EXTREME VALUES OF THE SEQUENCES OF INDEPENDENT
RANDOM VARIABLES WITH MIXED DISTRIBUTIONS

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Abstract. In this paper we consider some examples of the sequences of independent random variables with the same mixed distribution. In these cases we determine the type of extreme value distribution and the normalizing constants.

1. Introduction

Let \((X_n)\) be a sequence of independent random variables with the common distribution function \(F\). If for some constants \(a_n > 0\) and \(b_n\)

\[
P\left\{ \max_{1 \leq j \leq n} X_j \leq \frac{x}{a_n} + b_n \right\} \to x G(x),
\]

(1.1)

where \(G\) is non-degenerated distribution function, then the function \(G\) belongs to one of three classes of the maximum stable distributions, and the functions in these classes have the following forms (maybe after linear transformation of the argument):

Type I. \(G_1(x) = \exp(-e^{-x}), -\infty < x < +\infty;\)

Type II. \(G_2(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \exp(-x^{-\alpha}), & \text{if } x > 0, \end{cases}\) for some \(\alpha > 0;\)

Type III. \(G_3(x) = \begin{cases} \exp(-(x)^{\alpha}), & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}\) for some \(\alpha > 0.\)

These three types of distributions are called the extreme values distributions. If for some distribution functions \(F\) and \(G\) the relation (1.1) holds true, then we say that the common distribution function \(F\) of the random variables \(X_1, X_2, X_3, \ldots\) belongs to the domain of attraction of the function \(G\). We shall use the notation \(M_n = \max\{X_1, \ldots, X_n\}\). The constants \(a_n > 0\) and \(b_n\) from the relation (1.1) are called the normalizing constants. Note that for \(a_n > 0\), the inequality \(M_n \leq x/a_n + b_n\) is equivalent to \(a_n(M_n - b_n) \leq x.\)

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2. Necessary and sufficient conditions for convergence
to the extreme value distributions

If for some distribution function $F$ there exists a domain of attraction, then it
is determined by the asymptotic behaviour of the tail $1 - F(x)$, as $x \to +\infty$. The
following useful theorem 1.5.1 from [1] holds true:

**Theorem 1.** [1] Let $(X_n)$ be a sequence of independent random variables with
the common distribution function $F(x)$, $-\infty < x < +\infty$, $(u_n)$ a sequence of real
numbers, $0 \leq \tau \leq +\infty$ and $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Then, the equality
\[
\lim_{n \to \infty} P\{M_n \leq u_n\} = e^{-\tau},
\]
holds true if and only if $\lim_{n \to \infty} n(1 - F(u_n)) = \tau$.

Necessary and sufficient conditions for the function $F$ to belong to some domain
of attraction can also be formulated as in the following theorem 1.6.2 from [1]:

**Theorem 2.** [1] Let $(X_n)$ be a sequence of independent random variables with
the common distribution function $F$, and $x_F = \sup\{x|F(x) < 1\}$. Necessary and
sufficient conditions for the function $F$ to belong to the domain of attraction of
possible types are given by:

Type I. There exists a strictly positive function $g(t)$ defined on the set $(-\infty, x_F)$,
such that for every real number $x$ the equality $\lim_{t \to x_F} \frac{1 - F(tx)}{1 - F(t)} = e^{-x}$ holds
true.

Type II. $x_F = +\infty$ and $\lim_{t \to x_F} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$, for some $\alpha > 0$ and all $x > 0$.

Type III. $x_F < +\infty$ and $\lim_{h \to 0} \frac{1 - F(x_F - hx)}{1 - F(x_F - h)} = x^\alpha$, for some $\alpha > 0$ and all $x > 0$.

3. Mixed distributions

Let $X_1$ and $X_2$ be random variables with distribution functions $F_1(x)$ and
$F_2(x)$, respectively, and
\[
X = \begin{cases} 
X_1 & \text{with probability } p, \\
X_2 & \text{with probability } q,
\end{cases}
\]
where $p + q = 1$. The distribution function of the random variable $X$ is given by
\[
F(x) = P\{X \leq x\} = pP\{X_1 \leq x\} + qP\{X_2 \leq x\} = pF_1(x) + qF_2(x).
\]
The distribution of probability determined by the distribution function $F$ is called
the mixture of the distributions determined by the functions $F_1$ and $F_2$. 
We shall consider some examples of sequences of independent random variables with common mixed distribution. In these cases we are going to determine the type of extreme value distribution and the normalizing constants.

