TRACE FORMULAE ASSOCIATED WITH THE POLAR DECOMPOSITION
OF OPERATORS

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Abstract
Let \( T = X + iY \) be the Cartesian decomposition of an invertible operator \( T \) on a Hilbert space with trace class self-commutator \([T^*, T]\). Carey–Pincus introduced the principal function \( g \) and proved a trace formula associated with the Cartesian decomposition \( T = X + iY \). Applying the ordered \( C^\infty \)-functional calculus for \((X, Y)\) to their trace formula, we define the principal function \( g^P \) and prove a trace formula associated with the polar decomposition \( T = U|T| \). Using this formula, we show that \( g(x, y) = g^P(e^{i\theta}, r) \) almost everywhere \( x + iy = re^{i\theta} \) on \( \mathbb{C} \).

1. Introduction
Let \( B(\mathcal{H}) \) be the set of all bounded linear operators on a complex separable Hilbert space \( \mathcal{H} \), and let \( C_1 \) be the set of trace-class operators of \( B(\mathcal{H}) \). In [4], Carey–Pincus defined the principal function \( g \) and proved a trace formula associated with the Cartesian decomposition \( T = X + iY \) with \([T^*, T] \in C_1 \) (see also [12]). It is known that the principal functions are useful for the operator theory; for example, relating the size of the principal function to the existence of cyclic vectors, Berger [3] proved that, for a hyponormal operator \( T \), the operator \( T^n \) has a non-trivial invariant subspace for sufficiently high \( n \) (see other examples, [6; 9; 13; 14; 15; 16]). We also have two different trace formulae and the principal functions \( g \) and \( g^P \) associated with the decomposition \( T = X + iY \) and the polar decomposition \( T = U|T| \), respectively [4; 15; 16]. The relation between \( g \) and \( g^P \) is that if there exists a trace formula for the polar decomposition, then there exists \( g \) by a transformation of variables, and \( g \) essentially coincides with \( g^P \). An operator \( T \) is called \( p \)-hyponormal if \((T^*T)^p \geq (TT^*)^p \) [1]. If \( p = 1 \) and \( \frac{1}{2} \), then \( T \) is called hyponormal and semi-hyponormal, respectively. The principal function \( g \) has been studied well.

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For example, if $T$ is hyponormal, then $g \geq 0$ (see, for example [13; 16]). If $T$ is semi-hyponormal, then $g^P \geq 0$. Applying this property for $g^P$, we have that $g \geq 0$.

The existences of the trace formulae and $g$ and $g^P$ [4] have been shown separately (see also [15; 16]). In this paper, by the ordered $C^\infty$-functional calculus, we give a trace formula of $[T]$ and $U$ for an invertible operator $T = U[T]$ such that $[T^*, T] \in \mathcal{C}_1$. Using this result, we show a trace formula of a non-invertible semi-hyponormal operator $T = U[T]$ with unitary $U$ such that $[T, U] \in \mathcal{C}_1$. Finally, we show a relation between two principal functions $g$ and $g^P$ for such an operator $T$. We remark that for an operator $T = U[T]$, it is easy to see that if $[T, U] \in \mathcal{C}_1$, then $[T^*, T] \in \mathcal{C}_1$.

Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space of rapidly decreasing functions at infinity. For $T = X + iY$, let $\mathcal{E}$ and $\mathcal{F}$ be the spectral measures of self-adjoint operators $X$ and $Y$, respectively. We define $\tau$ on $\mathcal{S}(\mathbb{R}^2)$ by

$$
\tau(\phi) = \int \int \phi(x, y) d\mathcal{E}(x) d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbb{R}^2)).
$$

By a standard argument, we have

$$
\int \int e^{itX} e^{isY} \hat{\phi}(t, s) dtds = \int \int \hat{\phi}(x, y) d\mathcal{E}(x) d\mathcal{F}(y),
$$

where

$$
\hat{\phi}(t, s) = \frac{1}{2\pi} \int \int e^{-i(tx + sy)} \phi(x, y) dxdy
$$

is the Fourier transform of the function $\phi$ (see, for example, [13, p. 237]).

