CONFORMAL TRANSFORMATIONS OF DISSIPATIVE OPERATORS

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Abstract

In the first part of this paper we give an elementary proof of the Kreiss Theorem [6] and we slightly improve the Leveque–Trefethen–Spijker Theorem [8; 12] for the case of matrices. In the second part we use the previous results to prove that if $T$ is a quasi-dissipative matrix, and if $\varphi$ is a conformal transformation of the negative half-plane onto the unit disk, then $\varphi(T)$ has bounded powers.

In applied mathematics, it is very important to know when the powers of a matrix are bounded for a given norm, especially when solving a differential equation or a partial differential equation by discretisation. For more details the reader is referred to [10] and the recent textbook [5].

In the first section, we give an elementary proof of an old result of Kreiss [6], that is, $p(T) = \sup_{k \geq 0} \| T^k \|$ is finite if and only if the spectrum of $T$ is in the closed unit disk and if $T$ satisfies the Kreiss condition

$$ \|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda| - 1}, \quad \text{for } |\lambda| > 1. $$

Let $r(T)$ be the lower bound of all $C > 0$ that satisfy the Kreiss condition. A conjecture was made by Leveque and Trefethen [8], and was subsequently proved by Spijker [12], that

$$ p(T) \leq enr(T), \quad \text{if } T \in M_n(\mathbb{C}). $$

This is not the best inequality when the eigenvalues of $T$ are of modulus 1 because then $p(T) \leq m r_1(T)$, where $m$ denotes the number of distinct eigenvalues, and $r_1(T) \leq r(T)$ is the function defined after the proof of Theorem 1.1. Consequently, we give a slightly improved version of the above inequality.

In the second section, we use the previous results to show that if $H \in M_n(\mathbb{C})$ is a Hermitian matrix for a given norm, then its Cayley transformation $U = (1 + iH)(1 - iH)^{-1}$ has all its powers $U^k$ which are bounded ($k \in \mathbb{Z}$). We generalise this result by showing that if $T$ is a quasi-dissipative operator and if $\varphi$ is a conformal transformation of the half-plane $\Pi = \{ z : \Re(z) \leq 0 \}$ onto the unit disk $\Delta = \{ z : |z| \leq 1 \}$, then $\varphi(T)$ satisfies the Kreiss condition for some effective constant $C$. 

1. Power bounded operators

If \( T \) is in \( B(X) \) and satisfies \( \sup_k \| T^k \| \leq C < \infty \), then by Gelfand’s formula [1, theorem 3.2.8(iii)] the spectrum of \( T \) is in the unit disk and we have

\[
\| (\lambda - T)^{-1} \| = \left\| \frac{1}{\lambda} + \frac{T}{\lambda^2} + \frac{T^2}{\lambda^3} + \cdots \right\| \leq \frac{C}{|\lambda| - 1},
\]

for \( |\lambda| > 1 \). It is easy to see that if the spectrum is included in the unit disk then the powers of \( T \) are not necessarily bounded. To see this, take \( T = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C}) \), with \( a \neq 0 \). Then \( T^k = (I + N)^k = I + kN \), where \( N^2 = 0 \).

A natural question to ask is: do the two previous conditions imply the power-boundedness of \( T \)? Unfortunately, the answer is no for infinite-dimensional spaces. We can only arm that if \( \text{Sp}(T) \) and \( \| (\lambda - T)^{-1} \| \leq \frac{C}{|\lambda| - 1} \) for \( |\lambda| > 1 \), then

\[
\| T^k \| = O(k), \quad \text{for} \quad k \to \infty.
\]

