Novi Sad J. Math.
Vol. 33, No. 1, 2003, 93–107

ON MODIFIED SZASZ–MIRAKYAN OPERATORS

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Abstract. We consider certain modifications of Szasz-Mirakyan operators S_n in exponential weighted spaces C_q of continuous functions and operators T_n in L^p spaces of Lebesgue integrable functions.

We give theorems on approximation properties of these operators.

AMS Mathematics Subject Classification (2000): 41A36

Key words and phrases: Szasz-Mirakyan operator, degree of approximation, exponential weighted space.

1. Introduction

1.1. Let $q > 0$ be a fixed number,

$$v_p(x) := e^{-qx}, \qquad x \in R_0 := [0, +\infty). \tag{1}$$

and let C_q be the space of all real-valued functions f continuous on R_0 for which fv_q is uniformly continuous and bounded on R_0 and the norm

$$\|f\|_q \equiv \|f(\cdot)\|_q := \sup_{x \in R_0} v_q(x)\,|f(x)|\,. \tag{2}$$

Let $L^p(R_0)$, with a fixed $p \geq 1$, be the space of all real-valued functions f for which $|f|^p$ is Lebesgue integrable on R_0 and the norm

$$\|f\|_{L^p} := \left\{ \int_0^{+\infty} |f(x)|^p du \right\}^{\frac{1}{p}}. \tag{3}$$

In papers [1] and [2] are concerned with approximation properties of the Szasz-Mirakyan operators

$$S_n(f;x) := \sum_{k=0}^{\infty} \varphi_k(nx) f\left(\frac{k}{n}\right) \quad x \in R_0, \quad n \in N, \tag{4}$$

$(N = \{1, 2, ...\}$ for the functions $f \in C_q$, $q \geq 0$, where

$$\varphi_k(t) := e^{-t}\frac{t^k}{k!}, \qquad t \in R_0, \quad k \in N_0 = N \cup \{0\}. \tag{5}$$

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In [2], the authors was proved that S_n is a positive linear operator from the space C_q into C_r provided that $r > q > 0$ and $n > q\,(ln\,(r/q))^{-1}$. Also, they proved the direct and inverse approximation theorems for S_n and $f \in C_q$, $q > 0$, but by applying the norm of the space C_r, $r > q$.

In [3] and [4], the authors examined properties of the Szasz-Mirakyan-Kantorovitch operators

$$T_n(f;x) := n \sum_{k=0}^{\infty} \varphi_k(nx) \int_{k/n}^{(k+1)/n} f(t)dt, \qquad x \in R_0, \quad n \in N, \tag{6}$$

for $f \in L^p(R_0)$, $p \geq 1$. Theorems on convergence almost everywhere and convergence in L^p-norm of the sequence $(T_n(f))_1^{\infty}$ were proved in [3], while some approximation theorems for $T_n(f)$ and $f \in L^1(R_0)$ were given in [4].

1.2. In this paper we modify definitions (4) and (6). Let $q \geq 0$ be a fixed number. For $f \in C_q$ we define the operators

$$\begin{aligned} S_n[f;a_n,b_n,q](x) &\equiv S_n(f;a_n,b_n,q,x) := \\ &:= \sum_{k=0}^{\infty} \varphi_k(a_n x) f\left(\frac{k}{b_n+q}\right), \quad x \in R_0, \quad n \in N, \end{aligned} \tag{7}$$

where $(a_n)_1^{\infty}$, $(b_n)_1^{\infty}$ are given increasing and unbounded numerical sequences such that $b_n \geq a_n \geq 1$, and $(a_n/b_n)_1^{\infty}$ is non-decreasing and

$$\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right). \tag{8}$$

We shall prove that $S_n[f;a_n,b_n,q]$, $n \in N$, is a positive linear operator from the space C_q into C_q.

In the space $L^p(R_0)$ with a fixed $p \geq 1$ we define operators

$$\begin{aligned} T_n[f;a_n,b_n](x) &\equiv T_n(f;a_n,b_n,x) := \\ &:= b_n \sum_{k=0}^{\infty} \varphi_k(a_n x) \int_{k/b_n}^{(k+1)/b_n} f(t)dt, \quad x \in R_0, \quad n \in N, \end{aligned} \tag{9}$$

where φ_k is given by (5) and $(a_n)_1^{\infty}$, $(b_n)_1^{\infty}$ are sequences as in definition $S_n[f;a_n,b_n,q]$.