3.1. Mixture of normal distributions. Let \( (X_n) \) be a sequence of independent random variables with normal \( \mathcal{N}(0,1) \) distribution and \( M_n = \max \{X_1, \ldots, X_n\} \). As is well known, the limiting distribution of the maximum \( M_n \) is given by

\[
P\{a_n(M_n - b_n) \leq x\} \to e^{-e^{-x}}, \quad n \to \infty,
\]

where the normalizing constants \( a_n \) and \( b_n \) are

\[
a_n = \sqrt{2 \ln n}, \quad b_n = \sqrt{2 \ln n} - \frac{1}{2} \ln \ln n + \frac{4\pi}{
\sqrt{2 \ln n}}. \tag{3.1}
\]

**Corollary.** Let \( (Y_n) \) be a sequence of independent random variables with normal \( \mathcal{N}(m, \sigma^2) \) distribution. Then, we have \( X_n = (Y_n - m)/\sigma \in \mathcal{N}(0,1) \). If \( M_n = \max \{X_1, \ldots, X_n\} \) and \( \bar{M}_n = \max \{Y_1, \ldots, Y_n\} \), then

\[
P\{a_n(M_n - b_n) \leq x\} = P\left\{a_n \left(\frac{\bar{M}_n - m}{\sigma} - b_n\right) \leq x\right\} = P\{\bar{a}_n(\bar{M}_n - \bar{b}_n) \leq x\} \to \exp(-e^{-x}), \quad n \to \infty,
\]

where the constants \( \bar{a}_n \) and \( \bar{b}_n \) are given by:

\[
\bar{a}_n = \sigma \sqrt{2 \ln n}, \quad \bar{b}_n = m + \sigma b_n = m + \sigma \sqrt{2 \ln n} - \sigma \frac{\ln \ln n + 4\pi}{2\sqrt{2 \ln n}}.
\]

**Theorem 3.** Let \( (Z_n) \) be a sequence of independent random variables such that

\[
Z_n \in \begin{cases} 
\mathcal{N}(m_1, \sigma_1^2), \text{ with probability } p, \\
\mathcal{N}(m_2, \sigma_2^2), \text{ with probability } q,
\end{cases}
\]

for all \( n \), where \( p + q = 1 \). Let us denote \( M_n^* = \max \{Z_1, \ldots, Z_n\} \). If

(a) \( \sigma_1 > \sigma_2, m_1, m_2 \in R \) or (b) \( \sigma_1 = \sigma_2 \) and \( m_1 > m_2 \),

then for every real number \( x \) the equality

\[
\lim_{n \to \infty} P\{a_n^*(M_n^* - b_n^*) \leq x\} = e^{-e^{-x}} \tag{3.3}
\]

holds true, where the constants \( a_n^* \) and \( b_n^* \) are given by

\[
a_n^* = \frac{\sqrt{2 \ln n}}{\sigma_1}, \quad b_n^* = m_1 + \sigma_1 \sqrt{2 \ln n} - \frac{\sigma_1}{2\sqrt{2 \ln n}} \left(\ln \ln n + \frac{4\pi}{p^2}\right) \tag{3.4}
\]
Remark. Since \( M_n^* \) in distribution is the same as \( \hat{M}_{np} \), then all normalizing constants with \( a_n^* / \hat{a}_{np} \to 1 \) and \( a^*_n (b_{np} - b^*_n) \to 0 \) will work. The maximum \( M_n^* \) asymptotically comes from \( F_1 \).