If, moreover, \( C = 1 \), then \( \| T^k \| = O(\sqrt{k}) \). To see this, it is sufficient to apply Cauchy’s formula to \( T \) and to the circle \( \Gamma \) of radius \( 1 + \frac{1}{k} \), which gives \( T^k = \frac{1}{2\pi i} \int_{\Gamma} \mu^k (\mu - T)^{-1} d\mu \). Consequently,

\[
\| T^k \| \leq \frac{1}{2\pi} \left| 1 + \frac{1}{k} \right|^k \frac{C}{1 + \frac{1}{k} - 1} 2\pi \left( 1 + \frac{1}{k} \right) = Ck \left( 1 + \frac{1}{k} \right)^{k+1} \sim Cek.
\]

If \( C = 1 \), then we have \( e^zT = \lim_{k \to \infty} \left( \frac{k}{z} \right)^k \left( \frac{k}{z} - T \right)^{-k} \), and therefore by the resolvent condition satisfied for \( k > |z| \) we have

\[
\| e^zT \| \leq \lim_{k \to \infty} \left| \frac{k}{z} \right|^k \left| \left( \frac{k}{z} - T \right)^{-1} \right|^k \leq \lim_{k \to \infty} \left| \frac{k}{z} \right|^k \left( \frac{k}{z} - 1 \right) = e^{|z|}.
\]

Hence by Cauchy’s formula for the \( k \)th derivative, applied to \( e^zT \) and to the circle \( \Gamma' \) having centre at 0 and radius \( k \), we obtain \( T^k = \frac{k!}{2\pi i} \int_{\Gamma'} \frac{e^z}{z^{k+1}} dz \). Thus, by Sterling’s formula, \( \| T^k \| \leq k! \sqrt{2\pi k} \sim \sqrt{2\pi k} \).

Throughout this paper \( T \in B(X) \) is said to satisfy the Kreiss condition, with the constant \( C > 0 \) if \( \text{Sp}(T) \subset A \) and \( \| (\lambda - T)^{-1} \| \leq \frac{C}{|\lambda| - 1} \) for \( |\lambda| > 1 \). We note that if this condition is locally satisfied for \( 1 < |\lambda| < \rho \), then it is also satisfied for \( |\lambda| > 1 \) with a new larger constant. To see this we apply Holomorphic Functional Calculus to \( T \), to the function \( h(z) = \frac{1}{z-\lambda} \) and to the circle \( \Gamma = \{ z : |z| = \sqrt{\rho} \} \). Then we have

\[
\| (\lambda - T)^{-1} \| \leq \frac{1}{2\pi} \int_{\Gamma} |\lambda - z|^{-1} \frac{C}{|z| - 1} |dz| \\
\leq \frac{C \sqrt{\rho}}{(|\sqrt{\rho} - 1)|\lambda - \sqrt{\rho}| \sqrt{\rho} - 1 |\lambda| - 1},
\]

for \( |\lambda| > \rho \).
It is known that in the finite-dimensional case, the Kreiss condition implies that all the powers of $T$ are bounded. We shall give a more detailed outline later. For now we give an apparently unknown elementary proof of this result.

Let $T \in M_n(\mathbb{C})$, then $(\lambda - T)^{-1}$ is a matrix having coefficients $r_{ij}(\lambda)$ that are rational functions with numerators that are polynomials of degree inferior or equal to $n - 1$ and denominators that are the characteristic polynomials of $T$. Applying the decomposition theorem for rational functions in simple elements to each $r_{ij}$, it is easy to see that there exist matrices $A_{ij} \in M_n(\mathbb{C})$ such that

$$(\lambda - T)^{-1} = \sum_{i=1}^{\ell} \frac{A_{i1}}{\lambda - \lambda_i} + \frac{A_{i2}}{(\lambda - \lambda_i)^2} + \ldots + \frac{A_{in_i}}{(\lambda - \lambda_i)^{n_i}},$$

where $\lambda_1, \ldots, \lambda_\ell$ denote the roots of the characteristic polynomial and $n_1, \ldots, n_\ell$ their multiplicities ($n_1 + \ldots + n_\ell = n$).

**Theorem 1.1.** Let $T \in M_n(\mathbb{C})$. The following properties are equivalent:

(i) $\sup_{k \geq 0} \|T^k\| < \infty$.

