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GENERALIZED HAUSDORFF OPERATORS ON WEIGHTED HERZ SPACES

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Abstract. In this paper, we introduce new generalized Hausdorff operators. They include many famous operators as special cases. We obtain necessary and sufficient conditions for these operators to be bounded on the weighted Herz spaces. The corresponding new operator norm inequalities are obtained. They are significant improvements and generalizations of many known results. Several open problems are formulated.

1. Introduction

The classical Hardy operator T_0 is defined by

$$T_0(f,x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0,$$

and the classical Hardy inequality stated in Hardy et al. [4] is

$$\|T_0 f\|_p \le \frac{p}{p-1}\|f\|_p, \quad 1 < p < \infty, \tag{1.1}$$

where the best constant factor $\frac{p}{p-1}$ is the best value and it is the norm of the operator T_0, that is,

$$\|T_0\| = \frac{p}{(p-1)}.$$

As pointed out by Kufner et al. [6], the Hardy inequality has a fascinating past and will have (hopefully) also a fascinating future. These authors of [6] present some important steps of the development of (1.1), of its early weighted generalizations and of its various modifications and extensions. Another classical operator is the Hausdorff operator

$$T_1(f,x) = \int_0^\infty \frac{\psi(t)}{t} f\Big(\frac{x}{t}\Big)\,dt, \quad x > 0, \tag{1.2}$$

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where ψ is a local integrable function in $(0,\infty)$. This operator and its varieties have attracted many authors, for example, see [7–10]. With $u=\frac{x}{t}$, (1.2) yields

$$T_1(f,x)=\int_0^\infty \frac{\psi(\frac{x}{u})}{u}f(u)\,du,\quad x>0.$$

Choosing $\psi(u)=u^{-1}\varphi_{E_1}(u)$ and $\psi(u)=\varphi_{E_2}(u)$, where $E_1=(1,\infty)$, $E_2=(0,1]$, and φ_E denotes the characteristic function of the set E, we obtain the Hardy operator T_0 and the dual Hardy operator (or Cesáro operator) T_0^* defined by

$$T_0^*(f,x)=\int_x^\infty \frac{f(u)}{u}\,du,\quad x>0,$$

respectively. In addition, $T_2=T_0+T_0^*$ becomes the Calderòn maximal operator [1]:

$$T_2(f,x)=\frac{1}{x}\int_0^x f(t)\,dt+\int_x^\infty \frac{f(t)}{t}\,dt,\quad x>0.$$

The aim of this paper is to introduce the following new generalized Hausdorff operator

$$T(f,x)=\int_0^\infty \frac{\psi(t)}{t}f(g(t)x)\,dt,\quad x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n, \tag{1.3}$$

where $g(t)x=(g(t)x_1,g(t)x_2,\dots,g(t)x_n)$, $\psi:(0,\infty)\to(0,\infty)$ is a locally integrable function, $g:(0,\infty)\to(0,\infty)$ is a monotonic function (increasing or decreasing), and f is a measurable complex valued function on $\mathbb{R}^n$. If $g(t)=\frac{1}{t}$ and $n=1$, then T reduces to the Hausdorff operator T_1.

If we introduce other forms of g and ψ, it is possible to obtain other operators of interest. For example, if $g(t)=t$, $\psi(t)=t\omega(t)\varphi_E(t)$, where $E=(0,1]$, ω is a non-negative weight function, then T reduces to the weighted Hardy-Littlewood mean operator defined in [12]:

$$T_3(f,x)=\int_0^1 f(tx)\omega(t)\,dt,\quad x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n. \tag{1.4}$$

If $g(t)=\frac{1}{t}$, $\psi(t)=t^{-(n-1)}\omega(t)\varphi_E(t)$, ω, E are as in (1.4), then T reduces to the weighted Cesàro mean operator defined in [12]:

$$T_4(f,x)=\int_0^1 f\Big(\frac{x}{t}\Big)t^{-n}\omega(t)\,dt,\quad x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n.$$

If $g(t)=t$, $\psi(t)=te^{-\lambda t}$, $\lambda>0$, $n=1$, $x>0$, then T reduces to

$$T_5(f,x)=\int_0^\infty f(tx)e^{-\lambda t}\,dt=\frac{1}{x}\int_0^\infty f(u)e^{-\alpha u}\,du=\frac{1}{x}L(f,\alpha),$$

where $L(f,\alpha)=\int_0^\infty f(u)e^{-\alpha u}\,du$ is the Laplace transform of f, $u=tx$, $\alpha=\frac{\lambda}{x}>0$. Hence, (1.3) is a significant generalization of many famous operators.

It is well-known that the Herz spaces play an important role in characterizing the properties of functions and multipliers on the classical Hardy spaces.

In this paper, we obtain necessary and sufficient conditions for the generalized Hausdorff operator T defined by (1.3) to be bounded on the weighted Herz spaces. The corresponding new operator norm inequalities are obtained. Several open problems are formulated.

2. Definitions and statement of the main results

Let $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $D_k = B_k - B_{k-1}$ and let $\varphi_k = \varphi_{D_k}$ denote the characteristic function of the set D_k. Moreover, for a measurable function f on $\mathbb{R}^n$ and a non-negative weight function $\omega(x)$, we write

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx\right)^{1/p}.$$

In what follows, if $\omega \equiv 1$, then we will denote $L^p(\mathbb{R}^n, \omega)$ (in brief $L^p(\omega)$) by $L^p(\mathbb{R}^n)$.