Put $\nu(E) = \int \int \hat{\phi}(t, s) dtds$ for a measurable set $E \subset \mathbb{R}^2$. Since $\hat{\phi}(t, s) \in \mathcal{S}(\mathbb{R}^2)$, we have

$$
\int \int (1 + |t|)(1 + |s|) |\hat{\phi}(t, s)| dtds < \infty.
$$

Following Carey–Pincus [4], put $G(x, y) = \int \int e^{itx + isy} d\nu(t, s)$ and define

$$
G(X, Y) = \int \int G(x, y) d\mathcal{E}(x) d\mathcal{F}(y).
$$

Then

$$
\tau(\phi) = \int \int e^{itX} e^{isY} \nu(t, s) dtds = G(X, Y).
$$

Note here that we have $\tau(\psi) = \tau(\phi)$ for any smooth function $\psi(x, y)$ that coincides with $\phi(x, y)$ on $\text{supp}(\tau)$.

The map $\tau : \mathcal{S}(\mathbb{R}^2) \to B(\mathcal{H})$ has the following properties [13, chapter X, §2];

1. $\tau$ is linear, continuous and $\text{supp}(\tau) \subseteq \sigma(X) \times \sigma(Y)$,
2. $\tau(1) = I$, $\tau(p + q) = p(X) + q(Y)$ for polynomials $p$ and $q$ of one variable of $x$ and $y$, respectively,
3. $\tau(\phi) \tau(\psi) = \tau(\phi \psi) \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbb{R}^2)$,
4. $\tau(\phi)^* = \tau(\phi) \in \mathcal{C}_1$. 

By (3) we have an important property \([\tau(\phi), \tau(\psi)] \in C_1\) for \(\phi, \psi \in S(\mathbb{R}^2)\).

Let \(A\) be the linear space of all Laurent polynomials \(P(r, z)\) with polynomial coefficients such that \(P(r, z) = \sum_{k=-N}^{N} p_k(r)z^k\), where \(N\) is a non-negative integer and each \(p_k(r)\) is a polynomial. For the polar decomposition \(T = U|T|\) of \(T\), let 
\[
\mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|)U^k.
\]

For differentiable functions \(P, Q\) of two variables \((x, y)\), let 
\[
J(P, Q)(x, y) = \frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x}.
\]

For a trace-class operator \(T \in C_1\), we denote the trace of \(T\) by \(\text{Tr}(T)\).

In this paper, we prove the following trace formula of an invertible operator 
\[
T = X + iY = U|T|\text{ with } |[T], U| \in C_1 \text{ by the above Cartesian functional calculus of } \tau \text{ with } X \text{ and } Y.
\]

Let \(\mathcal{P}, \mathcal{Q} \in \mathcal{A}\),
\[
\text{Tr}([\mathcal{P}([T], U), \mathcal{Q}([T], U)]) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g^P(e^{i\theta}, r) dr d\theta.
\]

The function \(g^P\) in the above formula is called the principal function associated with the polar decomposition of \(T\). As a corollary of this result, we show that the same formula holds for a non-invertible semi-hyponormal operator \(T = U|T|\) with unitary \(U\) and \([|T], U| \in C_1\). For an operator \(T\), let \(\sigma(T)\) be the spectrum of \(T\). The following theorem [4, theorem 5.1] is a basis of this paper (see [12] also):

**Theorem 1** (Carey–Pincus). Let \(T = X + iY = U|T|\) be an operator with \([|T|, U] \in C_1\). Let \(\mathcal{E}, \mathcal{F}\) be the spectral measures of \(X\) and \(Y\), respectively, and \(\tau\) be given by (\(*\)). Then there exists a summable function \(g\) such that, for \(\phi, \psi \in S(\mathbb{R}^2)\),
\[
\text{Tr}([\tau(\phi), \tau(\psi)]) = \frac{1}{2\pi i} \int \int J(\phi, \psi)(x, y)g(x, y) dx dy.
\]

Moreover, if \(T\) is hyponormal, then \(g \geq 0\) and \(g(x, y) = 0\) for \(x + iy \notin \sigma(T)\).

The function \(g\) in Theorem 1 is called the principal function associated with the Cartesian decomposition of \(T\).

2. **Function calculus and trace**

Let \(|A|_1 = \text{Tr}(|A|)\) for \(A \in C_1\), that is, \(|A|_1\) is the trace norm of \(A\). Let \(A \in C_1\) and \(B\) be an operator. Then it holds that
\[
|\text{Tr}(A)| \leq |A|_1, \text{Tr}(AB) = \text{Tr}(BA), |AB|_1 \leq |A|_1|B|_1 \text{ and } |BA|_1 \leq |B|_1||A|_1.
\]

We use an elementary property that if operators \(A, B\) and \(C\) satisfy \([A, C], [B, C] \in C_1\) and \(A - B \in C_1\), then \([AB, C], [BA, C] \in C_1\) and
\[
\text{Tr}([AB, C]) = \text{Tr}([BA, C]).
\]

Our standard reference on trace is [11].
We begin with two lemmas that are key tools in this paper.