(ii) $T$ satisfies the Kreiss condition.

(iii) $Sp(T) \subset \Delta$ and all the eigenvalues of modulus 1 are simple poles of the resolvent.

**Proof.** (i) implies (ii) results from the remark given at the beginning. Let us show that (ii) implies (iii). Suppose for example that $T$ is not invertible. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $T$ with modulus 1, with multiplicities $n_1, \ldots, n_m$. Choose $r > 0$ such that the disks $B(\lambda_i, r)$ are disjoint, and $1 - 2r$ is greater than the moduli of the other eigenvalues. By applying Holomorphic Functional Calculus [1, theorem 3.3.3] to $T$ and to the functions $f_0, f_1$ defined by

$$f_0(\lambda) = \begin{cases} \lambda, & \text{on } B(0, 1 - 2r) \\ 0, & \text{on } \bigcup_{i=1}^{m} B(\lambda_i, r) \end{cases} \quad \text{and} \quad f_1(\lambda) = \begin{cases} 0, & \text{on } B(0, 1 - 2r) \\ \lambda, & \text{on } \bigcup_{i=1}^{m} B(\lambda_i, r) \end{cases},$$

we define $T_0 = f_0(T)$ and $T_1 = f_1(T)$. Then $T = T_0 + T_1$, $Sp(T_1) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \cup \{0\}$ and $Sp(T_0) = (Sp(T) \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_m\}) \cup \{0\}$. In particular, the spectral radius satisfies $\rho(T_0) \leq 1 - 2r < 1$. Hence, by Gelfand’s formula [1, theorem 3.2.8 (iii)], $\lim_{k \to \infty} \|T_0^k\| = 0$.

Also $T_0 T_1 = T_1 T_0 = 0$, and hence $T^k = T_0^k + T_1^k$. Since $\frac{1}{2}(\lambda - T)^{-1} = (\lambda - T_0)^{-1}(\lambda - T_1)^{-1}$, and $(\lambda - T_0)^{-1}$ has no poles on the unit circle, hypothesis (iii) implies that $(\lambda - T_1)^{-1}$ has only simple poles on the unit circle. In other words we have

$$(\lambda - T_1)^{-1} = \frac{A_1}{\lambda - \lambda_1} + \ldots + \frac{A_m}{\lambda - \lambda_m} + \frac{B}{\lambda},$$

It is easy to see that there exist matrices $A_{ij} \in M_n(\mathbb{C})$ such that

$$(\lambda - T)^{-1} = \sum_{i=1}^{\ell} \frac{A_{i1}}{\lambda - \lambda_i} + \frac{A_{i2}}{(\lambda - \lambda_i)^2} + \ldots + \frac{A_{in_i}}{(\lambda - \lambda_i)^{n_i}}.$$
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for some $A_1, \ldots, A_m, B \in M_n(\mathbb{C})$. This formula implies that

$$A_i = \frac{1}{2\pi i} \int_{T_i} (\lambda - T_1)^{-1} d\lambda,$$

where $T_i$ is the boundary of $\mathcal{B}(x_i, r)$, which proves that $A_i$ is the Riesz projection associated to $T$ and $x_i$ (see [1, pp 106–7]). By developing in series all the terms in (2), for $|\lambda| > 1$, we get

$$1 + \frac{T_1}{\lambda^2} + \ldots = \frac{A_1}{\lambda} \left(1 + \frac{x_1}{\lambda} + \ldots\right) + \frac{A_m}{\lambda} \left(1 + \frac{x_m}{\lambda} + \ldots\right) + \frac{B}{\lambda}.$$

Consequently $T_1 = x_1 A_1 + \ldots x_m A_m$. So, because the Riesz projections are orthogonal, we get

$$T_1^k = x_1^k A_1 + \ldots x_m^k A_m.$$  (3)

Therefore $\|T_1^k\| \leq \|A_1\| + \ldots + \|A_m\|$. This proves that $\|T^k\|$ is bounded. □