Definition 2.1. (see [11]) Let $\alpha \in \mathbb{R}^1$, $0 < p, q < \infty$ and ω_1 and ω_2 be non-negative weight functions. The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1, \omega_2) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n - \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \left\{\sum_{k\in\mathbb{Z}} [\omega_1(B_k)]^{\frac{\alpha p}{n}} \|f\varphi_k\|^p_{q,\omega_2}\right\}^{1/p}.$$

We can similarly define the non-homogeneous weighted Herz spaces $K_q^{\alpha,p}(\omega_1, \omega_2)$.

It is easy to see that when $\omega_1 = \omega_2 = 1$, we have

$$\dot{K}_q^{\alpha,p}(1,1) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n),$$
$$\dot{K}_p^{(\alpha/p),p}(\mathbb{R}^n) = L^p(|x|^\alpha\, dx), \quad \dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

Definition 2.2. (see [2]) A non-negative weight function ω satisfies Muckenhoupt's A_∞ condition or $\omega \in A_\infty$, if there is a constant C independent of the cube Q in $\mathbb{R}^n$, such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\, dx\right) \exp\left\{\frac{1}{|Q|}\int_Q \log\left(\frac{1}{\omega(x)}\right) dx\right\} \leq C, \ \forall Q \subset \mathbb{R}^n,$$

where $|Q|$ is the Lebesgue measure of Q.

Our main results are the following three theorems:

Theorem 2.1. *Let $\alpha \in \mathbb{R}^1$, $0 < p < \infty$, $1 \leq q < \infty$, $\omega_1 \in A_\infty$, and a non-negative weight function ω_2 satisfy*

$$\omega_2(tx) = t^\beta \omega_2(x), \quad t > 0, \quad \beta \in \mathbb{R}^1, \ x \in \mathbb{R}^n \tag{2.1}$$

Let $\psi : (0,\infty) \to (0,\infty)$ be a locally integrable function having the compact support on $(0,\infty)$; let $g : (0,\infty) \to (0,\infty)$ be an increasing function satisfying the submultiplicative condition

$$g(uv) \leq g(u)g(v), \quad u, v > 0.$$

Let $\|T\|$ be the norm of the operator T defined by (1.3) and mapping $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) \to \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$.

(1) *If* $g(t)^{-(\beta+n)/q}\frac{\psi(t)}{t}$ *is a concave function on* $(0,\infty)$ *and* $\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt<\infty$*, then*

$$\|T\|\le C(p,\alpha)\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt, \tag{2.2}$$

where

$$C(p,\alpha)=\begin{cases} C_0^{\frac{\alpha}{n}}2^{(1/p)-2}(1+p)^{1/p}(1+g(2)^{|\alpha|\delta}), & 0<p<1,\\ C_0^{\frac{\alpha}{n}}2^{1-(2/p)}(1+(1/p))(1+g(2)^{|\alpha|\delta}), & 1\le p<\infty. \end{cases} \tag{2.3}$$

(2) *If* $\|T\|<\infty$*, and* g *is a strictly increasing function on* $(0,\infty)$ *and the inverse* g^{-1} *of* g *satisfies* $g^{-1}(t)\to 0$ $(t\to 0^+)$*, then*

$$\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt\le\|T\|.$$

(C_0 *and* δ *are constants given in* (3.4)*, see Section 3 below.)*

Remark 1. ω_2 is an extension of the power weight $\omega_2(x)=|x|^\beta$, $(x\in\mathbb{R}^n)$.

We use the following notation:

$$KF=\Big\{f: f\in\dot K_q^{\alpha,p}(\omega_1,\omega_2),\ F(t)=\sup_{x\in\mathbb{R}^n}|f(g(t)x)|\Big(\frac{\psi(t)}{t}\Big)\ \text{is a concave function on } (0,\infty)\Big\}.$$

Then KF is a subspace of the space $\dot K_q^{\alpha,p}(\omega_1,\omega_2)$.

Theorem 2.2 *Let* $\alpha\in\mathbb{R}^1$, $0<p<\infty$, $0<q<1$, g, ψ, ω_1, ω_2 *be as in Theorem 2.1, and* $\|T\|$ *be the norm of the operator* T *defined by (1.3), mapping* $KF\to\dot K_q^{\alpha,p}(\omega_1,\omega_2)$.

(1) *If* $g(t)^{-(\beta+n)/q}\frac{\psi(t)}{t}$ *is a concave function on* $(0,\infty)$ *and* $\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt<\infty$*, then*

$$\|T\|\le C(p,q,\alpha,)\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt, \tag{2.4}$$

where $C(p,q,\alpha)$

$$=\begin{cases} C_0^{\alpha/n}2^{(1/p)-(1/q)-2}q^{-1/p}(1+q)^{1/q}(p+q)^{1/p}(1+g(2)^{|\alpha|\delta}), & 0<p\le q<1,\\ C_0^{\alpha/n}2^{(1/q)-2}(1+q)^{1/q}(1+g(2)^{|\alpha|\delta}), & 0<q<p<1,\\ C_0^{\alpha/n}2^{(1/q)-(2/p)-1}(1+q)^{1/q}(1+(1/p))(1+g(2)^{|\alpha|\delta}), & 0<q<1\le p<\infty. \end{cases} \tag{2.5}$$

(2) *If* $\|T\|<\infty$*, and* g *is a strictly increasing function on* $(0,\infty)$ *and the inverse* g^{-1} *of* g *satisfies* $g^{-1}(t)\to 0(t\to 0^+)$*, then*

$$\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt\le\|T\|.$$

(C_0 *and* δ *are constants given in* (3.4)*, see Section 3 below.)*

Remark 2. When g is a decreasing function, similar results with $g(2^{-1})$ and $g^{-1}(t) \to \infty (t \to \infty)$ instead of $g(2)$ and $g^{-1}(t) \to 0 (t \to 0)$ can be obtained from the previous two theorems.