**Lemma 2.** Let \( A \) be a positive invertible operator and operators \( D, E, F \) satisfying \([A, D], [E, D], [F, D] \in C_1\). Then for any real number \( \alpha \), we have

\[ [EA^\alpha F, D] \in C_1. \]

**Proof.** We use the following expansion known as the binomial series: For \(|z| < 1\), it holds

\[ (1 + z)^\alpha = \sum_{m=0}^{\infty} \left( \frac{\alpha}{m} \right) z^m, \]

where \( \left( \frac{\alpha}{m} \right) = \frac{\alpha(\alpha - 1) \cdots (\alpha - m + 1)}{m!} \). Considering \( ||\beta A|| < 1 \) with some positive number \( \beta \), we may assume that \( ||A|| < 1 \). Since \( A \) is an invertible positive operator and \( ||A|| < 1 \), we have \( ||A - I|| < 1 \) and

\[ A^\alpha = (I + (A - I))^\alpha = \lim_{n \to \infty} \sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) (A - I)^m. \]  

Let \( A_n = [\sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) (A - I)^m, D] \) for \( n = 1, 2, 3, \ldots \). Then \( \lim_{n \to \infty} A_n = [A^\alpha, D] \) with respect to the operator norm. By [12, p. 158 (3.3)], for a positive integer \( m \), it holds that

\[ ||[(A - I)^m, D]||_1 \leq m||A - I||^{m-1}||[A, D]||_1, \]

so that

\[ ||A_n||_1 \leq \left( \sum_{m=1}^{n} \left| \left( \frac{\alpha}{m} \right) m||A - I||^{m-1} \right| \right) ||[A, D]||_1. \]

Since \( ||A - I|| < 1 \), (1) converges absolutely. Hence \( \{A_n\} \) is a Cauchy sequence with respect to the norm \( \cdot \cdot || \cdot \cdot \). Let \( B \) denote the limit of the sequence \( \{A_n\} \) in \( C_1 \). For any unit vector \( \xi \in \mathcal{H} \), we define an operator \( C \) on \( \mathcal{H} \) by \( C\eta = (\eta, \xi)\xi \) for \( \eta \in \mathcal{H} \). Let \( \{e_j\} \) be a complete orthonormal basis of \( \mathcal{H} \) such that \( e_1 = \xi \). Since

\[ \text{Tr}(SC) = \sum_{j=1}^{\infty} (SCe_j, e_j) = (S\xi, \xi), \]

then

\[ (B\xi, \xi) = \text{Tr}(BC) = \lim_{n \to \infty} \text{Tr}(A_nC) = \lim_{n \to \infty} (A_n\xi, \xi) = ([A^\alpha, D]\xi, \xi). \]

Since \( \xi \) is an arbitrary vector, it follows that

\[ [A^\alpha, D] = B \in C_1. \]
We have
\[ [EA_n F, D] = [E, D]A_n F + E [A_n, D] F + E A_n F, \]
Since \( \lim_{n \to \infty} A_n = A^\alpha \) with respect to the operator norm,
\[
\lim_{n \to \infty} [E, D] A_n F = [E, D] A^\alpha F, \quad \lim_{n \to \infty} E [A_n, D] F = E [A^\alpha, D] F,
\]
and \( \lim_{n \to \infty} E A_n F = E A^\alpha F, \)
so that
\[
\lim_{n \to \infty} [E A_n F, D] = [E A^\alpha F, D]
\]
with respect to \( C_1 \).

The proof of Lemma 2 is based on an idea of [8, theorem 2].

Let \( T = X + i Y \) be the Cartesian decomposition of \( T \). For the spectral measures \( E \) and \( F \) of self-adjoint operators \( X \) and \( Y \), respectively, we recall
\[
\tau(\phi) = \int \int \phi(x, y) dE(x) dF(y) \quad (\phi \in S(\mathbb{R}^2)).
\]

**Lemma 3.** [4, p. 158] Let \( T = X + i Y \) be an invertible operator such that \( [T^*, T] \in C_1 \). Let \( \psi \in S(\mathbb{R}^2) \), \( D = \tau(\psi) \) and operators \( E, F \) satisfy \( [E, D], [F, D] \in C_1 \). Then,
\[
\text{Tr}([E \tau(\phi) F, D]) = \text{Tr}([E|T|^{2\alpha} F, D]).
\]