Let $T \in M_n(\mathbb{C})$ be such that the spectral radius satisfies $\rho(T) \leq 1$. Let us introduce the following three functions:

$$p(T) = \sup_{k \geq 0} \|T^k\|;
$$

$$r(T) = \sup_{|\lambda| > 1} (|\lambda| - 1)\|(|\lambda| - T)^{-1}\|;
$$

$$r_1(T) = \lim_{|\lambda| \to 1^+} (|\lambda| - 1)\|(|\lambda| - T)^{-1}\|.$$

We have $r_1(T) \leq r(T) \leq p(T)$. Theorem 1.1 implies that if $r(T)$ is finite then $p(T)$ is finite. Since $\lim_{|\lambda| \to 1^+} (|\lambda| - 1)\|(|\lambda| - T)^{-1}\| = 1$, then by the Maximum Theorem for continuous functions we conclude that if $r_1(T)$ is finite then $r(T)$ is also finite. If we take $\mathbb{C}^2$ with the Euclidean norm, then for

$$T = \begin{pmatrix} 0, & a \\ 0, & 0 \end{pmatrix},$$

we have $r_1(T) = 0$ and

$$r(T) = \sup_{|\lambda| > 1} (|\lambda| - 1)\left\|\frac{1}{\lambda} \left(1 + \frac{T}{\lambda}\right)\right\| \geq \frac{1}{4} \|2 + T\| > \frac{|a|}{4},$$

which can be as large as needed.

Many authors have been interested in knowing whether it is possible to estimate $p(T)$ in terms of $r(T)$. According to Tadmor [13] the original proof of Kreiss [6] gives a very bad estimate $p(T) \leq C r(T)^n$ for an effective constant $C$. Subsequently, Morton, Strang and Miller obtained better estimates of the type $p(T) \leq C n^4 r(T)$, $p(T) \leq C n^5 r(T)$, and $p(T) \leq C e^{n^2} r(T)$. Strang noticed that in the paper of Laptev [7] the estimate $p(T) \leq \frac{1}{\pi} n^2 r(T)$ is implicitly contained, an estimate which was
later improved by Tadmor [13] to $p(T) \leq \frac{32}{\pi}nr(T)$ by using a Cauchy integral. Using Bernstein’s inequality for the derivative of a rational function, Leveque and Trefethen [8] were able to show that $p(T) \leq 2enr(T)$, and they also conjectured that $p(T) \leq enr(T)$. The former result was slightly improved by Smith [11], who obtained $p(T) \leq (1 + \frac{2}{\pi})enr(T)$. Spijker [12] finally solved the conjecture and his proof is simple.

The example given in [8, pp 585–6] shows that in some way this is the best possible inequality, because we can find a series of matrices $T_k \in M_n(\mathbb{C})$ such that $T_k^n = 0$ and $\lim_{k \to \infty} \frac{p(T)}{\text{nr}(T)} = e$. This inequality, which is the best possible for most matrices, could be very bad for some extreme cases, for example matrices which have all their eigenvalues on the unit circle.

**Theorem 1.2.** Let $T \in M_n(\mathbb{C})$ have all its eigenvalues on the unit circle. Then $p(T) \leq mr_1(T)$, where $m \leq n$ is the number of distinct eigenvalues of $T$.

**Proof.** Suppose, without any restriction, that $r_1(T)$ is finite, because otherwise $p(T)$ is also infinite. By using the same argument as in the proof of Theorem 1.1, we conclude that

$$(\lambda - T)^{-1} = \frac{A_1}{\lambda - z_1} + \cdots + \frac{A_m}{\lambda - z_m},$$

where the $A_i$ are projections. If we take $\lambda$ converging to $z_i$, with $|\lambda| > 1$ and $\lambda$ colinear to $z_i$, then $(\lambda - z_i)(\lambda - T)^{-1}$ converges to $A_i$. Thus we obtain

$$||A_i|| = \lim_{|\lambda| \to 1} ||(\lambda - z_i)(\lambda - T)^{-1}|| \leq \lim_{|\lambda| \to 1} ||(\lambda - 1)(\lambda - T)^{-1}|| = r_1(T).$$

Hence, by (3), we obtain $\|T^k\| \leq mr_1(T)$. 