Formulas (7) and (9) for $q = 0$ and $a_n = b_n = n$, $n \in N$, yield (4) and (6).

It is obvious that the operators $T_n[f;a_n,b_n]$, $n \in N$, can be considered in the spaces C_q, $q \geq 0$. In Section 3 of this paper we shall consider these operators in the spaces $L^p(R_0)$.

Operators S_n defined by (7) we shall consider in Section 2. In particular, we shall prove theorems on degree of approximation of $f \in C_q$ by S_n using the modulus of continuity of f,

$$\omega_1(f; C_q; t) := \sup_{0\le h\le t} \|\Delta_h f(\cdot)\|_q, \qquad t \ge 0,$$

and the modulus of smoothness of f

$$\omega_2(f; C_q; t) := \sup_{0\le h\le t} \|\Delta_h^2 f(\cdot)\|_q, \qquad t \ge 0,$$

where

$$\Delta_h f(x) := f(x+h) - f(x); \qquad \Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h).$$

In this paper we shall denote by $M_k(\alpha, \beta), k = 1, 2, ...,$ suitable positive constants depending only on indicated parameters α, β.

2. Operators $S_n[f; a_n, b_n, q]$

We assume that $q \ge 0$ and sequences $(a_n)_1^\infty$, $(b_n)_1^\infty$ are fixed. We shall write $S_n(f; x)$ instead of $S_n(f; a_n, b_n, q; x)$.

2.1. First we shall give some auxiliary results. By elementary calculations we obtain the following two lemmas.

Lemma 1. *Let $q \ge 0$ be a fixed number. Then*

$$S_n(1; x) = \sum_{k=0}^{\infty} \varphi_k(a_n x) = 1, \tag{10}$$

$$S_n(t - x; x) = \left(\frac{a_n}{b_n + q} - 1\right) x, \tag{11}$$

$$S_n((t - x)^2; x) = \left(\frac{a_n}{b_n + q} - 1\right)^2 x^2 + \frac{a_n x}{(b_n + q)^2}, \tag{12}$$

$$S_n((t - x)^4; x) = \left(\frac{a_n}{b_n + q} - 1\right)^4 x^4 + \left(\frac{a_n}{b_n + q} - 1\right)^2 \frac{6a_n x^3}{(b_n + q)^2} + \\ + \left(\frac{7a_n}{b_n + q} - 4\right) \frac{a_n x^2}{(b_n + q)^3} + \frac{a_n x}{(b_n + q)^4},$$

$$S_n\left(e^{qt}; x\right) = e^{q_n x}, \tag{13}$$

$$S_n\left((t-x)^2e^{qt};x\right)=\left\{\left(\frac{a_n}{b_n+q}e^{q/(b_n+q)}-1\right)^2x^2+\right.\qquad(14)$$

$$\left.+\frac{a_nx}{(b_n+q)^2}e^{q/(b_n+q)}\right\}e^{q_nx},$$

for all $x\in R_0$ *and* $n\in N$*, where*

$$q_n:=a_n\left(e^{q/(b_n+q)}-1\right).\qquad(15)$$

Lemma 2. *For the operators* S_n *defined by (7) we have*

$$\lim_{n\to\infty}b_nS_n(t-x;x)=-qx,\quad \lim_{n\to\infty}b_nS_n((t-x)^2;x)=x,$$

$$\lim_{n\to\infty}b_n^2S_n((t-x)^4;x)=3x^2,$$

at every point $x\in R_0$.

Now we shall prove two main lemmas.

Lemma 3. *Let* $q\geq 0$ *be a fixed number. Then*

$$\left\|S_n\left[1/v_q\right]\right\|_q\leq 1\quad n\in N,\qquad(16)$$

and

$$\left\|S_n[f]\right\|_q\leq\left\|f\right\|_q,\qquad(17)$$

for every $f\in C_q$ *and* $n\in N$.

Formulas (7) and (5) and the inequality (17) show that S_n*,* $n\in N$*, defined by (7) is a positive linear operator from the space* C_q *into* C_q.

Proof. First we shall prove (16).

