}^{f^2(i)}\left(\frac{\varepsilon}{q(i)q(f(i))}\right) \geq \dots \geq \\ &\geq F_{Ax_1 - Ax_0}^{f^{2n-1}(i)}\left(\frac{\varepsilon}{\prod_{s=0}^{2n-2} q(f^s(i))}\right) \end{aligned}$$

for all $\varepsilon > 0$, $i \in I$, $n \in \mathbb{N}$. Also we have that

$$\begin{aligned} F_{Ax_{2n+1} - Ax_{2n}}^i(\varepsilon) &= F_{Ax_{2n} - Ax_{2n+1}}^i(\varepsilon) \geq \dots \geq \\ &\geq F_{Ax_1 - Ax_0}^{f^{2n}(i)}\left(\frac{\varepsilon}{\prod_{s=0}^{2n-1} q(f^s(i))}\right) \end{aligned}$$

for all $\varepsilon > 0$, $i \in I$, $n \in \mathbb{N}$. So for every $n \in \mathbb{N}$ we have

$$F_{Ax_n - Ax_{n-1}}^i(\varepsilon) \geq F_{Ax_1 - Ax_0}^{f^{n-1}(i)}\left(\frac{\varepsilon}{\prod_{s=0}^{n-2} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$.

We see that for all $m > k$

$$\begin{aligned} F_{Ax_{2k} - Ax_{2m+1}}^i(\epsilon) &\geq F_{Sx_{2m+1} - Tx_{2k}}^{f(i)}\left(\frac{\epsilon}{q(i)}\right) = \\ &= F_{Ax_{2m} - Ax_{2k-1}}^{f(i)}\left(\frac{\epsilon}{q(i)}\right) \geq \dots \geq F_{Ax_0 - Ax_{2m+1-2k}}^{f^{2k}(i)}\left(\frac{\epsilon}{\prod_{s=0}^{2k-1} q(f^s(i))}\right). \end{aligned}$$

If $2k > 2m+1$ then we have

$$\begin{aligned} F_{Ax_{2k} - Ax_{2m+1}}^i(\epsilon) &\geq F_{Ax_{2k-1} - Ax_{2m}}^{f(i)}\left(\frac{\epsilon}{q(i)}\right) \geq \dots \geq \\ &\geq F_{Ax_{2k-2m-1} - Ax_0}^{f^{2m+1}(i)}\left(\frac{\epsilon}{\prod_{s=0}^{2m} q(f^s(i))}\right) \end{aligned}$$

for every $\epsilon > 0$.

From the last two inequalities we can prove easily that for $n = 2k$, $p = 2m+1$ or for $n = 2k+1$, $p = 2m+1$ the following is satisfied

$$F_{Ax_{n+p} - Ax}^i(\epsilon) \geq F_{Ax_0 - Ax_p}^{f^n(i)}\left(\frac{\epsilon}{\prod_{s=0}^{n-1} q(f^s(i))}\right).$$

When $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$ we have

$$\begin{aligned} F_{Ax_{n+p} - Ax_n}^i(\epsilon) &= F_{Ax_{n+p} - Ax_{n+1} + Ax_{n+1} - Ax_n}^i\left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right) \geq \\ &\geq t\left(F_{Ax_{n+p} - Ax_{n+1}}^i\left(\frac{\epsilon}{2}\right), F_{Ax_{n+1} - Ax_n}^i\left(\frac{\epsilon}{2}\right)\right) \geq \dots \geq \\ &\geq t\left(F_{Ax_{p-1} - Ax_0}^{f^{n+1}(i)}\left(\frac{\epsilon}{2 \prod_{s=0}^n q(f^s(i))}\right), F_{Ax_1 - Ax_0}^{f^n(i)}\left(\frac{\epsilon}{2 \prod_{s=0}^{n-1} q(f^s(i))}\right)\right). \end{aligned}$$

Now we shall prove that for every $\epsilon > 0$, $\lambda \in (0, 1)$ and $i \in I$ there exists $N(i, \epsilon, \lambda)$ such that

$$F_{AX_{n+p}}^i - AX_n(\epsilon) > 1 - \lambda$$

for every $n \geq N(i, \epsilon, \lambda)$ and $p \in \mathbb{N}$.