**Example.** For $M_2(\mathbb{C})$ we give an example of a sequence of matrices $T_n$ having their two eigenvalues of modulus 1, such that $\lim_{n \to \infty} \frac{p(T_n)}{\text{nr}(T_n)} = 2$. Take $M_2(\mathbb{C})$ with the Euclidean norm $\|\cdot\|_2$, a sequence $(\theta_n)$ converging to 0, with $0 < \theta_n < \frac{\pi}{2}$ and

$$P_n = \begin{pmatrix} 1, & \cot \theta_n \\ 0, & 0 \end{pmatrix}.$$  

The $P_n$ are projections and $\|P_n\|_2 = \|1 - P_n\|_2 = \frac{1}{\sin \theta_n}$. Let $T_n = 1 - 2P_n$. Then we have $T_n^2 = 1$, and hence

$$T_n^k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ T_n, & \text{if } k \text{ is odd.} \end{cases}$$

Also

$$p(T_n) = \|T_n\|_2 = \|T_nT_n^*\|^{1/2} = \rho(T_nT_n^*)^{1/2} = \frac{1 + \cos \theta_n}{\sin \theta_n}.$$  

But since $T_n = 1 - P_n - P_n$, where $P_n$ and $1 - P_n$ are orthogonal projections, we have
\[ r_1(T_n) = \text{Max}(|P_n||1 - P_n|) = \frac{1}{\sin \theta_n} \] (see the proof of Theorem 2.2). Also
\[
\lim_{n \to \infty} \frac{p(T_n)}{r_1(T_n)} = \lim_{n \to \infty} (1 + \cos \theta_n) = 2.
\]

The proof of the fundamental inequality \( p(T) \leq enr(T) \) depends on an idea of Leveque and Trefethen [8, p. 587] and on the lemma proven by Spijker [12], which affirms the following property. Let \( R(z) = \frac{P(z)}{Q(z)} \) be a rational function such that the polynomials \( P, Q \) with complex coefficients are of degree at most \( n \), and \( R \) does not have poles on the unit circle. Then
\[
\int_{|z|=1} |R'(z)|dz \leq 2\pi n \max_{|z|=1} |R(z)|.
\]
We now give a slight improvement to the fundamental inequality.

**Theorem 1.3.** Let \( T \in M_n(\mathbb{C}) \) be such that \( \rho(T) \leq 1 \). Denote by \( m \) the number of distinct points of the spectrum of \( T \) on the unit circle. Then we have
\[
p(T) \leq mr_1(T) + e(n-m)r(T)||1 - A_1|| \ldots ||1 - A_m||, \tag{4}
\]
where \( A_1, \ldots A_m \) are the Riesz projections associated to the \( m \) eigenvalues on the unit circle.