**Proof.** We may assume that \( ||T|| < d < \frac{1}{2} \). Then \( ||X^2 + Y^2|| = |||T||^2 - \frac{1}{2}[T^*, T]|| < 1 \). Hence, \( X^2 + Y^2 < I \). Since \( T \) is invertible, we choose a positive number \( c \) such that \( 0 < c \leq X^2 + Y^2 \). Hence, we may assume that \( f \) of \( \tau(f) \) is a function on \( \{(x, y) | c \leq x^2 + y^2 < 1\} \). Also we choose \( \varphi \in C_0^\infty(\mathbb{R}^2) \) and \( d_1 \) such that \( d < d_1 < 1 \), \( \varphi(x, y) = 1 \) on \( \{(x, y) | c \leq x^2 + y^2 \leq d \} \) and \( \text{supp}(\varphi) \subset \{(x, y) | x^2 + y^2 < d_1 \} \). Then
\[
\tau(\phi \varphi) = \sum_{m=0}^{\infty} \left( \frac{\alpha}{m} \right) \int \int ((x^2 + y^2)^m - 1) dE(x) dF(y) = \sum_{m=0}^{\infty} \left( \frac{\alpha}{m} \right) \tau(((x^2 + y^2) - 1)^m)
\]
with respect to the operator norm. Since
\[
\tau((x^2 + y^2) - 1) = X^2 + Y^2 - I \quad \text{and} \quad |T|^2 = X^2 + Y^2 + \frac{1}{2}[T^*, T],
\]
we get
\[
\tau((x^2 + y^2) - 1) - (|T|^2 - I) \in C_1.
\]
Since by property (3) of \( \tau \) and the above it holds that
\[
\tau(((x^2 + y^2) - 1)^m) - \tau((x^2 + y^2) - 1)^m \in C_1,
\]

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we have for $m > 0$

$$
\tau((x^2 + y^2) - 1)^m - (|T|^2 - I)^m
= \tau((x^2 + y^2) - 1)^m - \tau((x^2 + y^2) - 1)^m + \tau((x^2 + y^2) - 1)^m - (|T|^2 - I)^m \in \mathcal{C}_1.
$$

Hence, it holds that

$$
\text{Tr}([\tau((x^2 + y^2) - 1)^m], D) = \text{Tr}([|T|^2 - I)^m], D])
$$

and

$$
\left[ \sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) \tau((x^2 + y^2) - 1)^m, D \right] \in \mathcal{C}_1.
$$

Therefore, we see

$$
\text{Tr}\left( \sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) \tau((x^2 + y^2) - 1)^m, D \right) = \text{Tr}\left( \sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) ((X^2 + Y^2) - I)^m, D \right).
$$

Let

$$
\varphi_\infty(r) = r^n = \sum_{m=0}^{\infty} \left( \frac{\alpha}{m} \right) (r - 1)^m \quad (0 < |r| < 1),
$$

$$
\varphi_n(r) = \sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) (r - 1)^m,
$$

$$
\phi_n(x, y) = \varphi_n(x^2 + y^2) = \sum_{m=0}^{n} \left( \frac{\alpha}{m} \right) ((x^2 + y^2) - 1)^m.
$$

Put $\tilde{\phi}_n = \phi_n \varphi$ and $\tilde{\phi} = \phi \varphi$. Then for some $f_k \in C^\infty$ with $\text{supp}(f_k) \subset \{(x, y) \mid x^2 + y^2 < 1\}$ ($k = 0, \cdots, m$), we have

$$
\frac{\partial^n}{\partial x^j \partial y^{m-j}}(\tilde{\phi}_n - \tilde{\phi})(x, y)
= (\varphi_n^{(m)}(r^2) - \varphi_\infty^{(m)}(r^2))f_m(x, y) + (\varphi_n^{(m-1)}(r^2) - \varphi_\infty^{(m-1)}(r^2))f_{m-1}(x, y)
+ \cdots + (\varphi_n(r^2) - \varphi_\infty(r^2))f_0(x, y),
$$

where $r^2 = x^2 + y^2$. We remark that each $f_k$ depends on $\frac{\partial^n}{\partial x^j \partial y^{m-j}}$ and is independent of $\tilde{\phi}_n$. Hence we obtain $\tilde{\phi}_n \to \tilde{\phi}$ in $S(\mathbb{R}^2)$. By [13, chapter X, corollary 2.3], it holds that

$$
[\tau(\tilde{\phi}_n), D] \to [\tau(\tilde{\phi}), D] \quad \text{in } \mathcal{C}_1.
$$

Since
Then there exists a summable function \( w \) such that, for \( x, y \) in the support of \( g \),
\[
|\exp(i\theta)x + \exp(i\phi)y| \leq r < \frac{\pi}{2},
\]
and \( g^P(e^{i\theta}, x, y) = g(x, y) \) almost everywhere \( x + iy = re^{i\theta} \) on \( \mathbb{C} \).