If $q=0$,then by (1) and (10) follows (16). If $q>0$, then by (1) and (13) and (15) we get

$$v_q(x)S_n\left(\frac{1}{v_q(t)};x\right)=e^{(q_n-q)x},\quad x\in R_0,\quad n\in N,$$

and

$$e^{q/(b_n+q)}-1=\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{q}{b_n+q}\right)^k<\sum_{k=1}^{\infty}\left(\frac{q}{b_n+q}\right)^k=\frac{q}{b_n},\quad n\in N,$$

which by $0<a_n/b_n\leq 1$ and by (15) implies

$$0<q_n<\frac{a_nq}{b_n}\leq q\quad\text{for } n\in N.$$

Hence

$$v_q(x)S_n\left(\frac{1}{v_q(t)};x\right) \le 1 \quad \text{for} \quad x\in R_0, \quad n\in N,$$

which yields (16) for $q>0$ and $n\in N$.

The inequality (17) follows by (16) and by the inequality

$$\|S_n[f]\|_q \le \|f\|_q \, \|S_n\left[1/v_q\right]\|_q\,, \quad n\in N. \qquad \square$$

Lemma 4. *Suppose that $q\ge 0$ and $(a_n)_1^\infty$, $(b_n)_1^\infty$ are fixed. Then there exists a positive constant $M_1(b_1,q)$ such that*

$$v_q(x)S_n\left(\frac{(t-x)^2}{v_q(t)};x\right) \le M_1(b_1,q)\left(\frac{x^2}{(b_n+q)^2}+\frac{x}{b_n+q}\right) \tag{18}$$

for all $x\in R_0$ and $n\in N$.

Proof. If $q=0$, then by (12) and properties of the sequences $(a_n)_1^\infty$ and $(b_n)_1^\infty$ we immediately obtain (18).

If $q>0$ then by (1) and (14) we have

$$v_q(x)S_n\left((t-x)^2/v_q(t);x\right) \le e^{(q_n-q)x}\left\{\left(\frac{a_n}{b_n+q}e^{q/(b_n+q)}-1\right)^2 x^2 + \right. \tag{19}$$

$$\left. +\frac{a_n x}{(b_n+q)^2}e^{q/(b_n+q)}\right\}, \quad x\in R_0, \quad n\in N.$$

In the proof of Lemma 3 it is proved that

$$e^{(q_n-q)x}\le 1 \quad \text{for} \quad x\in R_0, \quad n\in N. \tag{20}$$

Applying the inequality $e^t-1\le te^t$ for $t\ge 0$ and (8), we get

$$\left(\frac{a_n}{b_n+q}e^{q/(b_n+q)}-1\right)^2 = \left\{\left(\frac{a_n}{b_n+q}-1\right)e^{q/(b_n+q)} + \right.$$

$$\left. +e^{q/(b_n+q)}-1\right\}^2 \le 2e^{2q/(b_n+q)}\left\{\left(\frac{a_n}{b_n+q}-1\right)^2+\frac{q^2}{(b_n+q)^2}\right\} \le$$

$$\le M_2(b_1,q)\frac{1}{(b_n+q)^2}, \quad n\in N.$$

From this and by (19), (20) and $b_n\ge a_n\ge 1$, we obtain estimation (18) for $q>0$. $\square$

2.2. Now we shall prove approximation theorems.

Theorem 1. *Suppose that* $f \in C_q^2$, $q \geq 0$. *Then there exists a positive constant* $M_3(b_1, q)$ *such that*

$$v_q(x)\,|S_n(f;x) - f(x)| \leq M_3(b_1,q)\left\{\|f'\|_q \frac{x}{b_n+q} + \right. \tag{21}$$

$$\left. + \|f''\|_q \left(\frac{x^2}{(b_n+q)^2} + \frac{x}{b_n+q}\right)\right\}$$

for all $x \in R_0$ *and* $n \in N$.

Proof. From (7) and (5) we get

$$S_n(f;0) = f(0) \quad \text{for} \quad n \in N. \tag{22}$$

For a fixed $x > 0$ and $f \in C_q^2$ we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_x^s f''(u)\,du\,ds, \quad t \in R_0,$$

which yields

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u)\,du, \quad t \in R_0.$$

From this and by (10) we deduce that

$$S_n(f(t);x) = f(x) + f'(x) S_n(t-x;x) + S_n\left(\int_x^t (t-u) f''(u)\,du; x\right) \tag{23}$$

for $n \in N$. By (1) and (2) we can write

$$\left|\int_x^t (t-u) f''(u)\,du\right| \leq \|f''\|_q \left(\frac{1}{v_q(t)} + \frac{1}{v_q(x)}\right)(t-x)^2.$$