**Proof.** Suppose \( r(T) \) is finite, otherwise there is nothing to prove. By using the notation in the proof of Theorem 1.1 we can write
\[
(\lambda - T)^{-1} = Q(\lambda) + \frac{A_1}{\lambda - \lambda_1} + \ldots + \frac{A_m}{\lambda - \lambda_m},
\]
where the rational function \( Q \) has at most \( n-m \) poles inside the open unit disk. The same argument as in the proof of Theorem 1.2 shows that \( ||A_1||, \ldots, ||A_m|| \leq r_1(T) \).
Since \( T^k = T_0^k + T_1^k = T_0^k + \lambda_1^k A_1 + \ldots + \lambda_m^k A_m \), we obtain \( p(T) \leq p(T_0) + mr_1(T) \).
To estimate \( p(T_0) \) we use the identity
\[
\frac{1}{\lambda} + (\lambda - T)^{-1} = (\lambda - T_0)^{-1} + (\lambda - T_1)^{-1}, \quad \text{for} \quad \lambda \notin \text{Sp}(T). \tag{5}
\]
Let us denote by \( P = 1 - (A_1 + \ldots + A_m) = (1 - A_1) \ldots (1 - A_m) \) the Riesz projection associated to \( T_0 \); therefore \( PT_1 = T_1P = 0 \), \( PT_0 = T_0P = T_0 \) and of course \( P \) commutes with \( T \). Therefore
\[
P(\lambda - T)^{-1} = P(\lambda - T_0)^{-1}. \tag{6}
\]
Let us consider the closed subalgebra \( A = PM_n(\mathbb{C})P \), which has \( P \) as identity and which contains \( T_0 \). The resolvent of \( T_0 \) relative to the subalgebra \( A \) having identity \( P \) is therefore \( R(\lambda) = P(\lambda - T_0)^{-1} \) because \( (\lambda P - T_0)P(\lambda - T_0)^{-1} = P(\lambda - T_0)^{-1}(\lambda P - T_0) = P \). The poles of \( R(\lambda) \) are exactly all the poles of the resolvent of \( T \), which have modulus less than 1. Hence their number is at most \( n - m \). Since \( P(\lambda - T_0)^{-1} \)
converges to zero at infinity, then $R(\lambda)$ necessarily has at most $n - m - 1$ zeros. In addition, $|R(\lambda)| \leq |P| r(T)$ for $|\lambda| > 1$, from (6) and $|P| \leq ||1 - A_1|| \ldots ||1 - A_m||$. Using the same argument as Leveque and Trefethen [8] and Spijker [12], and applying it not to $M_n(\mathbb{C})$ but to the subalgebra $A = PM_n(\mathbb{C})P$ having the same norm and the identity $P$, we obtain $p(T_0) \leq e(n - m)||P||r(T)$. Hence we get the result.

Remarks. The beginning of the argument shows that there are only two possibilities. Either all the $A_i$ are zero, in which case $m = 0$, and hence $r_1(T) = 0$, or one of the $A_i$ is not zero, in which case $r_1(T) \geq ||A_i|| \geq 1$, because the norm of the projection is greater than or equal to 1. In practice it is difficult to estimate $||P||$. It is easy to see that $||P|| \leq 1 + mr_1(T)$. This gives a satisfactory formula for $k = 0$ or $k = m$. But for $1 \leq k \leq m - 1$, this bound does not give anything useful. For the case where $||1 - A_1|| = \ldots = ||1 - A_m|| = 1$, which occurs when the projections are Hermitian, (4) is better than the fundamental inequality. It would be interesting to find a good estimate for $||P||$.

2. Conformal transformation of quasi-dissipative operators

If $H$ is a self-adjoint operator on a Hilbert space, its Cayley transform $U = (1 + iH)(1 - iH)^{-1}$ gives a unitary operator, such that, for any $n \in \mathbb{Z}$, the norm of $U^n$ is always equal to 1. Is there a similar result if we take a Hermitian operator on a Banach space? More generally, if we take a quasi-dissipative operator $T$ on a Banach space and if $\psi$ is a conformal transformation of the half-plane $\{z: \Re(z) \leq 0\}$ onto the unit disk, can we conclude that the powers of $\psi(T)$ are bounded? We shall see that, in the case of matrices, the answer is yes. We suspect that the result is not true in the infinite-dimensional case, but we were not able to provide a counter-example.