Theorem 2.3. *Let $\beta \in \mathbb{R}^1$, $1 \leq p < \infty$, and g, ψ be two positive measurable functions defined on $(0,\infty)$, and g be a strictly monotonic function on $(0,\infty)$ and the inverse g^{-1} of g satisfies: (1) when g is increasing, $g^{-1}(t) \to 0$ $(t \to 0^+)$; (2) when g is decreasing, $g^{-1}(t) \to \infty$ $(t \to \infty)$, and a nonnegative weight function ω satisfies*

$$\omega(tx) = t^{\beta}\omega(x), \ t > 0, \quad \beta \in \mathbb{R}^1, \ x \in \mathbb{R}^n.$$

Then the operator T defined by (1.3), mapping $L^p(\omega) \to L^p(\omega)$, exists as a bounded operator if and only if

$$\int_0^{\infty} g(t)^{-(\beta+n)/p} \frac{\psi(t)}{t}\, dt < \infty. \tag{2.6}$$

Moreover, when (2.6) holds, the operator norm $\|T\|$ of T on $L^p(\omega)$ satisfies

$$\|T\| = \int_0^{\infty} g(t)^{-(\beta+n)/p} \frac{\psi(t)}{t}. \tag{2.7}$$

Remark 3. It follows from (2.7) and (1.3) that

$$\left\{\int_{\mathbb{R}^n} \left| \int_0^{\infty} \frac{\psi(t)}{t} f(g(t)x)\, dt \right|^p \omega(x)\, dx \right\}^{1/p} \\ \leq \left(\int_0^{\infty} g(t)^{-(\beta+n)/p} \frac{\psi(t)}{t}\, dt \right) \times \left\{ \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx \right\}^{1/p}, \tag{2.8}$$

where$\|T\| = \int_0^{\infty} g(t)^{-(\beta+n)/p} \frac{\psi(t)}{t}\, dt$ is the best possible constant. In particular, when $\psi(t) = t^{\lambda}\varphi_E(t)$, $E = (0,1]$, $g(t) = t$, $n = 1$, and $(-\infty,\infty)$ is substituted by $(0,\infty)$, and $\lambda > 0$, $\beta < \lambda p - 1$, then by (2.7), we get

$$\|T\| = \frac{p}{\lambda p - (\beta+1)}.$$

It follows from (2.8) that

$$\left\{\int_0^{\infty} \left| \frac{1}{x^{\lambda}} \int_0^{x} f(t)t^{\lambda-1}\, dt \right|^p \omega(x)\, dx \right\}^{1/p} \leq \frac{p}{\lambda p - (\beta+1)} \left\{ \int_0^{\infty} |f(x)|^p \omega(x)\, dx \right\}^{1/p}, \tag{2.9}$$

where $\|T\| = \frac{p}{\lambda p - (\beta+1)}$ is the best possible constant. If $\lambda = 1$, $\omega(x) = 1$, that is, $\beta = 0$, then (2.9) reduces to the Hardy inequality (1.1). If $\lambda = 1$, $\omega(x) = x^{\beta}$, then (2.9) reduces to the result of [6, p. 23]. When $\psi(t) = t^{-\lambda}\varphi_E(t)$, $E = (0,1]$, $\lambda \geq 0$, $g(t) = t^{-1}$, $n = 1$, and $(-\infty,\infty)$ is substituted by $(0,\infty)$, and $\beta > \lambda p - 1$, then by (2.7), we get $\|T\| = \frac{p}{(\beta+1) - \lambda p}$. It follows from (2.8) that

$$\left(\int_0^{\infty} | \frac{1}{x^{\lambda}} \int_x^{\infty} f(t)t^{\lambda-1}\, dt|^p \omega(x)\, dx \right)^{1/p} \leq \frac{p}{(\beta+1) - \lambda p} \left(\int_0^{\infty} |f(x)|^p \omega(x)\, dx \right)^{1/p},$$

where $\|T\| = \frac{p}{(\beta+1)-\lambda p}$ is the best possible constant. This is the weighted extension of the dual Hardy inequality in [5,6]. When $g(t) = t$, $\psi(t) = te^{-\lambda t}$, $\lambda > 0$, $n = 1$, and $(-\infty, \infty)$ is substituted by $(0, \infty)$, and $\beta < p - 1$, $x > 0$, then by (2.7), we get

$$\|T\| = \int_0^\infty t^{-(\beta+1)/p} e^{-\lambda t}\, dt = \frac{\Gamma(1 - (\beta+1)/p)}{\lambda^{1-(\beta+1)/p}}.$$

It follows from (2.8) that

$$\left\{\int_0^\infty |\frac{1}{x}\int_0^\infty f(t)e^{-(\lambda/x)t}\, dt|^p \omega(x)\, dx\right\}^{1/p} \leq \frac{\Gamma(1-(\beta+1)/p)}{\lambda^{1-(\beta+1)/p}} \left\{\int_0^\infty |f(x)|^p \omega(x)\, dx\right\}^{1/p}, \quad (2.10)$$

where $\|T\| = \frac{\Gamma(1-\frac{(\beta+1)}{p})}{\lambda^{1-\frac{(\beta+1)}{p}}}$ is the best possible constant. In particular, when $f(x) \geq 0$, $\omega(x) = 1$, that is, $\beta = 0$, then (2.10) reduces to the following Laplace transform inequality:

$$\left\{\int_0^\infty \left(\int_0^\infty f(t)e^{-(\lambda/x)t}\, dt\right)^p x^{-p}\, dx\right\}^{1/p} \leq \frac{\Gamma(1-(1/p))}{\lambda^{1-(1/p)}} \left(\int_0^\infty f^p(x)\, dx\right)^{1/p}. \quad (2.11)$$