Proof of Theorem 3. Let \( \varphi(x) = (2\pi)^{-1/2} e^{-x^2/2} \) and \( \Phi(x) = \int_{-\infty}^x \varphi(t) \, dt \). We shall use the following asymptotic relation

\[
1 - \Phi(x) \sim x^{-1/2} \varphi(x), \quad x \to \infty.
\]

(3.5)

If \( X_i \in \mathcal{N}(m_i, \sigma_i^2) \), \( i = 1, 2 \), then distribution function of the random variable \( X_i \) can be represented in the form

\[
F_i(x) = P\{X_i \leq x\} = P\left\{ \frac{X_i - m_1}{\sigma_1} \leq \frac{x - m_1}{\sigma_1} \right\} = \Phi\left( \frac{x - m_1}{\sigma_1} \right), \quad i = 1, 2
\]

Distribution function of the random variable \( Z_n \) has the form \( F(x) = p F_1(x) + q F_2(x) \). Using this representation of \( F(x) \) we obtain

\[
1 - F(t) = p \left[ 1 - \Phi\left( \frac{t - m_1}{\sigma_1} \right) \right] + q \left[ 1 - \Phi\left( \frac{t - m_2}{\sigma_2} \right) \right]
\]

\[
\sim \frac{p \sigma_1}{t - m_1} \varphi\left( \frac{t - m_1}{\sigma_1} \right) + \frac{q \sigma_2}{t - m_2} \varphi\left( \frac{t - m_2}{\sigma_2} \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left\{ \frac{p \sigma_1}{t - m_1} \exp\left[ -\frac{1}{2} \left( \frac{t - m_1}{\sigma_1} \right)^2 \right] + \frac{q \sigma_2}{t - m_2} \exp\left[ -\frac{1}{2} \left( \frac{t - m_2}{\sigma_2} \right)^2 \right] \right\}
\]

\[
= \frac{1}{\sqrt{2\pi} (t - m_1)} \exp\left[ -\frac{1}{2} \left( \frac{t - m_1}{\sigma_1} \right)^2 \right] (1 + o(1)), \quad t \to \infty;
\]

\[
1 - F(t) = \exp\left[ -\frac{(t - m_1 + xg(t))^2 - (t - m_1)^2}{2\sigma_1^2} \right] \frac{t - m_1 + xg(t)}{t - m_1 + xg(t)} \exp\left( -\frac{x^2 \sigma_1^2}{2\sigma_1^2} \right) \exp\left( -\frac{2x \sigma_1^2 (t - m_1)}{2\sigma_1^2} \right) \exp\left( -\frac{x \sigma_1^2 (t - m_1)}{2\sigma_1^2} \right)
\]

For \( g(t) = \sigma_1 / (t - m_1) \), we get

\[
1 - F(t + xg(t)) = \exp\left( -\frac{x^2 \sigma_1^2}{2(t - m_1)^2} \right) \frac{1}{1 + x \sigma_1^2 (t - m_1)^{-2} (1 + o(1))}
\]

\[
\to e^{-x}, \quad \text{as} \quad t \to +\infty.
\]

Using theorem 1 we conclude that the distribution function \( F(x) \) belongs to the domain of attraction of type I, i.e. there exist constants \( a^*_n \) and \( b^*_n \), such that the following equality holds true:

\[
\lim_{n \to \infty} P\left\{ M_n^* \leq \frac{x}{a^*_n} + b^*_n \right\} = e^{-x^{-2}}.
\]

The constants \( a^*_n \) and \( b^*_n \) can be determined as follows: let us first determine the constant \( u_n \), such that

\[
1 - F(u_n) \sim \frac{1}{n} e^{-x} \quad \text{as} \quad n \to \infty.
\]

\[
1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n} e^{-x}, \quad n \to \infty.
\]
Using the equalities \( F_1(u_n) = \Phi((u_n - m_1)/\sigma_1) \) and \( F_2(u_n) = \Phi((u_n - m_2)/\sigma_2) \) and asymptotic relation (3.5), we obtain

\[
1 - pF_1(u_n) - qF_2(u_n) = p \left[ 1 - \Phi\left(\frac{u_n - m_1}{\sigma_1}\right)\right] + q \left[ 1 - \Phi\left(\frac{u_n - m_2}{\sigma_2}\right)\right]
\]

\[
\sim \frac{p\sigma_1}{u_n - m_1} \varphi\left(\frac{u_n - m_1}{\sigma_1}\right) + \frac{q\sigma_2}{u_n - m_2} \varphi\left(\frac{u_n - m_2}{\sigma_2}\right) + \frac{1}{n}e^{-x}, \quad \text{as } n \to \infty.
\]