**Proof.** Since \( T \) is invertible, there exists a number \( c > 0 \) such that \( c \leq X^2 + Y^2 \). Therefore, \( \frac{\pi}{2} \leq X^2 \) or \( \frac{\pi}{2} \leq Y^2 \), so that, if \( \zeta \in \mathcal{S}(\mathbb{R}^2) \) satisfies \( \zeta(x, y) = 0 \) for \( \frac{\pi}{2} > |x|^2 \) or \( \frac{\pi}{2} > |y|^2 \), then \( \tau(\zeta) = 0 \). With \( g(x, y) \) in Theorem 1, we know that \( g(x, y) = 0 \) for \( x + iy \) with \( x^2 + y^2 < \frac{\pi}{2} \). Let \( w(x, y) \) and \( h(x, y) \) be in \( \mathcal{S}(\mathbb{R}^2) \) such that
\[
w(x, y) = (x + iy)(x^2 + y^2)^{-\frac{1}{2}} \quad \text{and} \quad h(x, y) = (x^2 + y^2)^{\frac{1}{2}} \text{ on the support of } g.
\]
For \( \psi, \phi, \phi_r \in \mathcal{S}(\mathbb{R}^2) \), let \( D = \tau(\psi), E = \tau(\phi) \) and \( F = \tau(\phi_r) \). By property (3) of \( \tau \) and Lemma 3, for a positive integer \( k \) we obtain
\[
\begin{align*}
\text{Tr}([EU^k F, D]) &= \text{Tr}([E(T[T]^{-1})^k F, D]) = \text{Tr}([E(T\tau(h^{-1}))^k F, D]) \\
&= \text{Tr}([E(\tau(x + iy)\tau(h^{-1}))^k F, D]) \\
&= \text{Tr}([E\tau((x + iy)(x^2 + y^2)^{-\frac{1}{2}})^k F, D]) = \text{Tr}([E\tau(w^k) F, D])
\end{align*}
\]
and

\[ \text{Tr}([E U^{-k} F, D]) = \text{Tr}([E |T|^{-1} k F, D]) = \text{Tr}([E(\tau(h)\tau((x + iy))^{-k} F, D]) \]

= \text{Tr}([E \tau((x - iy)(x^2 + y^2)^{-\frac{1}{2}}) F, D]) = \text{Tr}([E \tau(w^{-k}) F, D]). \]

Then for integers \( m, s \) and non-negative integers \( n, t \), we have

\[ \text{Tr}([U^m|T|^n, U^s|T|^t]) = \text{Tr}([\tau(w^m)\tau(h^n), \tau(w^s)\tau(h^t)]) \]

= \text{Tr}([\tau(w^m h^n), \tau(w^s h^t)]). \]

By Theorem 1, there exists a summable function \( g \) such that

\[ \text{Tr}([\tau(w^m h^n), \tau(w^s h^t)]) = \frac{1}{2\pi i} \int \int J(w^m h^n, w^s h^t)(x, y)g(x, y)dxdy. \]

By the transformation \( x = r \cos \theta \) and \( y = r \sin \theta \),

\[ \frac{1}{2\pi i} \int \int J(w^m h^n, w^s h^t)(x, y)g(x, y)dxdy \]

= \frac{1}{2\pi i} \int \int J(w^m h^n, w^s h^t)(r \cos \theta, r \sin \theta)g(r \cos \theta, r \sin \theta)rdrd\theta. \]

Hence we have, for Laurent polynomials \( \mathcal{P} \) and \( \mathcal{Q} \),

\[ \text{Tr}([\mathcal{P}([T], U), \mathcal{Q}([T], U)]) \]

= \frac{1}{2\pi i} \int \int J(\mathcal{P}(h, w), \mathcal{Q}(h, w))(r \cos \theta, r \sin \theta)g(r \cos \theta, r \sin \theta)rdrd\theta. \]

For \( x + iy \in \sigma(T) \), let \( x = r \cos \theta \) and \( y = r \sin \theta \). Since \( w(x, y) = (x + iy)(x^2 + y^2)^{-\frac{1}{2}} \)
and \( h(x, y) = (x^2 + y^2)^{\frac{1}{2}} \), then \( w(r \cos \theta, r \sin \theta) = e^{i\theta} \), \( h(r \cos \theta, r \sin \theta) = r \),

\[ \frac{\partial(h, w)}{\partial(r, \theta)} = \frac{\partial(r, e^{i\theta})}{\partial(r, \theta)} = ie^{i\theta} \text{ and } \frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial(r \cos \theta, r \sin \theta)}{\partial(r, \theta)} = r. \]

