SYSTEM-PERMUTABLE FISCHER SUBGROUPS ARE INJECTORS

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Abstract

In 1973 Dark provided the first example of a group and a Fitting set D such that the D-injectors are not normally embedded, and the first example of a group with Fischer D-subgroups that are not D-injectors, though they are injectors for another Fitting set, F. In their 1992 book Finite soluble groups, Doerk and Hawkes point out that in this second example the F-injectors are not even system-permutable, a weaker condition than normally embedded. Here we work with system-permutable Fischer F-subgroups. First, we show that a system-permutable Fischer F-subgroup that is also pronormal must be an F-injector. Then we prove that we can drop the requirement of pronormality and reach the same conclusion. Thus in Dark’s example the existence of Fischer D-subgroups that are not system-permutable is necessary for any Fischer D-subgroups not to be D-injectors.

Several authors have studied the question of when the Fischer F-subgroups and F-injectors of a finite solvable group are the same subgroups. Fischer [5] showed that if F is what is now called a Fischer set, then the F-subgroups of G are indeed F-injectors of G. A somewhat more general result of Anderson [1] is that when the Fischer F-subgroups of H are normally embedded in H for each subgroup H of G, the Fischer F-subgroups and F-injectors of G coincide. In [4], we came to the same conclusion if all the Fischer F subgroups of G are either subnormally embedded or locally pronormal in G. In [2], Dark provides the first example of a group and a Fitting set D such that the D-injectors are not normally embedded, and the first example of a group with Fischer D-subgroups that are not D-injectors, though they are injectors for another Fitting set, F. Doerk and Hawkes [3, 646] point out that in this latter example the F-injectors are not even system-permutable, a weaker condition than normally embedded. Doerk and Hawkes [3, VIII(4.9)] have also provided an example in which the Fischer F-subgroups are not injectors for any Fitting set. Here we work with system-permutable Fischer F-subgroups. First we show that a system-permutable Fischer F-subgroup that is also pronormal must be an F-injector. Then we prove that we can drop the requirement of pronormality and reach the same conclusion. Thus, in Dark’s example, the existence of Fischer D-subgroups that are not system-permutable is necessary for the Fischer D-subgroups not to be D-injectors. We begin with some key definitions. Definitions and notation will be as in [3], and all groups will be finite and solvable.

A subgroup A is pronormal in G if, for each g ∈ G, A is conjugate to Ag by an element of ⟨A, Ag⟩.

A subgroup A is system-permutable in G if there exists a Hall system Σ such that, for every subgroup B of Σ, AB is a subgroup of G.

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A subgroup $A$ is *normally embedded* in $G$ if, for each prime $r$ dividing $A$, a Sylow $r$-subgroup of $A$ is a Sylow $r$-subgroup of some normal subgroup of $G$.

If $F$ is a Fitting set of $G$, a subgroup $U$ of $G$ is a *Fischer $F$-subgroup* of $G$ if $U$ contains every $F$-subgroup of $G$ that it normalises.

If $F$ is a Fitting set of $G$ and $H$ is a subgroup of $G$, then $F_H$ will denote the Fitting set $\{X \leq H : X \in F\}$. The subscript will be suppressed in all cases unless its inclusion clarifies the argument.

**Lemma 1.** If $U$ is a proper subgroup of a finite group $G$, then there exists a maximal subgroup $M$ of $G$ such that $M = UK$, where $K$ is normal in $G$.

**Proof.** Let $1 = N_0 < N_1 < N_2 < \ldots < G$ be a $U$-composition series for $G$, and let $k$ be the smallest index such that $UN_k = G$. Let $M = UN_{k-1}$. Note that $N_{k-1}$ is $U$-invariant and normal in $N_k$, so $N_{k-1}$ is normal in $G = UN_k$. Now suppose $M \leq X \leq G$. Then $U \leq M \leq X$, so $X$ is $U$-invariant. Also, $N_{k-1} \leq X$, so $N_k = X \cap N_k \leq N_{k-1}$. Because $X \cap N_k$ is $U$-invariant and $N_k/N_{k-1}$ is $U$-irreducible, $X \cap N_k = N_k$ or $X \cap N_k = N_{k-1}$. Thus $X = X \cap G = X \cap UN_k = U(X \cap N_k) = G$ or $M$, and $M$ is maximal in $G$. With $K = N_{k-1}$, $M$ is the subgroup we seek. 

**Lemma 2.** Suppose that $G$ is a finite solvable group, $U \leq H \leq G$, and $U$ is system-permutable and pronormal in $G$. Then $U$ is system-permutable and pronormal in $H$.

**Proof.** $U$ is pronormal in $H$ by [3, I(6.3)(a)]. Now choose any Hall system of $U$. By [3, I(4.16)], it extends to a Hall system of $H$, which similarly extends to a Hall system $\Sigma$ of $G$. Then $\Sigma$ reduces into $H$ by definition, and, because $U$ is system-permutable and pronormal in $G$, $U$ permutes with $\Sigma$ by [3, I(6.7)]. Then, by [3, I(4.25)(c)], $U$ permutes with $\Sigma \cap H$ in $H$.