Let us denote: \( v_n = (u_n - m_1)/\sigma_1 \) and \( w_n = (u_n - m_2)/\sigma_2 \). For large values of \( n \) the inequality \( v_n < w_n \) holds true, and

\[
p \frac{\varphi(v_n)}{v_n} + q \frac{\varphi(w_n)}{w_n} = \frac{1}{\sqrt{2\pi}} \left( \frac{p}{v_n} e^{-v_n^2/2} + \frac{q}{w_n} e^{-w_n^2/2} \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \frac{p}{v_n} e^{-v_n^2/2} \left( 1 + \frac{v_n}{p} \frac{q}{w_n} e^{-(w_n^2-v_n^2)/2} \right).
\]

Let \( \Delta_n = w_n^2 - v_n^2 = (u_n - m_1)^2/\sigma_1^2 - (u_n - m_2)^2/\sigma_2^2 \). If \( \sigma_1 > \sigma_2 > 0 \), then

\[
\Delta_n = \left( \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2} \right) u_n^2 + Au_n + B \to +\infty, \quad \text{as } n \to \infty.
\]

If \( \sigma_1 = \sigma_2 = \sigma \) and \( m_1 > m_2 \), then \( \Delta_n = [2u_n(m_1 - m_2) + m_1^2 - m_2^2]/\sigma^2 \to +\infty \), as \( n \to \infty \). In both cases we have the following asymptotic equality

\[
p \frac{\varphi(v_n)}{v_n} + q \frac{\varphi(w_n)}{w_n} = \frac{1}{\sqrt{2\pi}} \frac{p}{v_n} e^{-v_n^2/2} (1 + o(1)), \quad \text{as } n \to \infty.
\]

Hence, the constant \( u_n \) should be determined from the conditions \( u_n = \sigma_1 v_n + m_1 \) and

\[
p \frac{\varphi(v_n)}{v_n} \sim \frac{1}{n}e^{-x}, \quad \text{as } n \to \infty. \quad (3.6)
\]

Asymptotic relation (3.6) can be transformed in the following way:

\[
\frac{1}{p^n} e^{-x} \frac{v_n}{\varphi(v_n)} \to 1,
\]

\[
- \ln n - \ln p - x + \ln v_n - \ln \varphi(v_n) \to 0,
\]

\[
- \ln n - \ln p - x + \ln v_n + \frac{1}{3} \ln 2\pi + \frac{v_n^2}{2} \to 0. \quad (3.7)
\]

It follows from (3.7) that \( v_n^2/(2\ln n) \to 1 \) as \( n \to \infty \), and

\[
\ln v_n = \frac{1}{2}(\ln 2 + \ln \ln n) + o(1). \quad (3.8)
\]

The relation (3.7) can also be written in the form

\[
\frac{v_n^2}{2} = x + \ln n + \ln p - \frac{1}{2} \ln 2\pi - \ln v_n + o(1). \quad (3.9)
\]
Now let us substitute the value of $\ln v_n$ from (3.8) into (3.9). We obtain

$$
\frac{v_n^2}{2} = x + \ln n + \ln p - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln 2 - \frac{1}{2}\ln n + o(1)
$$

$$
= x + \ln n + \ln p - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln n + o(1),
$$

$$
v_n^2 = 2\ln n \left\{ 1 + \frac{x + \ln p - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln n}{\ln n} + o\left(\frac{1}{\ln n}\right) \right\}.
$$

Note that $\ln p - \frac{1}{2}\ln 4\pi = -\frac{1}{2}\ln \frac{4\pi}{n^2}$. Using the formula $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$ as $x \to 0$, we get

$$
v_n = \sqrt{2\ln n} \left\{ 1 + \frac{x + \ln p - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln n}{2\ln n} + o\left(\frac{1}{\ln n}\right) \right\}
$$

$$
= \sqrt{2\ln n} \left\{ 1 + \frac{1}{2\ln n} \left( x - \frac{1}{2} \ln \frac{4\pi}{p^2} - \frac{1}{2}\ln n \right) + o\left(\frac{1}{\ln n}\right) \right\}.
$$