Remark 4. Hardy [4, Theorems 350, 352] proved the following three inequalities:

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) \geq 0$, $\omega(x) = x^{p-2}$. Then

$$\|L(f)\|_p \leq \Gamma(1/p)\|f\|_{p,\omega}, \quad (2.12)$$

$$\|L(f)\|_{p,\omega} \leq \Gamma(1 - \frac{1}{p})\|f\|_p. \quad (2.13)$$

For $1 < p \leq 2$,

$$\|L(f)\|_q \leq (\frac{2\pi}{q})^{1/q}\|f\|_p. \quad (2.14)$$

These inequalities are "the Laplace transforms analogues" of inequalities in the theory of Fourier series. As pointed out by Hardy [3,4], it is not asserted that the constant in (2.14) is the best possible and it may be difficult to find the best possible value. Here we prove that the constants in (2.8)–(2.11) are the best possible. Hence, our main results are significant generalization of many known results.

Remark 5. There are some similar results for the non-homogeneous weighted Herz spaces $K_q^{\alpha,p}(\omega_1, \omega_2)$. We omit the details here.

Several open problems. In Theorem 2.3, we solve the best value for $C(p, \beta)$ in the following inequality

$$\|Tf\|_{p,\omega} \leq C(p, \beta)\|f\|_{p,\omega},$$

that is, $C(p,\beta) = \|T\| = \int_0^\infty g(t)^{-(\beta+n)/p} \frac{\psi(t)}{t}\, dt$ is the best possible constant, but yet in Theorems 2.1 and 2.2, the best value for $C(p,\alpha)$ and $C(p,q,\alpha)$ in the following inequalities

$$\|Tf\|_K \leq C(p,\alpha)\|f\|_K$$

and

$$\|Tf\|_K \leq C(p,q,\alpha)\|f\|_{KF}$$

are not solved. It is not asserted that the constants $C(p,\alpha)$ and $C(p,q,\alpha)$ in Theorems 2.1 and 2.2 are the best possible.

3. Proofs of the theorems

We require the following lemmas to prove our results.

Lemma 3.1. *Let f be a nonnegative measurable function on $[0,b]$, $0 < b < \infty$. If $1 \leq p < \infty$, then*

$$\left(\int_0^b f(x)\, dx\right)^p \leq b^{(p-1)} \int_0^b f^p(x)\, dx. \tag{3.1}$$

Lemma 3.1 is an immediate consequences of Hölder inequality.

Lemma 3.2. (see [5]) *Let f be a nonnegative measurable and concave function on $[a,b]$, $0 < \alpha \leq \beta$. Then*

$$\left\{\frac{\beta+1}{b-a}\int_a^b [f(x)]^\beta\, dx\right\}^{1/\beta} \leq \left\{\frac{\alpha+1}{b-a}\int_a^b [f(x)]^\alpha\, dx\right\}^{1/\alpha}. \tag{3.2}$$

Setting $a = 0$, for $\alpha = p$, $\beta = 1$, that is, $0 < p \leq 1$, we obtain from (3.2)

$$\left(\int_0^b f(x)\, dx\right)^p \leq \frac{p+1}{2^p} \times b^{p-1} \int_0^b f^p(x)\, dx \qquad (3.3).$$

By the properties of A_∞ weights, we have

Lemma 3.3 (see [2]) *If $\omega \in A_\infty$, then there exist $\delta > 0$, $C_0 > 0$, such that for each ball $\mathbb{B}$ in $\mathbb{R}^n$ and measurable subset E of $\mathbb{B}$,*

$$\frac{\omega(E)}{\omega(B)} \leq C_0\left(\frac{|E|}{|B|}\right)^\delta, \tag{3.4}$$

where $|E|$ is the Lebesgue measure of E and $\omega(E) = \int_E \omega(x)\, dx$.

Lemma 3.4 (see [5]) (C_p inequality) *Let $a_1, a_2, \ldots, a_n$ be arbitrary real (or complex) numbers. Then*

$$\left(\sum_{k=1}^n |a_k|\right)^p \leq C_p\left(\sum_{k=1}^n |a_k|^p\right), \quad 0 < p < \infty, \tag{3.5}$$

where

$$C_p = \begin{cases} 1, & 0 < p < 1, \\ n^{p-1}, & 1 \le p < \infty. \end{cases}$$

In what follows, we shall write simply $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ to denote K and $B(2^k)$ to denote $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$.

Proof of Theorem 2.1. Since ψ has a compact support on $(0,\infty)$, there exists $b > 0$, such that $\operatorname{supp}\psi(t) \subset (0,b]$. First, we prove (2.2). Using Minkowski's inequality for integrals and (2.1), and setting $u = g(t)x$, we get

$$\begin{aligned} \|(Tf)\varphi_k\|_{q,\omega_2} &\le \int_0^b \left\{ \int_{D_k} |f(g(t)x)|^q \omega_2(x)\,dx \right\}^{1/q} \frac{\psi(t)}{t}\,dt \\ &= \int_0^b \left\{ \int_{2^{k-1}g(t)<|u|\le 2^k g(t)} |f(u)|^q \omega_2(u)\,du \right\}^{1/q} g(t)^{-(\beta+n)/q} \frac{\psi(t)}{t}\,dt. \end{aligned}$$