The set AX is probabilistic bounded which means that

$$\sup_{\epsilon} D_{AX}^i(\epsilon) = 1$$

for all $i \in I$ and so for each $\lambda \in (0, 1)$, there exists $\epsilon_i(\lambda) > 0$ such that

$$D_{AX}^i(\epsilon_i(\lambda)) > 1 - \lambda.$$

Since

$$D_{AX}^i(\epsilon_i) = \sup_{\delta < \epsilon_i} \inf_{u, v \in AX} F_{u-v}^i(\delta)$$

it follows that:

$$\sup_{\delta < \epsilon_i} \inf_{u, v \in AX} F_{u-v}^i(\delta) > 1 - \lambda,$$

and we have that

$$\inf_{u, v \in AX} F_{u-v}^i(\epsilon_i(\lambda)) \geq \inf_{u, v \in AX} F_{u-v}^i(\delta)$$

which implies

$$\inf_{u, v \in AX} F_{u-v}^i(\epsilon_i(\lambda)) \geq \sup_{\delta < \epsilon_i} \inf_{u, v \in AX} F_{u-v}^i(\delta) > 1 - \lambda.$$

Let for every $i \in I$: $\epsilon(i) = \frac{Q^{n_i+1}(i)\epsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$, for every $\epsilon > 0$.

Let $n > n_i$. Then we have

$$\begin{aligned} \frac{\epsilon}{\prod_{s=0}^{n-1} q(f^s(i))} &= \frac{\epsilon}{\prod_{s=0}^{n_i} q(f^s(i)) q(f^{n_i+1}(i)) \dots q(f^{n-1}(i))} \geq \\ &> \frac{\epsilon}{\prod_{s=0}^{n_i} q(f^s(i)) Q^{n-n_i-1}(i)} = \frac{\epsilon(i)}{Q^n(i)}. \end{aligned}$$

The mapping t is continuous and since $t(1,1)=1$, for $\lambda \in (0,1)$ there exists $r \in (0,1)$ so that for all $x \geq r$ and all $y \geq r$ we have $t(x,y) > 1 - \lambda$. Let $\varepsilon_i(r)$ be such that

$$\inf_{u,v \in AX} F_{u,v}^{g(i)}(\varepsilon_i(r)) > r .$$

Suppose that $r > 1 - \lambda$. Further, let $n(i,r) \in \mathbb{N}$ be such that

$$\frac{\varepsilon(i)}{2Q^n(i)} \geq \varepsilon_i(r) ,$$

for every $n \geq n(i,r)$.

If $n = 2k$, $p = 2m+1$ or $n = 2k+1$, $p = 2m+1$ and $n \geq n(i,r) + n_i$ it follows that:

$$F_{Ax_{n+p}-Ax_n}^i(\varepsilon) \geq F_{Ax_0-Ax_p}^{f^n(i)}\left(\frac{\varepsilon(i)}{Q^n(i)}\right)$$

where $\varepsilon(i) = \frac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$.

Furthermore

$$F_{Ax_0-Ax_p}^{f^n(i)}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F_{Ax_0-Ax_p}^{f^n(i)}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right) \geq F_{Ax_0-Ax_p}^{g(i)}(\varepsilon_i(r)) > r > 1 - \lambda .$$

If $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$ and $n > n(i,r) + n_i$ then

$$F_{Ax_{n+p}-Ax_n}^i(\varepsilon) \geq t(F_{Ax_1-Ax_0}^{g(i)}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right), F_{Ax_{p-1}-Ax_0}^{g(i)}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right)) \geq t(r,r) > 1 - \lambda .$$

So we have proved that the sequence $\{Ax_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and since X is sequentially complete, there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} Ax_n = x^* .$$

According to the construction of the sequence $\{x_n\}_{n \in \mathbb{N}}$ we have

$$\lim_{n \rightarrow \infty} Sx_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n} = x^* .$$

Since

$$\lim_{n \rightarrow \infty} Ax_n = x^*$$

we have the following implications

$$\begin{aligned} \lim_{n \rightarrow \infty} Ax_{2n} = x^* &\Rightarrow T(\lim_{n \rightarrow \infty} Ax_{2n}) = Tx^* \Rightarrow \lim_{n \rightarrow \infty} TAx_{2n} = Tx^* \Rightarrow \\ &\Rightarrow A(\lim_{n \rightarrow \infty} Tx_{2n}) = Tx^* \Rightarrow A(\lim_{n \rightarrow \infty} Tx_{2n-1}) = Tx^* \Rightarrow Ax^* = Tx^* . \end{aligned}$$

Also

$$\begin{aligned} \lim_{n \rightarrow \infty} Ax_{2n-1} = x^* &\Rightarrow S(\lim_{n \rightarrow \infty} Ax_{2n-1}) = Sx^* \Rightarrow \lim_{n \rightarrow \infty} Sx_{2n-1} = \\ &= Sx^* \Rightarrow A(\lim_{n \rightarrow \infty} Sx_{2n-1}) = Sx^* \Rightarrow A(\lim_{n \rightarrow \infty} Sx_{2n-2}) = \\ &= Sx^* \Rightarrow Ax^* = Sx^* , \end{aligned}$$

and it follows that

$$Ax^* = Sx^* = Tx^* .$$

Now we shall show that Ax^* is a fixed point of the mapping

A.