Since $u_n = m_1 + \sigma_1 v_n$, it follows that

$$
u_n = \frac{\sigma_1 x}{\sqrt{2\ln n}} + m_1 + \sigma_1 \sqrt{2\ln n} - \frac{\sigma_1}{2\sqrt{2\ln n}} \left( \ln \ln n + \ln \frac{4\pi}{p^2} \right) + o\left(\frac{1}{\ln n}\right).
$$

On the other hand $u_n \sim \frac{x}{a_n^*} + b_n^*$, as $n \to \infty$. Consequently it is easy to obtain the normalizing constants $a_n^*$ and $b_n^*$ in the form (3.4).

### 3.2. Mixture of Cauchy distributions

Let $(X_n)$ be a sequence of independent random variables with the Cauchy distribution $K(1,0)$, determined by the distribution function

$$
F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x.
$$

Let us denote $M_n = \max \{ X_1, \ldots, X_n \}$. For $x > 0$ we have

$$
1 - F(tx) = \frac{\pi}{2} - \arctan(tx) \to \frac{1}{x}, \quad t \to \infty.
$$

Indeed, on substituting $\arctan t = \pi/2 - \vartheta$, we obtain $t = \tg(\pi/2 - \vartheta) = \ctg \vartheta$,

$$
\lim_{t \to \infty} \left( \frac{\pi}{2} - \arctg t \right) \cdot t = \lim_{\vartheta \to 0} \vartheta \ctg \vartheta = 1.
$$

Similarly, $\lim_{t \to \infty} \left( \frac{\pi}{2} - \arctg(tx) \right) tx = 1$, and (3.10) follows easily. In this case, the distribution function $F$ belongs to the domain of attraction of the function $G_2(x)$, and we have the type II of limiting distribution. The normalizing constants are $a_n = 1/\gamma_n$ and $b_n = 0$, where the constant $\gamma_n$ can be determined from the equality

$$
1 - F(\gamma_n) = \frac{1}{2} - \frac{1}{\pi} \arctg \gamma_n = \frac{1}{n}.
Hence, \( \gamma_n = \tan\left(\frac{\pi}{2} - \frac{x}{n}\right) = \cot \frac{x}{n} \). For every \( x > 0 \) we have:

\[
\lim_{n \to \infty} P\left\{ M_n \cdot \tan \frac{\pi}{n} \leq x \right\} = e^{-x^{-1}}.
\]

**Corollary.** Let \( (Y_n) \) be a sequence of independent random variables with the Cauchy distribution \( K(\lambda, 0) \), which is determined by the distribution function

\[
F(x) = \frac{1}{\pi} + \frac{1}{\pi} \arctg \frac{x}{\lambda}.
\]

Let us denote \( \tilde{M}_n = \max\{Y_1, \ldots, Y_n\} \) and \( Y_n/\lambda = X_n \). Then, \( M_n = \lambda M_n \). It is easy to see that the random variables \( Y_n/\lambda \) has \( K(1,0) \) distribution. Since the inequality \( M_n \cdot \tan \frac{\pi}{n} \leq x \) is equivalent to \( \frac{1}{\lambda} \tilde{M}_n \cdot \tan \frac{\pi}{n} \leq x \), it follows that the normalizing constants in this case are given by \( a_n = \frac{1}{\lambda} \tan \frac{\pi}{n} \) and \( b_n = 0 \). For these values of normalizing constants and every \( x > 0 \) we have

\[
\lim_{n \to \infty} P\left\{ \frac{1}{\lambda} \tilde{M}_n \cdot \tan \frac{\pi}{n} \leq x \right\} = e^{-x^{-1}}.
\]

**Theorem 4.** Let \( K(\lambda_i, 0) \) be the class of random variables with the distribution function

\[
F_i(x) = \frac{1}{2} + \frac{1}{\pi} \arctg \frac{x}{\lambda_i}, \quad i = 1, 2.
\]

