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BANACH SPACES OVER TOPOLOGICAL SEMIFIELDS AND COMMON FIXED POINTS

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Abstract. In this paper we shall prove the existence of a common fixed point of two mappings of a Banach space over a topological semifield.

INTRODUCTION. The notion of topological semifield has been introduced by M. Antonovskiĭ, V. Boltyanskiĭ and T. Sarymsakov in [1]. Let E be a topological semifield and K the set of all its positive elements. Take any two elements x, y in E. If $y - x$ is in $\overline{K}$ (in K), this is denoted by $x \ll y$ ($x < y$). As proved in [1], every topological semifield E contains a subsemifield, called the axis of E, isomorphic to the field $\mathbf{R}$ of real numbers. Consequently, by identifying the axis and $\mathbf{R}$, each topological semifield can be regarded as a topological linear space over the field $\mathbf{R}$.

The ordered triple (X, d, E) is called a metric space over the topological semifield if there exists a mapping $d\colon X \times X \to \overline{K}$ satisfying the usual axioms for a metric (see [1] and [3]).

Linear spaces considered in this paper are defined over the field $\mathbf{R}$. Let X be a linear space. The ordered triple $(X, \|\ \|, E)$ is called a feeble normed space over the topological semifield if there exists a mapping $\|\ \|\colon X \to \overline{K}$ satisfying the usual axioms for a norm (see [1] and [2]).

Let $(X, \|\ \|, E)$ be a feeble normed space over the topological semifield E and let $d(x, y) = \|x - y\|$ for all x, y in X. The space $(X, \|\ \|, E)$ is said to be a Banach space over the topological semifield E if (X, d, E) is a sequentially complete metric space over the topological semifield E.

We shall prove the following theorem.

Key words and phrases: Banach space over a topological semifield, common fixed point, Cauchy sequence and sequentially complete metric space

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THEOREM 1. *Let X be a Banach space over the topological semifield E and $S, T\colon X \to X$ two maps. If there exist real numbers b, c, q and t such that*

$$0 \leqslant qt + |b|(1-t) - c < b + c, \quad 0 < t < 1, \tag{1}$$

$$b\,\|Sx - Ty\| + c(\|x - Sx\| + \|y - Ty\|) \ll q\|x - y\| \tag{2}$$

for all x, y in X, then the sequence $\{x_n\}$, the members of which are

$$\begin{aligned} x_{2n+1} &= (1-t)x_{2n} + t\,Sx_{2n}, \\ x_{2n+2} &= (1-t)x_{2n+1} + t\,Tx_{2n+1}, \quad x_0 \in X, \quad n = 0,1,2,\dots \end{aligned} \tag{3}$$

converges to the common fixed point of S and T in X.

Proof. Let x_0 in X be an arbitrary point. Using the sequence (3) we have

$$\begin{aligned} \|x_{2n+1} - x_{2n}\| &= t\,\|Sx_{2n} - x_{2n}\|, \\ \|x_{2n+2} - x_{2n+1}\| &= t\,\|Tx_{2n+1} - x_{2n+1}\| \end{aligned} \tag{4}$$

and hence

$$\begin{aligned} \|x_{2n+1} - Sx_{2n}\| &= \|(1-t)x_{2n} + t\,Sx_{2n} - Sx_{2n}\| \\ &= (1-t)\|x_{2n} - Sx_{2n}\| = \frac{1-t}{t}\,\|x_{2n} - x_{2n+1}\|, \\ \|x_{2n+2} - Tx_{2n+1}\| &= \|(1-t)x_{2n+1} + t\,Tx_{2n+1} - Tx_{2n+1}\| \\ &= (1-t)\|x_{2n+1} - Tx_{2n+1}\| = \frac{1-t}{t}\,\|x_{2n+1} - x_{2n+2}\|. \end{aligned}$$