If $T \in B(X)$, where $X$ is a Banach space, the numerical range of $T$ is defined by

$$V(T) = \{f(Tx): x \in X, f \in X^*, ||x|| = ||f|| = f(x) = 1\}.$$  

Then $T$ is a Hermitian operator on this Banach space if and only if $V(T) \subset \mathbb{R}$. If $X$ is a Hilbert space, with the standard norm, then this concept coincides with that of the self-adjoint operator. But, in general, the two notions are different. The only things we can say are that $Sp(T) \subset \mathbb{R}$ and $||e^{\alpha T}|| = 1$, for all $t \in \mathbb{R}$ (see [2, lemma 5.2, p. 46]). Unfortunately, the set of Hermitian elements does not have interesting algebraic properties. In general, $T$ Hermitian does not imply that $T^2$ is Hermitian. For more details, see [2; 3; 4].

We say that $T \in B(X)$ is dissipative if $V(T) \subset \{z: \Re(z) \leq 0\}$. By the Lumer–Phillips theorem [2, p. 30], $T$ is dissipative, if and only if $||e^{tT}|| \leq 1$, for all $t \geq 0$. If $H$ is Hermitian then $iH, -iH$ and $-H^2$ are dissipative. Below are more examples of dissipative matrices. We say that $T$ is quasi-dissipative if there exists a constant $C > 0$ such that $||e^{tT}|| \leq C$, for all $t \geq 0$. In this case, there exists an equivalent norm on $B(X)$ for which $T$ becomes dissipative (see [2, lemma 2.7, p.21]).
If we take the nilpotent matrix
\[ T = \begin{pmatrix} 0, & a \\ 0, & 0 \end{pmatrix} \in M_2(\mathbb{C}) \]
with \( a > 0 \), then \( \text{Sp}(T) \subset \{ z : \Re(z) \leq 0 \} \) and \( T \) is not quasi-dissipative because \( e^{tT} = 1 + tT \), and hence \( \lim_{t \to \infty} ||e^{tT}|| = \infty \). By the conformal transformation \( \varphi_0(z) = \frac{1+z}{1-z} \), \( U = \varphi_0(T) = 1 + 2T \) and \( U^n = 1 + 2nT \). In this case the powers of \( U \) are not bounded. In other words, the fact that \( \text{Sp}(T) \subset \{ z : \Re(z) \leq 0 \} \) is not sufficient to conclude that the powers of \( U \) are bounded. We need more, for example the hypothesis of quasi-dissipativity. Let us now look at two small interesting examples.

**Example 1.** Take
\[ T = \begin{pmatrix} -a, & c \\ 0, & -b \end{pmatrix}, \]
with \( 0 \leq a < b \) and \( 0 < c \leq b - a \), then
\[ e^{tT} = \begin{pmatrix} e^{-at}, & e^{-at} - e^{-bt} \\ 0, & e^{-bt} \end{pmatrix}. \]

If we take in \( M_2(\mathbb{C}) \) the norm \( || \cdot ||_1 \) defined by
\[ \left\| \begin{pmatrix} x, & \beta \\ \gamma, & \delta \end{pmatrix} \right\|_1 = \max \{|x| + |\gamma|, |\beta| + |\delta|\}, \]
then
\[ ||e^{tT}||_1 = \begin{cases} e^{-at}, \text{ for } t \geq 0, \\ \frac{e^{-at}}{\pi t}(e^{-bt} - e^{-at}) + e^{-bt}, \text{ for } t < 0. \end{cases} \]
In particular, \( T \) is dissipative and \( ||e^{tT}|| \) increases rapidly on the negative half-line \( \mathbb{R}_- \). If we take \( a = 0, c = b \) and \( U = (1 + T)(1 - T)^{-1} \), then
\[ U^n = \begin{pmatrix} 1, & 1 - \frac{b^2}{\pi^2}n^2 \\ 0, & \left(1 - \frac{b^2}{4n^2}\right) \end{pmatrix}. \]
Therefore, if we take \( b > 1 \), then
\[ ||U^n|| = \begin{cases} 1, \text{ if } n \text{ is even,} \\ 1 + 2\left(\frac{b^2}{n^2} - 1\right)^n > 1, \text{ if } n \text{ is odd.} \end{cases} \]