Applying the above inequality and (11), (12) and (18), we derive from (23)

$$v_q(x)\,|S_n(f;x) - f(x)| \leq \|f'\|_q \left|\frac{a_n}{b_n+q} - 1\right| x +$$

$$\|f''\|_q \left\{ v_q(x) S_n\left(\frac{(t-x)^2}{v_q(t)}; x\right) + S_n\left((t-x)^2; x\right)\right\} \leq$$

$$\leq M_3(b_1,q)\left\{\|f'\|_q \frac{x}{b_n+q} + \|f''\|_q \left(\frac{x^2}{(b_n+q)^2} + \frac{x}{b_n+q}\right)\right\} \quad \text{for } n \in N.$$

Thus the proof of (21) is completed. □

Theorem 2. *Suppose that $f \in C_q$, with a fixed $q \geq 0$. Then there exists a positive constant $M_4(b_1, q)$ such that*

$$v_q(x)\,|S_n(f;x) - f(x)| \leq \tag{24}$$

$$\leq M_4(b_1,q)\left\{e^{q\Psi_n(x)}\sqrt{\frac{x}{b_n+q}}\omega_1(f;C_q;\Psi_n(x)) + \omega_2(f;C_q;\Psi_n(x))\right\},$$

for all $x \in R_0$ and $n \in N$, where

$$\Psi_n(x) = \left(\frac{x^2}{(b_n+q)^2} + \frac{x}{b_n+q}\right)^{\frac{1}{2}}. \tag{25}$$

Proof. Let $x > 0$. Similarly as in [1] and [2] we apply the Stieklov function of $f \in C_q$:

$$f_h(x) := \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[f(x+s+t) - f(x+2(s+t))]dsdt \tag{26}$$

for $x \in R_0, h > 0$. From (26) we get

$$f_h'(x) = \frac{1}{h^2}\int_0^{\frac{h}{2}}[8\Delta_{h/2}f(x+s) - 2\Delta_h f(x+2s)]ds,$$

$$f_h''(x) = \frac{1}{h^2}\left[8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)\right].$$

Consequently

$$\|f_h - f\|_q \leq \omega_2\left(f, C_q; h,\right), \tag{27}$$

$$\|f_h'\|_q \leq 5h^{-1}e^{qh}\omega_1\left(f, C_q; h\right), \tag{28}$$

$$\|f_h''\|_q \leq 9h^{-2}\omega_2\left(f, C_q; h\right), \tag{29}$$

for $h > 0$. We see that $f_h \in C_q^2$ if $f \in C_q$. Hence, for $x > 0$ and $n \in N$, we can write

$$v_q(x)\,|S_n(f;x) - f(x)| \leq v_q(x)\left\{|S_n\left(f - f_h; x\right)| + \right. \tag{30}$$

$$\left. + |S_n\left(f_h; x\right) - f_h(x)| + |f_h(x) - f(x)|\right\} := A_1 + A_2 + A_3.$$

By (17) and (27) we have

$$A_1 \leq \|f - f_h\|_q \leq \omega_2\left(f, C_q; h\right), \quad A_2 \leq \omega_2\left(f, C_q; h\right).$$

Applying Theorem 1 and (28) and (29), we get

$$A_3 \le M_3(b_1,q)\left\{\|f_h'\|_q \frac{x}{b_n+q} + \|f_h''\|_q \left(\Psi_n(x)\right)^2\right\} \le$$

$$\le M_4(b_1,q)\left\{e^{qh}h^{-1}\frac{x}{b_n+q}\omega_1(f;C_q;h)+\right.$$

$$\left.+h^{-2}\left(\Psi_n(x)\right)^2\omega_2(f;C_q;h)\right\}.$$

Combining these and setting $h=\Psi_n(x)$, for fixed $x>0$ and $n\in N$, we obtain (24) for $x>0$. The estimation (24) follows for $x=0$ by (22). □

Let

$$\lambda(x) := \left(1+x^2\right)^{-1}, \quad x\in R_0. \tag{31}$$

Theorem 3. *Assuming as in Theorem 1, we obtain*

$$\|[S_n(f)-f]\lambda\|_q \le M_5(b_1,q)\frac{1}{b_n+q}\left(\|f'\|_q+\|f''\|_q\right) \quad for \quad n\in N, \tag{32}$$

where $M_5(b_1,q)$ is a suitable positive constant.