**Theorem 1.** Suppose that $G$ is a finite solvable group and $F$ is a Fitting set of $G$. If $V$ is a pronormal, system-permutable Fischer $F$-subgroup of $G$, then $V$ is an $F$-injector of $G$.

**Proof.** Suppose that $G$ is a counterexample to the theorem of minimal order, and $V$ is a pronormal, system-permutable Fischer $F$-subgroup that is not an $F$-injector of $G$. If $V \leq H < G$, then $V$ is a Fischer $F$-subgroup of $H$, and, by Lemma 2, $V$ is system-permutable and pronormal in $H$. Hence $V$ is an $F$-injector of $H$ by minimality of $G$. Also, if $N$ is normal in $G$, $VN/N$ is pronormal and system-permutable in $G/N$ by [3, I(4.25)(b), I(6.3)(c)]. These facts make it possible to follow exactly steps (1)–(5) of the proof of Anderson’s result related in [3, VIII(4.7)].

Thus we know the following: the radical, $G_F$, is trivial; $G = VSN$, where $N$ is minimal normal in $G$; $S$ is an $F$-injector of $SN$; $V$ is an $F$-injector of $VN$; $SN$ is normal in $G$; $VN$ does not contain $SN$; and $G/N$ is primitive with unique minimal normal subgroup $SN/N$ and core-free maximal subgroup $VN/N$.

Now let $\Sigma$ be the Hall system of $G$ with which $V$ permutes, let $p$ be the prime dividing $|N|$, and let $Y$ be the Hall $p'$-subgroup of $G$ in $\Sigma$. Note that $S \cap N$ is
an $F$-injector of $N$, and is normal in the abelian $N$, so $S \cap N = 1$ because, by [3, VIII(2.4)(d)], $N_F = N \cap G_F = 1$. Thus $S$ is isomorphic to $SN/N$ and is an elementary abelian $q$-group. If $q = p$, then $S$ is subnormal in the $p$-group $SN$, which is normal in $G$, and so, by [3, VIII(2.4)(c)], $S \leq G_F = 1$, a contradiction. Hence $q \neq p$. Because $SN$ is normal in $G$, $Y \cap SN$ is some Sylow $q$-subgroup of $SN$, which is of the form $S^x$, where $x \in N$.

Now because $V$ is $\Sigma$-permutable, $VY$ is a subgroup of $G$, and $VY \cap SN = VY \cap S^x N = S^x (VY \cap N)$. Clearly, $VY \cap N \leq O_p(VY) \leq V$. But $V \cap N = 1$ for the same reason that $S \cap N = 1$, so $VY \cap N = 1$. Hence $VY \cap SN = S^x$. But this means that the $F$-subgroup $S^x$ is normalised by $VY$ and therefore by $V$, so $S^x \leq V$ because $V$ is a Fischer $F$-subgroup. Hence $SN = S^x N \leq VN$, contradicting our assumption and establishing the theorem.

**Theorem 2.** Suppose that $G$ is a finite solvable group and $F$ is a Fitting set of $G$. If $U$ is a system-permutable Fischer $F$-subgroup of $G$, then $U$ is an $F$-injector of $G$.

**Proof.** Suppose that $G$ is a minimal counterexample to the theorem, and $U$ is a system-permutable Fischer $F$-subgroup of $G$ that is not an $F$-injector of $G$. Then, by Lemma 1, there exists a maximal subgroup $M$ of $G$ such that $M = UK$, where $K$ is normal in $G$. Now we know that $U$ is $\Sigma$-permutable for some Hall system $\Sigma$ of $G$, so $\Sigma$ reduces into $U$ by [3, I(4.25)(a)], and $\Sigma$ reduces into $M = UK$ by [3, I(4.17)]. Thus $U$ is $\Sigma \cap M$-permutable in $M$ by [3, I(4.25)(c)], and $U$ is a Fischer $F$-subgroup of $M$. By minimality of $G$, $U \in \text{Inj}_F(M)$. If $M$ is not normal in $G$, $N_G(M) = M$, so $U$ is pronormal in $N_G(M)$ because it is an $F$-injector of $M$ by [3, VIII(2.14)(a)]. But $M = UK$ is pronormal in $G$ because $M$ is maximal in $G$, so $U$ being pronormal in $M = N_G(UK)$ implies that $U$ is pronormal in $G$ by [3, I(6.4)]. But then $U \in \text{Inj}_F(G)$ by Theorem 1, a contradiction. Hence $M$ is normal in $G$.

Let $g \in G$ and consider $\langle U, U^x \rangle$. Because $U \in \text{Inj}_F(M)$, $U^x \in \text{Inj}_F(M^x) = \text{Inj}_F(M)$ because $M$ is normal in $G$. But this means that $U^x = U^m$ for some $m \in M$ by [3, VIII(2.9)]. But $U$ is pronormal in $M$, so $U^m = U^x$ for some $x \in \langle U, U^m \rangle = \langle U, U^x \rangle$. Hence $U^x = U^x$ for $x \in \langle U, U^x \rangle$, and $U$ is pronormal in $G$. Thus, by Theorem 1, $U \in \text{Inj}_F(G)$, the final contradiction.

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**References**