Then the inequalities

$$\|x_{2n+1} - Tx_{2n+1}\| - \|x_{2n+1} - Sx_{2n}\| \ll \|Sx_{2n} - Tx_{2n+1}\| \tag{5}$$

and

$$\|x_{2n+2} - Sx_{2n+2}\| - \|x_{2n+2} - Tx_{2n+1}\| \ll \|Sx_{2n+2} - Tx_{2n+1}\| \tag{6}$$

respectively become

$$\frac{1}{t}\|x_{2n+2} - x_{2n+1}\| - \frac{1-t}{t}\,\|x_{2n} - x_{2n+1}\| \ll \|Sx_{2n} - Tx_{2n+1}\| \tag{7}$$

and

$$\frac{1}{t}\,\|x_{2n+3} - x_{2n+2}\| - \frac{1-t}{t}\,\|x_{2n+1} - x_{2n+2}\| \ll \|Sx_{2n+2} - Tx_{2n+1}\|. \tag{8}$$

Let $b \geqslant 0$. If we put in (2) $x = x_{2n}$ and $y = x_{2n+1}$, then from (4), (5) and (7) we get

$$\begin{aligned} \frac{b}{t}\,\|x_{2n+2} - x_{2n+1}\| - |b|\frac{1-t}{t}\,\|x_{2n} - x_{2n+1}\| + \frac{c}{t}\,\|x_{2n+1} - x_{2n}\| \\ + \frac{c}{t}\,\|x_{2n+2} - x_{2n+1}\| \ll q\|x_{2n} - x_{2n+1}\| \end{aligned} \tag{9}$$

and hence

$$\|x_{2n+2} - x_{2n+1}\| \ll k\,\|x_{2n+1} - x_{2n}\|, \tag{10}$$

where $k = (qt + (1-t)|b| - c)/(b+c)$ and $b = |b|$. Now, if we put in (2) $x = x_{2n+2}$ and $y = x_{2n+1}$, and use (4), (6) and (8), we get

$$\frac{b}{t}\,\|x_{2n+3} - x_{2n+2}\| - |b|\frac{1-t}{t}\,\|x_{2n+2} - x_{2n+2}\| + \frac{c}{t}\,\|x_{2n+3} - x_{2n+2}\| \\ + \frac{c}{t}\,\|x_{2n+2} - x_{2n+1}\| \ll q\|x_{2n+2} - x_{2n+1}\| \tag{11}$$

and hence

$$\|x_{2n+3} - x_{2n+2}\| \ll k\,\|x_{2n+2} - x_{2n+1}\|. \tag{12}$$

Now, if $b < 0$, then we use the inequalities

$$\|x_{2n+1} - Tx_{2n+1}\| + \|x_{2n+1} - Sx_{2n}\| \gg \|Sx_{2n} - Tx_{2n+1}\| \tag{13}$$

and

$$\|x_{2n+2} - Sx_{2n+2}\| + \|x_{2n+2} - Tx_{2n+1}\| \gg \|Sx_{2n+2} - Tx_{2n+1}\|. \tag{14}$$

If in (2) we put $x = x_{2n}$ and $y = x_{2n+1}$, then by (13) we obtain (9), since then $-|b| = b$. If in (2) we put $x = x_{2n+2}$ and $y = x_{2n+1}$, then by (14) we obtain (11).

From (10) and (12) then we obtain

$$\|x_n - x_{n+1}\| \ll k\,\|x_{n-1} - x_n\|,$$

which implies

$$\|x_n - x_{n+1}\| \ll k^n\|x_0 - x_1\|.$$

Since (1) implies $0 \leqslant k < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X. Because X is a Banach space over the topological semifield E, we deduce that $\{x_n\}$ converges to a point u in X. Then from (4) we have $\lim_n Tx_{2n+1} = u$.

If in (2) we put $x = u$ and $y = x_{2n+1}$, then by (4) we have

$$b\,\|Su - Tx_{2n+1}\| + c\|u - Su\| + \frac{c}{t}\,\|x_{2n+2} - x_{2n+1}\| \ll q\|u - x_{2n+1}\|.$$

Letting $n \to +\infty$, we get $(b+c)\|Su - u\| \ll 0$. Then, as $b + c > 0$, it follows that $Su = u$. Hence, u is a fixed point for S. Similarly, $Tu = u$. Thus, u is a common fixed point of S and T. This completes the proof. ■

REMARK. In case $S = T$, $b = 0$ and $c = 1$ in Theorem 1, we obtain the Theorem 1 of Nešić [4].

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