For each $t \in (0,b)$, there exists an integer m such that $2^{m-1} < t \le 2^m$. Setting

$$\begin{aligned} A_1 &= \{u \in \mathbb{R}^n : 2^{k-1}g(2^{m-1}) < |u| \le 2^k g(2^{m-1})\}, \quad E_1 = B(2^k g(2^{m-1})), \\ A_2 &= \{u \in \mathbb{R}^n : 2^k g(2^{m-1}) < |u| \le 2^k g(2^m)\}, \quad E_2 = B(2^k g(2^m)), \end{aligned}$$

we obtain

$$\begin{aligned} \|(Tf)\varphi_k\|_{q,\omega_2} &\le \int_0^b \left\{ \int_{A_1} |f(u)|^q \omega_2(u)\,du \right. \\ &\qquad \left. + \int_{A_2} |f(u)|^q \omega_2(u)\,du \right\}^{1/q} g(t)^{-(\beta+n)/q} \frac{\psi(t)}{t}\,dt \\ &\le \int_0^b (\|f\varphi_{A_1}\|_{q,\omega_2} + \|f\varphi_{A_2}\|_{q,\omega_2}) g(t)^{-(\beta+n)/q} \frac{\psi(t)}{t}\,dt. \end{aligned}$$

It follows that

$$\begin{aligned} &\|Tf\|_K \le \\ &\left\{ \sum_{k\in\mathbb{Z}} [\omega_1(B_k)]^{\frac{\alpha p}{n}} \left[\int_0^b (\|f\varphi_{A_1}\|_{q,\omega_2} + \|f\varphi_{A_2}\|_{q,\omega_2}) g(t)^{-(\beta+n)/q} \frac{\psi(t)}{t}\,dt \right]^p \right\}^{1/p}. \end{aligned} \tag{3.6}$$

Now, we consider two cases for p:

Case 1. $0 < p < 1$. In this case, it follows from (3.6), (3.3) and (3.5) that

$$\begin{aligned} \|Tf\|_K &\le \frac{(1+p)^{1/p}}{2b^{(1/p)-1}} \left\{ \sum_{k\in\mathbb{Z}} [\omega_1(B_k)]^{\alpha p/n} \right. \\ &\qquad \left. \times \int_0^b (\|f\varphi_{A_1}\|_{q,\omega_2}^p + \|f\varphi_{A_2}\|_{q,\omega_2}^p) g(t)^{-(\beta+n)p/q} \left(\frac{\psi(t)}{t}\right)^p dt \right\}^{1/p} \\ &\le 2^{(1/p)-2}(1+p)^{1/p} b^{1-(\frac{1}{p})} \left\{ \left[\int_0^b \sum_{k\in\mathbb{Z}} \omega_1(E_1)^{\alpha p/n} \|f\varphi_{A_1}\|_{q,\omega_2}^p \left(\frac{\omega_1(B_k)}{\omega_1(E_1)}\right)^{\alpha p/n} \right.\right. \end{aligned}$$

$$\times\, g(t)^{-(\beta+n)p/q}(\frac{\psi(t)}{t})^p\,dt\Big]^{1/p} + \Big[\int_0^b \sum_{k\in(Z)} \omega_1(E_2)^{\alpha p/n}\|f\varphi_{A_2}\|^p_{q,\omega_2}$$

$$\times\left(\frac{\omega_1(B_k)}{\omega_1(E_2)}\right)^{\alpha p/n} g(t)^{-(\beta+n)p/q}\left(\frac{\psi(t)}{t}\right)^p dt\Big]^{1/p}\Big\}. \quad (3.7)$$

By (3.4) and $|B_k| = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \times 2^{kn}$, we have

$$\frac{\omega_1(B_k)}{\omega_1(E_1)} \le C_0\left(\frac{|B_k|}{|E_1|}\right)^{\delta} = C_0 g(2^{m-1})^{-n\delta} \quad (3.8)$$

and

$$\frac{\omega_1(B_k)}{\omega_1(E_2)} \le C_0 g(2^m)^{-n\delta}. \quad (3.9)$$

It follows from (3.7)–(3.9) that

$$\|Tf\|_K \le C_0^{\frac{\alpha}{n}} 2^{(1/p)-2}(1+p)^{1/p}\|f\|_K$$
$$\times \int_0^b (g(2^{m-1})^{-\alpha\delta} + g(2^m)^{-\alpha\delta}) g(t)^{-(\beta+n)/q}\frac{\psi(t)}{t}\,dt. \quad (3.10)$$

By the submultiplicativity of g, we have $g(2^m) \le g(2)g(2^{m-1})$. If $\alpha > 0$, then

$$[g(2^{m-1})]^{-\alpha\delta} \le [g(2^m)]^{-\alpha\delta}[g(2)]^{\alpha\delta}.$$

Since g is a increasing function, thus $t \le 2^m$ implies that $g(t) \le g(2^m)$, and therefore that $g(2^m)^{-\alpha\delta} \le g(t)^{-\alpha\delta}$. This implies

$$g(2^{m-1})^{-\alpha\delta} + g(2^m)^{-\alpha\delta} \le g(2^m)^{-\alpha\delta}\{1+g(2)^{\alpha\delta}\} \le g(t)^{-\alpha\delta}\{1+g(2)^{\alpha\delta}\}. \quad (3.11)$$

Similarly, if $\alpha < 0$, then

$$g(2^{m-1})^{-\alpha\delta} + g(2^m)^{-\alpha\delta} \le g(t)^{-\alpha\delta}\{1+g(2)^{-\alpha\delta}\}. \quad (3.12)$$

(3.11) and (3.12) imply that

$$g(2^{m-1})^{-\alpha\delta} + g(2^m)^{-\alpha\delta} \le g(t)^{-\alpha\delta}\{1+g(2)^{|\alpha|\delta}\}. \quad (3.13)$$

Thus, by (3.10) and(3.13), we get

$$\|Tf\|_K \le C_0^{\alpha/n} 2^{(1/p)-2}(1+p)^{1/p}(1+g(2)^{|\alpha|\delta})\|f\|_K \int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt. \quad (3.14)$$