Similarly as in Theorem 2 we obtain

Theorem 4. *Let $f\in C_q$ with a fixed $q>0$. Then there exists a positive constant $M_6(b_1,q)$ such that*

$$\|[S_n(f)-f]\lambda\|_q \le M_6(b_1,q)\left\{\frac{1}{\sqrt{b_n+q}}\omega_1\left(f;C_q;1/\sqrt{b_n+q}\right)+\right. \tag{33}$$

$$\left.+\omega_2\left(f;C_q;1/\sqrt{b_n+q}\right)\right\}$$

for all $n\in N$.

Proof. Arguing as in the proof of Theorem 2 and applying (30), (31) and the estimations for A_i, $\quad i=1,2,3$, given above, we obtain

$$\lambda(x)v_q(x)\left|S_n(f;x)-f(x)\right| \le 2\omega_2\left(f;C_q;h\right)+\lambda(x)A_2 \le \tag{34}$$

$$\le 2\omega_2\left(f;C_q;h\right)+M_7(b_1,q)\left\{e^{qh}\frac{x}{h(b_n+q)}\omega_1(f;C_q;h)\right.$$

$$\left.+\frac{1}{h^2(b_n+q)}\omega_2(f;C_q;h)+\right\}$$

for $x\in R_0$, $n\in N$ and $h>0$. Now, for fixed $n\in N$ setting $h=\frac{1}{\sqrt{b_n+q}}$, we derive (33) from (34) and (22). □

From Theorem 2 or Theorem 4 we obtain the following

Corollary 1. *Let $f \in C_q$ with a fixed $q \geq 0$. Then for the operators S_n defined by (7) we have*

$$\lim_{n\to\infty} S_n(f;x) = f(x), \quad x \in R_0. \tag{35}$$

The convergence (35) is uniform on every interval $[x_1, x_2]$, $\quad x_2 > x_1 \geq 0$.

2.3. In this section we shall prove the Voronovskaya type theorem for S_n.

Theorem 5. *Suppose that $f \in C_q^2$ with a fixed $q > 0$. Then for S_n defined by (7) we have*

$$\lim_{n\to\infty} b_n \{S_n(f;x) - f(x)\} = -qxf'(x) + \frac{x}{2} f''(x) \tag{36}$$

for every $x \in R_0$.

Proof. The equality (22) implies (36) for $x = 0$. Let $x > 0$ be a fixed point. Then by the Taylor formula for $f \in C_q^2$ we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \varepsilon_1(t;x)(t-x)^2, \quad t \in R_0,$$

where $\varepsilon_1(t) \equiv \varepsilon_1(t;x)$ is a function such that $\varepsilon_1 \in C_q$ and $\varepsilon_1(0) = 0$. From this and by (10) we get

$$S_n(f(t);x) = f(x) + f'(x)S_n(t-x;x) + \frac{1}{2} f''(x)S_n\left((t-x)^2;x\right) +$$
$$+ S_n\left(\varepsilon_1(t)(t-x)^2;x\right), \quad n \in N,$$

and next by Lemma 2

$$\lim_{n\to\infty} b_n \{S_n(f;x) - f(x)\} = -qxf'(x) +$$
$$+ \frac{x}{2} f''(x) + \lim_{n\to\infty} b_n S_n\left(\varepsilon_1(t)(t-x)^2;x\right).$$

Applying Hölder inequality, we have

$$\left|S_n\left(\varepsilon_1(t)(t-x)^2;x\right)\right| \leq \left\{S_n\left(\varepsilon_1^2(t);x\right)\right\}^{\frac{1}{2}} \left\{S_n\left((t-x)^4;x\right)\right\}^{\frac{1}{2}}, \quad n \in N.$$

By Theorem 2 and $\varepsilon_1^2 \in C_{2q}$ we have

$$\lim_{n\to\infty} S_n\left(\varepsilon_1^2(t);x\right) = \varepsilon_1^2(x) = 0.$$

From the above and from Lemma 2 we deduce that

$$\lim_{n\to\infty} b_n S_n\left(\varepsilon_1(t)(t-x)^2;x\right) = 0.$$

Combining these, we obtain (36) for $x > 0$. □

2.4. Now we shall give some properties of derivatives of operators (7).

Theorem 6. *Suppose that $f \in C_q$ with a fixed $q \geq 0$. Then for every $r \in N$ and $n \in N$ we have*

$$\left\| (S_n[f])^{(r)} \right\|_q \leq a_n^r \left\| \Delta_{1/(b_n+q)}^r f(\cdot) \right\|_q, \tag{37}$$

where

$$\Delta_h^r f(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh). \tag{38}$$

The formula (7) and the inequality (37) show that $S_n[f] \in C_q^\infty$, $n \in N$, if $f \in C_q$.