$$\begin{aligned} F_{Ax^*-A^2x^*}^i(\varepsilon) &\geq F_{Sx^*-TAx^*}^{f(i)}\left(\frac{\varepsilon}{q(i)}\right) = F_{Sx^*-ATx^*}^{f(i)}\left(\frac{\varepsilon}{q(i)}\right) = \\ &= F_{Ax^*-A^2x^*}^{f(i)}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq F_{Ax^*-A^2x^*}^{f^n(i)}\left(\frac{\varepsilon}{\prod_{s=0}^{n-1} q(f^s(i))}\right) \geq \\ &\geq F_{Ax^*-A^2x^*}^{f^n(i)}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F_{Ax^*-A^2x^*}^{g(i)}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \end{aligned}$$

$$\text{where } \varepsilon(i) = \frac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))} , \quad (n > n_i) .$$

We have that

$$F_{Ax^*-A^2x^*}^{g(i)} \left(\frac{\varepsilon}{Q^n(i)} \right) + 1, \text{ for } n \rightarrow \infty$$

and so

$$F_{Ax^*-A^2x^*}^i(\varepsilon) = 1$$

for all $\varepsilon > 0$ and all $i \in I$. This implies that

$$Ax^* = A^2x^* = A(Ax^*),$$

i.e. Ax^* is a fixed point of the mapping A .

It is easy to see that Ax^* is a fixed point of the mappings S and T

$$S(Ax^*) = A(Sx^*) = A(Ax^*) = Ax^*$$

and

$$T(Ax^*) = A(Tx^*) = A(Ax^*) = Ax^* .$$

Now we shall show that Ax^* is a unique fixed point of the mappings A , S and T . If we suppose that x_1 is another common fixed point of the mappings A , S and T , we have that

$$\begin{aligned} F_{Ax^*-Ax_1}^i(\varepsilon) &\geq F_{Sx^*-Tx_1}^{f(i)} \left(\frac{\varepsilon}{q(i)} \right) = F_{Ax^*-x_1}^{f(i)} \left(\frac{\varepsilon}{q(i)} \right) \geq \\ &\geq \dots \geq F_{Ax^*-Ax_1}^{f^n(i)} \left(\frac{\varepsilon(i)}{Q^n(i)} \right) \geq F_{Ax^*-Ax_1}^{g(i)} \left(\frac{\varepsilon(i)}{Q^n(i)} \right) + 1 \end{aligned}$$

for $n \rightarrow \infty$ and $\varepsilon(i) = \frac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$ and for all $i \in I$, which means

that

$$F_{Ax^*-x_1}^i(\varepsilon) = F_{Ax^*-Ax_1}^i(\varepsilon) = 1$$

for all $\varepsilon > 0$ and for $i \in I$. From the last equality we have that

$$Ax^* = x_1,$$

which means that Ax^* is the unique fixed point for the mappings A , S and T .

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REZIME

TEOREME O ZAJEDNIČKOJ NEPOKRETNOSTI TAČKI FAMILIJE
 PRESLIKAVANJA U VEROVATNOSNIM LOKALNO KONVEKSNIM
 PROSTORIMA

$(X, \{F_i\}_{i \in I}, t)$ je sekvencijalno kompletan verovatnosno lokalno konveksni prostor sa neprekidnom T-normom t . U radu je dokazana teorema u kojoj je dat potreban i dovoljan uslov za postojanje jedinstvene nepokretne tačke za dva preslikavanja S i T gde je $S: X \rightarrow X$, $T: X \rightarrow X$. Teorema glasi: Neprekidna preslikavanja S i T imaju jedinstvenu zajedničku nepokretnu tačku u X ako i samo ako postoji neprekidno preslikavanje $A: X \rightarrow SX \cap TX$ koje je komutativno sa S i T , AX je verovatnosno ograničen podskup od X i zadovoljeni su sledeći uslovi:

1. Za svako $i \in I$ postoji $q(i) > 0$ i $f(i) \in I$ tako da je

$$F_{Ax-Ay}^i(\epsilon) \geq F_{Tx-Sy}^{f(i)}\left(\frac{\epsilon}{q(i)}\right)$$

za svako $\epsilon > 0$ i svako $x, y \in X$.

2. Za svako $i \in I$ postoje brojevi $n_i \in \mathbb{N}$ i $Q(i) \in (0, 1)$ tako da je

$$q(f^n(i)) \leq Q(i) \leq 1 \quad \text{za svako } n > n_i$$

3. Za svako $i \in I$ postoji $g(i) \in I$ tako da je

$$F_X^{f^n(i)}(\epsilon) > F_X^{g(i)}(\epsilon)$$

za svako $\epsilon > 0$, svako $x \in X$ i svako $n \in \mathbb{N}$.

Tada postoji jedan i samo jedan elemenat $x^* \in X$ takav da je Ax^* jedinstvena nepokretna tačka za preslikavanja A , S i T .