Also, it holds that

\[ \frac{\partial(\mathcal{P}(r, e^{i\theta}), \mathcal{Q}(r, e^{i\theta}))}{\partial(r, \theta)} = \frac{\partial(\mathcal{P}(h, w), \mathcal{Q}(h, w))}{\partial(h, w)}(r, e^{i\theta}) \cdot \frac{\partial(h, w)}{\partial(r, \theta)} \]

= \( ie^{i\theta} \cdot \frac{\partial(\mathcal{P}(h, w), \mathcal{Q}(h, w))}{\partial(h, w)}(r, e^{i\theta}) = ie^{i\theta} \cdot J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) \)
and

\[
\begin{align*}
\frac{\partial}{\partial(r, \theta)} \left( \frac{\partial}{\partial(r, \theta)} \left( P(h(r \cos \theta, r \sin \theta), w(r \cos \theta, r \sin \theta)), Q(h(r \cos \theta, r \sin \theta), w(r \cos \theta, r \sin \theta)) \right) \right) \\
= \frac{\partial}{\partial(x, y)} \left( P(h(x, y), w(x, y)), Q(h(x, y), w(x, y)) \right) \frac{\partial(x, y)}{\partial(r, \theta)} \\
= r \cdot \frac{\partial}{\partial(x, y)} \left( P(h(x, y), w(x, y)), Q(h(x, y), w(x, y)) \right) \\
= r \cdot J(P(h, w), Q(h, w)) \left( r \cos \theta, r \sin \theta \right).
\end{align*}
\]

Hence we have

\[
\text{Tr}([P(|T|, U), Q(|T|, U)]) = \frac{1}{2\pi i} \int \int \frac{i}{\partial(h, w)} \frac{\partial(h, w)}{\partial(r, \theta)} \left( r, e^{i\theta} \right) e^{i\theta} g(r \cos \theta, r \sin \theta) drd\theta.
\]

Put \( g^P(e^{i\theta}, r) = g(r \cos \theta, r \sin \theta). \) Then

\[
\text{Tr}([P(|T|, U), Q(|T|, U)]) = \frac{1}{2\pi} \int \int J(P, Q) \left( r, e^{i\theta} \right) e^{i\theta} g^P(e^{i\theta}, r) drd\theta.
\]

The function \( g^P \) in Theorem 4 is called the principal function associated with the polar decomposition \( T = U|T| \) of \( T \). An invertible operator \( T \) is said to be log-hyponormal if \( \log T^* T \geq \log TT^* \) [10]. Lemma 2 and Theorem 4 give another proof of a trace formula of log-hyponormal operators in [5].

For the proof of the next result, we need the following two lemmas. For an operator \( T \), let \( \sigma_{ap}(T) \) and \( \sigma_p(T) \) be the approximate point spectrum and the point spectrum of \( T \), respectively.

**Lemma 5.** Let \( T = U|T| \) be an invertible semi-hyponormal operator with \([|T|, U] \in C_1\). Then the principal function \( g^P \) associated with the polar decomposition \( T = U|T| \) of \( T \) satisfies \( g^P(e^{i\theta}, r) = 0 \) for \( re^{i\theta} \notin \sigma(T) \).

**Proof.** Put \( S = U|T|^{\frac{1}{2}}. \) Then \( S \) is hyponormal and \( [S^*, S] = [|T|, U]U^* \in C_1 \). Let \( g^S \) be the principal function associated with the polar decomposition of \( S = U|T|^{\frac{1}{2}}. \) Then by Theorems 1 and 4 it holds that \( g^S(e^{i\theta}, r) = 0 \) for \( re^{i\theta} \notin \sigma(S) \). By [16, lemma VI 3.6] and \( S = U|T|^{\frac{1}{2}}, \) we also have

\[
\sigma(T) = \{ r^2 e^{i\theta} : re^{i\theta} \in \sigma(S) \} \quad \text{and} \quad g^P(e^{i\theta}, r) = g^S(e^{i\theta}, r^2).
\]

Hence, \( g^P \) has the desired property. \( \blacksquare \)
Lemma 6. Let $T = U[T]$ be an operator with unitary $U$ and put $S = U([T] + I)$. Let $z \in \partial \sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.