Proof. From (7) we deduce that

$$\frac{d}{dx} S_n(f(t);x) = -a_n S_n(f(t);x) +$$

$$+ a_n S_n(f(t+1/(b_n+q);x) = a_n S_n(\Delta_{1/(b_n+q)} f(t);x)$$

and next for every $r \in N$

$$\frac{d^r}{dx^r} = a_n^r S_n(\Delta_{1/(b_n+q)}^r f(t);x), \qquad x \in R_0, \quad n \in N, \tag{39}$$

where $\Delta_h^r f(\cdot)$ is defined by (38). Applying Lemma 3, we derive from (39)

$$\left\| (S_n[f])^{(r)} \right\|_q \leq a_n^r \left\| \Delta_{1/(b_n+q)}^r f(\cdot) \right\|_q$$

for all $n \in N$ and $r \in N$. □

Corollary 2. *If assumptions of Theorem 6 are satisfed, then*

$$\left\| (S_n[f])^{(r)} \right\|_q \leq \left(1 + e^{q/(b_n+q)}\right)^r a_n^r \left\| f(\cdot) \right\|_q$$

for every $n \in N$ and $r \in N$.

From formulas (7) and (39) and by classical theorems of mathematical analysis we obtain

Corollary 3. *Let $f \in C_q$ a with fixed $q \geq 0$. Then:*

(i) if f is an increasing (decreasing) function on R_0, then $S_n[f;a_n,b_n,q]$, $n \in N$, is also increasing (decreasing) function on R_0;

(ii) if f is a convex (concave) function on R_0, then $S_n[f;a_n,b_n,q]$, $n \in N$, is also a convex (concave) on R_0.

Theorem 7. *Suppose that $f \in C_q$ with a fixed $q \geq 0$ and $x_0 > 0$ is a point where there exists $f'(x_0)$. Then*

$$\lim_{n\to\infty} \left(S_n[f]\right)'(x_0) = f'(x_0). \tag{40}$$

Proof. By assumptions for f we can write

$$f(t) = f(x_0) + f'(x_0)(t - x_0) + \varepsilon_2(t, x_0)(t - x_0) \qquad \text{for} \quad t \in R_0,$$

where ε_2 is function continuous at x_0 and $\varepsilon_2 \in C_q$. From (7) we get

$$\left(S_n[f]\right)'(x) = -a_n S_n(f(t); x) + \frac{b_n + q}{x} S_n(tf(t); x) =$$

$$= (b_n - a_n + q) S_n(f(t); x) + \frac{b_n + q}{x} S_n((t - x)f(t); x)$$

for $x > 0$ and $n \in N$. Consequently, we obtain

$$\left(S_n[f(t)]\right)'(x_0) = f(x_0)\left\{b_n - a_n + q + \frac{b_n + q}{x_0} S_n(t - x_0; x_0)\right\} + \tag{41}$$

$$+f'(x_0)\left\{(b_n - a_n + q) S_n(t - x_0; x_0) + \frac{b_n + q}{x_0} S_n\left((t - x_0)^2; x_0\right)\right\} +$$

$$+(b_n - a_n + q) S_n\left(\varepsilon_2(t)(t - x_0; x_0\right) + \frac{b_n + q}{x_0} S_n\left(\varepsilon_2(t)(t - x_0)^2; x_0\right).$$

Properties of ε_2 and Corollary 1 imply

$$\lim_{n\to\infty} S_n\left(\varepsilon_2(t)(t - x_0); x_0\right) = 0. \tag{42}$$

Analogously as in the proof of Theorem 5 we obtain

$$\lim_{n\to\infty} b_n S_n\left(\varepsilon_2(t)(t - x_0)^2; x_0\right) = 0. \tag{43}$$

Applying (8),(11), (12), (42) and (43) we immediately obtain (40) from (41). □

3. Operators $T_n[f; a_n, b_n]$

We shall assume that the sequences $(a_n)_1^\infty$ and $(b_n)_1^\infty$ given in formula (9) for $T_n[f; a_n, b_n]$ are fixed. For these operators we shall give analogies of some results proved in [3].