Let \( (Z_n) \) be a sequence of independent random variables such that for every \( n \),

\[
Z_n \in \begin{cases} 
K(\lambda_1, 0), & \text{with probability } p, \\
K(\lambda_2, 0), & \text{with probability } q,
\end{cases}
\]

where \( p + q = 1 \), and \( M^*_n = \max\{Z_1, \ldots, Z_n\} \). Then, for all \( x > 0 \) we have

\[
\lim_{n \to \infty} P\left\{ \frac{\pi}{n(p\lambda_1 + q\lambda_2)} M^*_n \leq x \right\} = e^{-x^{-1}}. \tag{3.11}
\]

**Remark.** For Cauchy variables with different scale parameters, \( M^*_n \) will come from either of the two parent distributions, which explains why the extremal distribution is also a mixture. More specifically, for pure Cauchy variables, \( nM^*_n/n\lambda \to U \) with \( F_U(x) = e^{-x^{-1}} \). In a sample of size \( n \) from the mixture of two Cauchy distributions one observes approximately \( np \) and \( nq \) variables of the two types, and hence \( M^*_n \approx \max(M_1^{(1)}, M_2^{(2)}) \) in distribution. Since \( F_U \) is a max-stable distribution, this explains the result.

**Proof of Theorem 4.** The distribution function of the random variable \( Z_n \) is

\[
F(x) = \frac{1}{2} + \frac{1}{\pi} \arctg \frac{x}{\lambda_1} + \frac{1}{\pi} \arctg \frac{x}{\lambda_2}.
\]

It is easy to prove that \( \lim_{t \to \infty} t \left( \frac{\pi}{2} - \arctg(\alpha t) \right) = \frac{1}{\alpha} \) for \( \alpha > 0 \). Consequently, for
every $x > 0$ we obtain the following relations:

$$
\frac{1 - F(tx)}{1 - F(t)} = \frac{\frac{1}{2} - \frac{p}{\pi} \arctg \frac{tx}{\lambda_1} - \frac{q}{\pi} \arctg \frac{tx}{\lambda_2}}{\frac{1}{2} - \frac{p}{\pi} \arctg \frac{t}{\lambda_1} - \frac{q}{\pi} \arctg \frac{t}{\lambda_2}} = pt\left(\frac{\pi}{2} - \arctg \frac{tx}{\lambda_1}\right) + qt\left(\frac{\pi}{2} - \arctg \frac{tx}{\lambda_2}\right) = \frac{p\lambda_1/tx + q\lambda_2/tx}{p\lambda_1 + q\lambda_2} = x, \quad \text{as } t \to \infty.
$$

It follows from theorem 2 that the function $F$ belongs to the domain of attraction of the function $G_2$, i.e., in this case we have the type II of extreme value distribution. The normalizing constants are $a_n^* = 1/\gamma_n$ and $b_n^* = 0$, where $1 - F(\gamma_n) = 1/n$. This equation can be transformed equivalently as follows:

$$
\frac{1}{2} - \frac{p}{\pi} \arctg \frac{\gamma_n}{\lambda_1} - \frac{q}{\pi} \arctg \frac{\gamma_n}{\lambda_2} = \frac{1}{n},
$$

$$
\gamma_n p \left(\frac{\pi}{2} - \arctg \frac{\gamma_n}{\lambda_1}\right) + \gamma_n q \left(\frac{\pi}{2} - \arctg \frac{\gamma_n}{\lambda_2}\right) = \frac{\pi}{n} \gamma_n.
$$

From the last equality we obtain $p\lambda_1 + q\lambda_2 = \frac{\pi}{n} \gamma_n \to 0$, as $\gamma_n \to \infty$. Consequently, we have $\gamma_n \sim \frac{n(p\lambda_1 + q\lambda_2)}{\pi}$ and $a_n^* \sim \frac{\pi}{n(p\lambda_1 + q\lambda_2)}$. Now it is easy to derive the equality (3.11).]