Proof. Since $U$ and $[T] + I$ are invertible, so is $S$. Since $z \in \partial \sigma(S)$ and $\partial \sigma(S) \subseteq \sigma_{ap}(S)$, we have $z \in \sigma_{ap}(S)$. Hence, let $\pi : B(H) \rightarrow B(K)$ denote the Berberian representation [2]. Since $\sigma_{ap}(S) = \sigma_p(\pi(S))$, there exists $x \in K$ such that $\pi(S)x = \pi(U)\pi([T] + I)x$.

Since $\pi(U)$ is unitary, there exists $y \in K$ such that $\pi(U)^*y = x$. Hence

$$||y||^2 = (y, y) \leq (\pi(U)\pi([T] + I)\pi(U)^*y, y) = (zx, y) \leq |z||x|| ||y|| = |z||y||^2,$$

so that $1 \leq |z|$. Let $z_0 \in \sigma(S)$ such that $|z_0| = \inf\{|\mu| : \mu \in \sigma(S)\}$. Since $S$ is invertible, we have $z_0 \in \partial \sigma(S)$.

By the above argument, we obtain $1 \leq |z_0|$. \hfill \blacksquare

Now we give another proof of [7, theorem 9].

Theorem 7. Let $T = U[T]$ be a semi-hyponormal operator with unitary $U$ and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function $g^P$ such that, for $P, Q \in A$,

$$\text{Tr}([P(|T|, U), Q(|T|, U)]) = \frac{1}{2\pi} \int \int J(P, Q)(r, e^{i\theta})g^P(e^{i\theta}, r)drd\theta.$$

Proof. Since by the assumption $[|T|, U] \in \mathcal{C}_1$ it holds that $[T^*, T] \in \mathcal{C}_1$, by Theorem 4 we may only prove the theorem when $T$ is not invertible. Put $[\tilde{T}] = [T] + I$ and $\tilde{T} = U|T|$. Then $\tilde{T}$ is semi-hyponormal. For Laurent polynomials $P$ and $Q$, put $\tilde{P}(r, z) = P(r - 1, z)$ and $\tilde{Q}(r, z) = Q(r - 1, z)$. Then

$$\text{Tr}([P(|T|, U), Q(|T|, U)]) = \text{Tr}([P([\tilde{T}] - I, U), Q([\tilde{T}] - I, U)]) = \text{Tr}([\tilde{P}([\tilde{T}], U), \tilde{Q}([\tilde{T}], U)])].$$

Since $\tilde{T}$ is invertible and $[|\tilde{T}|, U] = [|T|, U] \in \mathcal{C}_1$, by Theorem 4 there exists a summable function $\tilde{g}^P$ such that

$$\text{Tr}([\tilde{P}(|\tilde{T}|, U), \tilde{Q}(|\tilde{T}|, U)]) = \frac{1}{2\pi} \int \int J(\tilde{P}, \tilde{Q})(r, e^{i\theta})g^P(e^{i\theta}, r)drd\theta.$$

By Lemma 5, it holds that $g^P(e^{i\theta}, r) = 0$ for $re^{i\theta} \not\in \sigma(\tilde{T})$. We have

$$\frac{1}{2\pi} \int \int J(\tilde{P}, \tilde{Q})(r, e^{i\theta})g^P(e^{i\theta}, r)drd\theta = \frac{1}{2\pi} \int_{\sigma(\tilde{T})} J(\tilde{P}, \tilde{Q})(r, e^{i\theta})g^P(e^{i\theta}, r)drd\theta.$$
\[
\frac{1}{2\pi} \int_{\sigma(\tilde{T})} J(\mathcal{P}, Q)(r-1, e^{i\theta}) d\rho d\theta = \frac{1}{2\pi} \int_{A} J(\mathcal{P}, Q)(\rho, e^{i\theta}) g^{P}(e^{i\theta}, \rho + 1) d\rho d\theta \quad \text{(by the transformation } \rho = r-1),
\]
where \(A = \{(r-1)e^{i\theta} : re^{i\theta} \in \sigma(\tilde{T})\}\). We remark that, by Lemma 6, \(r-1 \geq 0\) for \(re^{i\theta} \in \sigma(\tilde{T})\). We define \(g^{P}\) by \(g^{P}(e^{i\theta}, r) = \tilde{g}^{P}(e^{i\theta}, r + 1)\). Then \(g^{P}\) is the desired function.

Finally, we show a relation between \(g\) and \(g^{P}\).