3.1. First we shall give some elementary properties of T_n. From (9) we get

$$T_n(1; a_n, b_n; x) = 1 \qquad \text{for} \quad x \in R_0,\ n \in N.$$

Lemma 5. *Let $p \ge 1$ be a fixed number. $T_n[f;a_n,b_n]$, $n \in N$, is a positive linear operator from the space $L^p(R_0)$ into C_0, i.e. C_q with $q=0$. Moreover*

$$\|T_n[f;a_n,b_n]\|_0 \le b_n^{1/p}\,\|f\|_{L^p} \qquad n \in N. \tag{44}$$

Proof. We shall prove only (44). From (9) it follows that

$$|T_n(f;a_n,b_n;x)| \le b_n^{1/p}\sum_{k=0}^{\infty}\varphi_k(a_nx)\left\{\int_{k/b_n}^{(k+1)/b_n}|f(t)|^p dt\right\}^{1/p} \le$$

$$\le \|f\|_{L^p}\, b_n^{1/p}\sum_{k=0}^{\infty}\varphi_k(a_nx) = \le \|f\|_{L^p}\, b_n^{1/p},$$

for every $f \in L^p(R_0)$, $p \ge 1$, $x \in R_0$ and $n \in N$, which implies (44). □

Lemma 6. *Let $p \ge 1$ be a fixed number. $T_n[f;a_n,b_n]$, $n \in N$, is a positive linear operator from the space $L^p(R_0)$ into $L^p(R_0)$. Moreover*

$$\|T_n[f;a_n,b_n]\|_{L^p} \le \frac{b_n}{a_n}\|f\|_{L^p} \le \frac{b_1}{a_1}\|f\|_{L^p} \tag{45}$$

for every $f \in L^p(R_0)$ and $n \in N$

Proof. Let $p=1$. Then, applying the equality

$$\int_0^{+\infty}\varphi_k(a_nx) = \frac{1}{a_n}, \qquad k \in N_0, \quad n \in N, \tag{46}$$

we get

$$\|T_n[f;a_n,b_n]\|_{L^1} = \int_0^{+\infty}\left|b_n\sum_{k=0}^{\infty}\varphi_k(a_nx)\int_{k/b_n}^{(k+1)/b_n}f(t)dt\right|dx \le$$

$$\le b_n\sum_{k=0}^{\infty}\left(\int_{k/b_n}^{(k+1)/b_n}|f(t)|dt\right)\int_0^{+\infty}\varphi_k(a_nx)dx = \frac{b_n}{a_n}\|f\|_{L^1}, \qquad n \in N.$$

If $p>1$, then by (3), (10) and (46) and by Jensen inequalities we get

$$\|T_n[f;a_n,b_n]\|_{L^p}^p = \int_0^{+\infty}\left|\sum_{k=0}^{\infty}\varphi_k(a_nx)b_n\int_{k/b_n}^{(k+1)/b_n}f(t)dt\right|^p dx \le$$

$$\int_0^{+\infty}\sum_{k=0}^{\infty}\left(\varphi_k(a_nx)\left|b_n\int_{k/b_n}^{(k+1)/b_n}f(t)dt\right|^p\right)dx \le$$

$$\le b_n\sum_{k=0}^{\infty}\int_{k/b_n}^{(k+1)/b_n}|f(t)|^p dt\left(\int_0^{+\infty}\varphi_k(a_nx)dx\right) \le \frac{b_n}{a_n}\|f\|_{L^p}^p, \qquad n \in N.$$

By properties of $(b_n/a_n)_1^\infty$ the proof of (45) is completed. □

Lemma 7. *Let $f \in L^1(R_0)$ and let*

$$F(x) := \int_0^x f(t)dt, \qquad x \in R_0. \tag{47}$$

Then $F \in C_0$, i.e. $F \in C_q$ with $q = 0$, and there exist operators $S_n[F; a_n, b_n; 0]$, $n \in N$, defined by (7). Moreover

$$\left(S_n[F; a_n, b_n, 0]\right)'(x) = \frac{a_n}{b_n} T_n(f; a_n, b_n; x) \tag{48}$$

for every $x \in R_0$ and $n \in N$.