Case 2. $1 \le p < \infty$. In this case, by (3.6), (3.1), (3.3), (3.8) and (3.9), we similarly obtain

$$\|Tf\|_K \le (2b)^{1-(1/p)}\Big\{\sum_{k\in\mathbb{Z}}[\omega_1(B_k)]^{\frac{\alpha p}{n}}$$
$$\times \int_0^b (\|f\varphi_{A_1}\|^p_{q,\omega_2} + \|f\varphi_{A_2}\|^p_{q,\omega_2}) g(t)^{(-(\beta+n)/q)p}\left(\frac{\psi(t)}{t}\right)^p dt\Big\}^{1/p}$$
$$\le C_0^{\frac{\alpha}{n}} 2^{1-(2/p)}(1+(1/p))(1+g(2)^{|\alpha|\delta})\|f\|_K \int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt. \quad (3.15)$$

Hence, by (3.14) and (3.15), we get

$$\|T\| \le C(p,\alpha)\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt,$$

where $C(p,\alpha)$ is defined by (2.3).

To prove the opposite inequality, putting $\varepsilon \in (0,1)$, we set $\omega_1(B_k) = 2^{kn\delta}$, $\omega_2(x) = |x|^\beta$, and

$$f_\varepsilon(x) = \begin{cases} 0, & |x| \le 1, \\ |x|^{-\alpha\delta-(\beta+n)/q-\varepsilon}, & |x| > 1. \end{cases}$$

Then for $k = 0, -1, -2, \dots$, $\|f_\varepsilon\varphi_k\|_{q,\omega_2} = 0$ and for $k \in \mathbb{Z}^+$, we have

$$\begin{aligned} \|f_\varepsilon\varphi_k\|_{q,\omega_2} &= \left(\int_{2^{k-1}<|x|\le 2^k} |x|^{-(\alpha\delta+(\beta+n)/q+\varepsilon)q}\omega_2(x)\,dx\right)^{1/q} \\ &= \left\{\frac{2\pi^{n/2}}{\Gamma(n/2)}\int_{2^{k-1}}^{2^k} r^{-(\alpha\delta+\varepsilon)q-1}dr\right\}^{1/q} = C_n^{1/q}2^{-k(\alpha\delta+\varepsilon)}, \end{aligned}$$

where

$$C_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\left|\frac{2^{(\alpha\delta+\varepsilon)q}-1}{(\alpha\delta+\varepsilon)q}\right|.$$

It follows that

$$\begin{aligned} \|f_\varepsilon\|_K &= \left\{\sum_{k=1}^\infty \omega_1(B_k)^{\alpha p/n}\left(\int_{2^{k-1}<|x|\le 2^k} |x|^{-(\alpha\delta+(\beta+n)/q+\varepsilon)q}\omega_2(x)\,dx\right)^{p/q}\right\}^{1/p} \\ &= C_n^{1/q}\left\{\sum_{n=1}^\infty 2^{-kp\varepsilon}\right\}^{1/p} = C_n^{1/q}\frac{2^{-\varepsilon}}{(1-2^{-p\varepsilon})^{1/p}}. \end{aligned} \tag{3.16}$$

Since $g(t)|x| > 1$ implies that $t > g^{-1}(\frac{1}{|x|})$, we have

$$T(f_\varepsilon, x) = |x|^{-\alpha\delta-(\beta+n)/q-\varepsilon}\int_{g^{-1}(\frac{1}{|x|})}^\infty g(t)^{-\alpha\delta-(\beta+n)/q-\varepsilon}\frac{\psi(t)}{t}\,dt.$$

Let ε $(0 < \varepsilon < 1)$ be given. Since g is a strictly increasing function, there exists $m \in \mathbb{N}$, such that $2^{m-1} \le \frac{1}{\varepsilon} < 2^m$ and $2^{-m\varepsilon} \to 1$ $(\varepsilon \to 0^+)$. Note that $2^{k-1} < |x| \le 2^k$, so that if $k \ge m+1$, then $g^{-1}(\frac{1}{|x|}) \le g^{-1}(\frac{1}{2^{k-1}}) \le g^{-1}(\frac{1}{2^m}) \le g^{-1}(\varepsilon)$. Thus,

$$\begin{aligned} \|Tf_\varepsilon\|_K^p &= \sum_{k\in\mathbb{Z}} \omega_1(B_k)^{\alpha p/n}\left\{\int_{|x|>1}[T(f_\varepsilon,x)\varphi_k(x)]^q\omega_2(x)\,dx\right\}^{p/q} \\ &= \sum_{k=1}^\infty \omega_1(B_k)^{\alpha p/n}\left\{\int_{2^{k-1}<|x|\le 2^k}|x|^{-(\alpha\delta+(\beta+n)/q+\varepsilon)q}\right. \\ &\quad\times\left.\left(\int_{g^{-1}(\frac{1}{|x|})}^\infty g(t)^{-\alpha\delta-(\beta+n)/q-\varepsilon}\frac{\psi(t)}{t}\,dt\right)^q\omega_2(x)\,dx\right\}^{p/q} \end{aligned}$$

$$\geq \left(\int_{g^{-1}(\varepsilon)}^{\infty} g(t)^{-\alpha\delta-(\beta+n)/q-\varepsilon}\frac{\psi(t)}{t}\,dt\right)^p$$

$$\times\sum_{k=m+1}^{\infty}\omega_1(B_k)^{\alpha p/n}\left\{\int_{2^{k-1}<|x|\leq 2^k}|x|^{-(\alpha\delta+(\beta+n)/q+\varepsilon)q}\omega_2(x)\,dx\right\}^{p/q}$$

$$=\left(\int_{g^{-1}(\varepsilon)}^{\infty} g(t)^{-\alpha\delta-(\beta+n)/q-\varepsilon}\frac{\psi(t)}{t}\,dt\right)^p C_n^{p/q}\frac{2^{-(m+1)p\varepsilon}}{(1-2^{-p\varepsilon})}. \quad (3.17)$$