**Theorem 8.** Let \(T = X + iY = U|T|\) be a semi-hyponormal operator with unitary \(U\) and \([|T|, U] \in \mathcal{C}_{1}\). If \(g\) and \(g^{P}\) are the principal function associated with the Cartesian decomposition of \(T\) and the summable function in Theorem 7, respectively, then

\[
g(x, y) = g^{P}(e^{i\theta}, r)
\]

almost everywhere \(x + iy = re^{i\theta}\) on \(\mathbb{C}\).

**Proof.** Since \([|T|, U] \in \mathcal{C}_{1}\), by Lemma 2 we have \([|T|^{2}, U] \in \mathcal{C}_{1}\). Hence

\[
2i[X, Y] = T^{*}T - TT^{*} = |T|^{2} - U|T|^{2}U^{*} = [|T|^{2}, U]U^{*} \in \mathcal{C}_{1}.
\]

Let \(Q_{0}(x, y) = y\). For the polynomial \(Q_{0}(x, y) = y\) and an arbitrary polynomial \(\mathcal{P}(x, y)\), by Theorem 1 and [4, theorem 5.2] we have

\[
\text{Tr}([\mathcal{P}(X, Y), Q_{0}(X, Y)]) = \frac{1}{2\pi i} \int_{\sigma(\tilde{T})} J(\mathcal{P}, Q_{0}) g(x, y) dxdy
= \frac{1}{2\pi i} \int_{\sigma(\tilde{T})} \mathcal{P}_{x}(x, y) g(x, y) dxdy
= \frac{1}{2\pi i} \int_{M} \mathcal{P}_{x}(r \cos \theta, r \sin \theta) g(r \cos \theta, r \sin \theta) rdrd\theta,
\]

where \(M = \{(r, \theta) : re^{i\theta} \in \sigma(T), 0 \leq \theta < 2\pi\}\). Let

\[
\tilde{\mathcal{P}}(r, z) = \mathcal{P}\left(\frac{zr + rz^{-1}}{2}, \frac{zr - rz^{-1}}{2i}\right) \quad \text{and} \quad \tilde{Q}_{0}(r, z) = \frac{zr - rz^{-1}}{2i}.
\]

Then

\[
J(\tilde{\mathcal{P}}, \tilde{Q}_{0}) = \left(\mathcal{P}_{x} \cdot \frac{z + z^{-1}}{2} + \mathcal{P}_{y} \cdot \frac{z - z^{-1}}{2i}\right) \left(\frac{r}{2i}(1 + \frac{1}{z^{2}})\right)
- \frac{r}{2} \left(\mathcal{P}_{x} \cdot \left(1 - \frac{1}{z^{2}}\right) + \frac{1}{i} \mathcal{P}_{y} \cdot \left(1 + \frac{1}{z^{2}}\right)\right) \frac{z - z^{-1}}{2i}.
\]
Hence

\[ J(\tilde{P}, \tilde{Q}_0)(r, e^{i\theta}) = (P_x \cdot \cos \theta + P_y \cdot \sin \theta)(-ir \cos \theta) - r(iP_x \cdot \sin \theta - iP_y \cdot \cos \theta) \sin \theta \]

\[ = -irP_x. \]

Theorem 7 implies

\[
\text{Tr}\left( \left[ P \left( \frac{U|T| + |T|U^{-1}}{2}, \frac{U|T| - |T|U^{-1}}{2i} \right) \right] \right)
= \frac{1}{2\pi} \int \int_M J(\tilde{P}, \tilde{Q}_0)(r, e^{i\theta})g^P(e^{i\theta}, r)drd\theta
= \frac{1}{2\pi} \int \int_M -irP_x(r \cos \theta, r \sin \theta)g^P(e^{i\theta}, r)drd\theta
= \frac{1}{2\pi i} \int \int_M P_x(r \cos \theta, r \sin \theta)g^P(e^{i\theta}, r)rd\theta.
\]

Since

\[
\text{Tr}([P(X, Y), Q_0(X, Y)]) = \text{Tr}\left( \left[ P \left( \frac{U|T| + |T|U^{-1}}{2}, \frac{U|T| - |T|U^{-1}}{2i} \right) \right] \right),
\]

we have (2) = (3) and

\[
\int \int_M P_x(r \cos \theta, r \sin \theta)g(r \cos \theta, r \sin \theta)rd\theta
= \int \int_M P_x(r \cos \theta, r \sin \theta)g^P(e^{i\theta}, r)rd\theta.
\]

Since \( P \) is an arbitrary polynomial, we obtain the desired relation between \( g \) and \( g^P \). \( \blacksquare \)

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References