3.3. Mixture of uniform and truncated exponential distribution. Let $(X_n)$ be a sequence of independent random variables with the uniform $U[0,c]$ distribution. The distribution function of the random variable $X_n$ is given by $F_1(x) = x/c$, $0 \leq x \leq c$. For $n \geq \tau > 0$ it follows from the inequality $1 - F_1(u_n) = \tau/n$ that $u_n = c(1 - \tau/n)$. Using theorem 1 we obtain $P\{M_n \leq c(1 - \tau/n)\} \to e^{-\tau}$ as $n \to \infty$. For $x < 0$ and $\tau = -x$ we obtain $P\{M_n \leq c\} \to e^x$, as $n \to \infty$. Hence, in this case we have the extreme value distribution of type III, where $\alpha = 1$. The normalizing constants are $a_n = \frac{\pi}{\tau}$ and $b_n = c$.

Let $(Y_n)$ be a sequence of independent random variables with the same truncated exponential distribution $E(\lambda, c)$. This distribution is determined by the distribution function $F_2$ which is given by $F_2(x) = 0$ for $x \leq 0$, $F_2(x) = 1$ for $x \geq c$ and

$$
F_2(x) = \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}, \quad \text{if } 0 \leq x \leq c.
$$

For $x < 0$ and $\tau = -x$, it follows from theorem 1 that

$$
\lim_{n \to \infty} P\left\{M_n \leq c + \frac{x(e^\lambda - 1)}{\lambda n} + o\left(\frac{1}{n}\right)\right\} = e^x.
$$

In this case we also obtain the limiting distribution of type III. The normalizing constants are given by $\bar{a}_n = \lambda n(e^\lambda - 1)^{-1}$ and $b_n = c$. 

Theorem 5. Let $U[0,c]$ be the class of random variables with the uniform distribution on the interval $[0,c]$ and $\mathcal{E}(\lambda,c)$ the class of random variables with the truncated exponential distribution. Let $(Z_n)$ be a sequence of independent random variables such that for every $n$,

$$Z_n \in \begin{cases} U[0,c], & \text{with probability } p, \\ \mathcal{E}(\lambda,c), & \text{with probability } q, \end{cases}$$

where $p + q = 1$. If $M_n^* = \max\{Z_1, \ldots, Z_n\}$, then for every $x < 0$ the following equality holds true:

$$\lim_{n \to \infty} P\left\{ M_n^* \leq c + \frac{x}{n} \left( \frac{p}{c} + \frac{\lambda q}{e^{\lambda c} - 1} \right)^{-1} \right\} = e^x. \quad (3.12)$$

Proof. The distribution function of the random variable $Z_n$ is given by

$$F(x) = \frac{px}{c} + q \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}, \quad 0 \leq x \leq c.$$ 

It is easy to prove that for every $x > 0$ the equality

$$\lim_{h \to 0} \frac{1 - F(c - h x)}{1 - F(c - x)} = x,$$

holds true. Using theorem 2 we conclude that the function $F$ belongs to the domain of attraction of the function $G_3$. The normalizing constants are $b_n^* = c$ and $a_n^* = n/k$, where $k$ should be determined. Using theorem 1 we obtain that for $x < 0$ the condition $P\{M_n \leq c + x k / n\} \to e^x$ as $n \to \infty$, can be written in the form:

$$1 - F\left( c + \frac{k x}{n} \right) \sim -\frac{x}{n}, \quad n \to \infty,$$

$$1 - \frac{p}{c} \left( c + \frac{k x}{n} \right) - q \frac{1 - e^{-\lambda c} e^{-\lambda k x / n}}{1 - e^{-\lambda c}} \sim -\frac{x}{n}, \quad n \to \infty,$$

$$1 - p \frac{p k x}{c n} - \frac{q}{1 - e^{-\lambda c}} \left\{ 1 - e^{-\lambda c} + e^{-\lambda c} \frac{\lambda k x}{n} + o\left( \frac{1}{n} \right) \right\} \sim -\frac{x}{n}, \quad n \to \infty.$$

Now, we obtain that

$$1 - p \frac{p k x}{c n} - \frac{e^{-\lambda c} q \lambda k x}{n(1 - e^{-\lambda c})} \sim -\frac{x}{n} \quad \text{as } n \to \infty.$$ 

Consequently, the constant $k$ is given by

$$k = \left( \frac{p}{c} + \frac{\lambda q}{e^{\lambda c} - 1} \right)^{-1},$$

and the equality (3.12) holds true. \hfill \blacksquare

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References


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