Thus, by (3.16) and (3.17), we get

$$\|T\|\geq\frac{\|Tf_\varepsilon\|_K}{\|f_\varepsilon\|_K}\geq 2^{-mp\varepsilon}\left\{\int_{g^{-1}(\varepsilon)}^{\infty} g(t)^{-\alpha\delta-(\beta+n)/q-\varepsilon}\frac{\psi(t)}{t}\,dt\right\}. \quad (3.18)$$

Taking limit as $\varepsilon\to 0$ in (3.18), we obtain

$$\|T\|\geq\int_0^{\infty} g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt.$$

This finishes the proof of Theorem 2.1. ■

Proof of Theorem 2.2. First, we prove (2.4). Using (3.3) and the notations in the proof of Theorem 2.1, we obtain

$$\|(Tf)\varphi_k\|_{q,\omega_2}\leq\left\{\int_{D_k}\left(\int_0^b(\sup_{x\in\mathbb{R}^n}|f(g(t)x)|)\frac{\psi(t)}{t}\,dt\right)^q\omega_2(x)\,dx\right\}^{1/q}$$

$$\leq 2^{-1}(1+q)^{1/q}b^{1-(1/q)}\left\{\int_{D_k}\left(\int_0^b(\sup_{x\in\mathbb{R}^n}|f(g(t)x)|)^q(\frac{\psi(t)}{t})^q\,dt\right)\omega_2(x)\,dx\right\}^{1/q}$$

$$\leq 2^{-1}(1+q)^{1/q}b^{1-(1/q)}$$

$$\times\left\{\int_0^b\left(\int_{2^{k-1}g(t)<|u|\leq 2^k g(t)}|f(u)|^q\omega_2(u)\,du\right)g(t)^{-(\beta+n)}(\frac{\psi(t)}{t})^q\,dt\right\}^{1/q}$$

$$\leq 2^{-1}(1+q)^{1/q}b^{1-(1/q)}\left\{\int_0^b(\|f\varphi_{A_1}\|_{q,\omega_2}^q+\|f\varphi_{A_2}\|_{q,\omega_2}^q)g(t)^{-(\beta+n)}(\frac{\psi(t)}{t})^q\,dt\right\}^{1/q}.$$

It follows that

$$\|Tf\|_K\leq 2^{-1}(1+q)^{1/q}b^{1-(1/q)}\left\{\sum_{k\in\mathbb{Z}}\omega_1(B_k)^{\alpha p/n}\right.$$

$$\left.\times\left[\int_0^b(\|f\varphi_{A_1}\|_{q,\omega_2}^q+\|f\varphi_{A_2}\|_{q,\omega_2}^q)g(t)^{-(\beta+n)}(\frac{\psi(t)}{t})^q\,dt\right]^{p/q}\right\}^{1/p}. \quad (3.19)$$

Now,we consider three cases:

Case 1. $0<p\leq q<1$. In this case, it follows from (3.19), (3.3), (3.5), (3.1), (3.8) and (3.9) that

$$\|Tf\|_K\leq 2^{-(1+(1/q))}q^{-(1/p)}(1+q)^{1/q}(p+q)^{1/p}b^{1-(1/p)}$$

$$\times\left\{\sum_{k\in\mathbb{Z}}\omega_1(B_k)^{\alpha p/n}\int_0^b(\|f\varphi_{A_1}\|_{q,\omega_2}^p+\|f\varphi_{A_2}\|_{q,\omega_2}^p)g(t)^{-(\beta+n)p/q}(\frac{\psi(t)}{t})^p\,dt\right\}^{1/p}$$

$$\le 2^{(1/p)-(1/q)-2}q^{-(1/p)}(1+q)^{1/q}(p+q)^{1/p}\|f\|_K$$

$$\times\int_0^b\Big\{(\frac{\omega_1(B_k)}{\omega_1(E_1)})^{\alpha/n}+(\frac{\omega_1(B_k)}{\omega_1(E_2)})^{\alpha/n}\Big\}g(t)^{-(\beta+n)/q}\frac{\psi(t)}{t}\,dt$$

$$\le C_0^{\alpha/n}2^{(1/p)-(1/q)-2}q^{-(1/p)}(1+q)^{1/q}(p+q)^{1/p}\|f\|_K$$

$$\times\int_0^b(g(2^{m-1})^{-\alpha\delta}+g(2^m)^{-\alpha\delta})g(t)^{-(\beta+n)/q}\frac{\psi(t)}{t}\,dt$$

$$\le C_0^{\alpha/n}2^{(1/p)-(1/q)-2}q^{-(1/p)}(1+q)^{1/q}(p+q)^{1/p}(1+g(2)^{|\alpha|\delta})\|f\|_K$$

$$\times\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt. \quad (3.20)$$

Case 2. $0<q\le p<1$. In this case, by (3.19), (3.5), (3.1), (3.8) and (3.9), we similarly obtain

$$\|Tf\|_K\le\frac{(1+q)^{1/q}}{2}b^{1-(1/p)}\Big\{\sum_{k\in\mathbb{Z}}\omega_1(B_k)^{\alpha p/n}$$

$$\times\int_0^b(\|f\varphi_{A_1}\|_{q,\omega_2}^q+\|f\varphi_{A_2}\|^q)^{p/q}g(t)^{-(\beta+n)p/q}(\frac{\psi(t)}{t})^p\,dt\Big\}^{1/p}$$

$$\le 2^{(1/q)-2}(1+q)^{1/q}\|f\|_K\int_0^b\Big\{(\frac{\omega_1(B_k)}{\omega_1(E_1)})^{\alpha/n}+\left(\frac{\omega_1(B_k)}{\omega_1(E_2)}\right)^{\alpha/n}\Big\}g(t)^{-(\beta+n)/q}\frac{\psi(t)}{t}\,dt$$

$$\le C_0^{\alpha/n}2^{(1/q)-2}(1+q)^{1/q}(1+g(2)^{|\alpha|\delta})\|f\|_K\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt. \quad (3.21)$$