Proof. It is well know that F defined by (47) is continuous and bounded function on R_0 if $f \in L^1(R_0)$, i.e. $F \in C_0$ if $f \in L^1(R_0)$. From this and by Lemma 3 and Theorem 6 we deduce that there exists $S_n[F; a_n, b_n; 0]$, $n \in N$, defined by (7) and

$$\frac{d}{dx} S_n(F(t); a_n, b_n, 0; x) = a_n S_n\left(\Delta_{1/b_n} F(t); a_n, b_n, 0; x\right) =$$

$$= \frac{a_n}{b_n} T_n(f(t); a_n, b_n; x), \qquad x \in R_0,\ n \in N.$$

□

3.2. In [3] the operator $T_n[f]$ defined by (6) for $f \in L^1(R_0)$ was written by the formula

$$T_n(f; x) = \int_0^{+\infty} K_n(x; s) f(s) ds, \qquad x \in R_0,\ n \in N, \tag{49}$$

where

$$K_n(x; s) = n e^{-nx} \frac{(nx)^k}{k!}$$

for $k/n < s \le (k+1)/n$, $k \in N_0$; $K_n(x; 0) = 0$, $x \ge 0$. For the operators (49) it was proved in the following [3]:

Lemma 8. *If $f \in L^1(R_0)$, then*

$$\sup_{n \in N} |T_n(f; x)| \le 3\Theta(f; x), \qquad x \in R_0,$$

where

$$\Theta(f; x) := \sup_{0 < s < \infty, s \ne x} \frac{1}{s - x} \int_x^s |f(y)| dy. \tag{50}$$

3.3. It is obvious that the operator $T_n[f; a_n, b_n]$ defined by (9) can be written as:

$$T_n(f; a_n, b_n; x) = \int_0^{+\infty} W_n(x; s; a_n, b_n) f(s) ds \tag{51}$$

for $f \in L^1(R_0)$, $x \in R_0$, $n \in N$, where

$$W_n(x; s; a_n, b_n) := b_n e^{-a_n x} \frac{(a_n x)^k}{k!}$$

for $k/b_n < s \leq (k+1)/b_n$, $k \in N_0$; $W_n(x; 0; a_n, b_n) = 0$ for $x \in R_0$.

Applying (51) and arguing similarly as in the proof of Lemma 8 (see [3], p.p. 550, 551 - Lemma 4 and Lemma 5) we can prove

Lemma 9. *Let $f \in L^1(R_0)$. Then there exists a positive constant $M_8(a_1, b_1)$ such that*

$$\sup_{n \in N} |T_n(f; a_n, b_n; x)| \leq M_8(a_1, b_1)\Theta(f; x), \qquad x \in R_0,$$

where $\Theta(f; \cdot)$ is defined by (50).

3.4. Now we shall prove the main theorems for $T_n[f; a_n, b_n]$, which are analogies of the Butzer theorems given in [3].

Theorem 8. *Suppose that $f \in L^1(R_0)$. Then*

$$\lim_{n\to\infty} T_n(f; a_n, b_n; x) = f(x) \tag{52}$$

at every point $x \in R_0$ where

$$F'(x) = f(x). \tag{53}$$

Hence (52) follows almost everywhere on R_0.

Proof. The properties of F given in Lemma 7 and by Theorem 7 imply that

$$\lim_{n\to\infty} (S_n(F; a_n, b_n, 0))'(x) = F'(x)$$

at every $x \in R_0$, where $F'(x)$ there exists. From this and by (48) and (8) we obtain

$$\lim_{n\to\infty} T_n(f; a_n, b_n; x) = F'(x) = f(x)$$

at every $x \in R_0$ where (53) follows. Since (53) follows almost everywhere on R_0 for $f \in L^1(R_0)$, we have (52) almost everywhere on R_0. □

Theorem 9. *Suppose that* $f \in L^1(R_0)$ *and* $f \in L^p(R_0)$ *with a fixed* $p > 1$. *Then*

$$\lim_{n\to\infty} \|T_n[f; a_n, b_n] - f\|_{L^p} = 0. \tag{54}$$

Proof. It is known ([5], [3]) that if $f \in L^p(R_0)$, $p > 1$, then the function $\Theta(f; \cdot)$ defined by (50) belongs also to $L^p(R_0)$ and

$$\int_0^{+\infty} (\Theta(f; x))^p \, dx \le 2\left(\frac{p}{p-1}\right)^p \int_0^{+\infty} |f(x)|^p dx.$$

From this and by Lemma 6, Lemma 9 and Theorem 8 and by the Lebesgue theorem on convergence of sequence in L^p -space we immediately derive the desired assertion (54). □

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Received by the editors September 30, 2002