Case 3. $0<q\le 1\le p<\infty$. In this case, by (3.19), (3.1), (3.5), (3.3), (3.8) and (3.9), we obtain

$$\|Tf\|_K\le\frac{(1+q)^{1/q}b^{1-(1/p)}}{2^{(1/p)+1-(1/q)}}\Big\{\sum_{k\in\mathbb{Z}}\omega_1(B_k)^{\alpha p/n}$$

$$\times\int_0^b(\|f\varphi_{A_1}\|_{q,\omega_2}^p+\|f\varphi_{A_2}\|_{q,\omega_2}^p)g(t)^{-(\beta+n)p/q}(\frac{\psi(t)}{t})^p\,dt\Big\}^{1/p}$$

$$\le C_0^{\alpha/n}2^{(1/q)-(2/p)-1}(1+q)^{1/q}(1+(1/p))(1+g(2)^{|\alpha|\delta})\|f\|_K$$

$$\times\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt. \quad (3.22)$$

Hence, by (3.20)–(3.22), we get

$$\|T\|\le C(p,q,\alpha)\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt,$$

where $C(p,q,\alpha)$ is defined by (2.5). By the same technique used in Theorem 2.1 one can show that the opposite inequality:

$$\|T\|\ge\int_0^\infty g(t)^{-\alpha\delta-(\beta+n)/q}\frac{\psi(t)}{t}\,dt.$$

This finishes the proof of Theorem 2.2. ∎

Proof of Theorem 2.3. By Minkowski inequality for integrals and setting $u = g(t)x$, we get

$$\begin{aligned}\|Tf\|_{p,\omega} &\le \left\{\int_{\mathbb{R}^n}\Big(\int_0^\infty |f(g(t)x)|\frac{\psi(t)}{t}\,dt\Big)^p\omega(x)\,dx\right\}^{1/p}\\ &\le \int_0^\infty\left\{\int_{\mathbb{R}^n}|f(g(t)x)|^p\omega(x)\,dx\right\}^{1/p}\frac{\psi(t)}{t}\,dt\\ &= \int_0^\infty\left\{\int_{\mathbb{R}^n}|f(u)|^p\omega(u)\,du\right\}^{1/p} g(t)^{-(\beta+n)/p}\frac{\psi(t)}{t}\,dt\\ &= \|f\|_{p,\omega}\int_0^\infty g(t)^{-(\beta+n)/p}\frac{\psi(t)}{t}\,dt.\end{aligned}$$

It follows that

$$\|T\| = \sup_{f\neq 0}\frac{\|Tf\|_{p,\omega}}{\|f\|_{p,\omega}} \le \int_0^\infty t^{-(\beta+n)/p}\frac{\psi(t)}{t}\,dt.$$

To prove the opposite inequality, putting $\varepsilon \in (0,1)$, we set $\omega(x) = |x|^\beta$ and

$$f_\varepsilon(x) = \begin{cases} 0, & |x| \le 1,\\ |x|^{-(\beta+n)/p-\varepsilon}, & |x| > 1,\end{cases}$$

thus

$$\|f_\varepsilon\|_{p,\omega}^p = \int_{|x|>1}|x|^{-(\beta+n+\varepsilon p)}\omega(x)\,dx = \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_1^\infty r^{-n-p\varepsilon}r^{n-1}dr = \frac{2\pi^{n/2}}{p\varepsilon\Gamma(n/2)}.$$

If g is strictly increasing, putting $|x| < \frac{1}{\varepsilon}$, $g(t)|x| > 1$ implies that $t > g^{-1}(\frac{1}{|x|}) > g^{-1}(\varepsilon)$. It follows that

$$\begin{aligned}\|Tf_\varepsilon\|_{p,\omega} &= \left\{\int_{|x|>1}\Big(\int_{g^{-1}(\frac{1}{|x|})}^\infty (g(t)|x|)^{-(\beta+n)/p-\varepsilon}\frac{\psi(t)}{t}\,dt\Big)^p\omega(x)\,dx\right\}^{1/p}\\ &\ge \left(\int_{g^{-1}(\varepsilon)}^\infty g(t)^{-(\beta+n)/p-\varepsilon}\frac{\psi(t)}{t}\,dt\right)\Big\{\frac{2\pi^{n/2}}{p\varepsilon\Gamma(n/2)}\Big\}^{1/p}.\end{aligned}$$

This implies

$$\|T\| \ge \frac{\|Tf_\varepsilon\|_{p,\omega}}{\|f_\varepsilon\|_{p,\omega}} \ge \int_{g^{-1}(\varepsilon)}^\infty g(t)^{-(\beta+n)/p-\varepsilon}\frac{\psi(t)}{t}\,dt. \tag{3.23}$$

Taking limits as $\varepsilon \to 0$ in (3.23), we get

$$\|T\| \ge \int_0^\infty g(t)^{-(\beta+n)/p}\frac{\psi(t)}{t}\,dt. \tag{3.24}$$

Then by (3.23) and (3.24), we have

$$\|T\| = \int_0^\infty g(t)^{-(\beta+n)/p}\frac{\psi(t)}{t}\,dt.$$

The proof for the decreasing case is similar. The theorem is proved. ■

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