**Example 2.** Let \( P \) be a projection in \( B(X) \). Then we have \( e^{-tP} = 1 - P(1 - e^{-t}) = 1 - P + Pe^{-t} \). Hence \( ||e^{-tP}|| \leq ||1 - P|| + ||P|| \) for \( t \geq 0 \) implies that \( P \) is quasi-dissipative. For this case we have \( U = \frac{1 - P}{1-t^2} = 1 - P \), hence \( U^k = U \), for any \( k \geq 1 \).
Lemma 2.1. If $T \in B(X)$ is quasi-dissipative with constant $C > 0$, then we have $\|(\lambda - T)^{-1}\| \leq \frac{C}{\Re(\lambda)}$, for $\Re(\lambda) > 0$.

Proof. By the Laplace transform, we have $(\lambda - T)^{-1} = \int_{0}^{\infty} e^{-\lambda t} e^{T} dt$. Therefore

$$\|(\lambda - T)^{-1}\| \leq \int_{0}^{\infty} |e^{-\lambda t}||e^{T}| dt \leq C \int_{0}^{\infty} e^{-\Re(\lambda) t} dt = \frac{C}{\Re(\lambda)}.$$

According to theorem 2 in [8] the converse is true in finite dimension.

By the Riesz theory, we know that all compact self-adjoint operators $T$ on a Hilbert space can be written in the form $T = \sum \lambda_{n} P_{n}$, where $\lambda_{n}$ are the eigenvalues of $T$ and the orthogonal projections $P_{n}$ are self-adjoint and of finite rank. This theorem was generalised for compact Hermitian operators on Banach spaces (see [3, theorem 1, p. 82] and [4, theorem 11, p. 33]), except that in general, even in finite dimensions, the $P_{n}$ are not Hermitian (see the remark in [3, p. 84] and Crabb’s example [2, p. 58]). The proof of this theorem is difficult because it is based on a fixed-point theorem (Schauder–Tychonoff or Markov–Kakutani). We now give an elementary proof of this result for matrices.

Theorem 2.2. (Spectral Theorem for Hermitian Matrices.) Let $H \in M_{n}(\mathbb{C})$ be a Hermitian matrix for some norm. Then

$$H = \lambda_{1} P_{1} + \ldots + \lambda_{m} P_{m},$$

where $\lambda_{1}, \ldots, \lambda_{m}$ are the distinct eigenvalues of $H$ and $P_{1}, \ldots, P_{m}$ are the corresponding Riesz projections. These projections satisfy $P_{1} + \ldots + P_{m} = I$, $P_{i} P_{j} = 0$ for $i \neq j$ and $\|P_{i}\| = 1$ for $i = 1, \ldots, m$.

Proof. Since $H$ is Hermitian, $T = iH$ is dissipative, then by the previous lemma we have

$$\|(\lambda - T)^{-1}\| \leq \frac{1}{\Re(\lambda)}.$$

By the argument used in the proof of (ii) implies (iii) of Theorem 1.1, and by taking $\lambda$ converging to $i \lambda_{r}$ ($r = 1, \ldots, m$), with $\Re(\lambda) > 0$ and $\Im(\lambda) = \lambda_{r}$, we prove that the poles $i \lambda_{1}, \ldots, i \lambda_{m}$ are simple. By formula (1), we have $(\lambda - T)^{-1} = \frac{A_{1}}{\lambda - \lambda_{1}} + \ldots + \frac{A_{m}}{\lambda - \lambda_{m}}$, for all $\lambda \neq i \lambda_{1}, \ldots, i \lambda_{m}$. Changing $\lambda$ to $i \lambda$ we obtain

$$(\lambda - H)^{-1} = \frac{A_{1}}{\lambda - \lambda_{1}} + \ldots + \frac{A_{m}}{\lambda - \lambda_{m}}.$$

The same argument as the one used in the proof of Theorem 1.1 shows that $A_{i} = P_{i}$, the Riesz projection associated to $H$ and $\lambda_{i}$. Hence we have $P_{1} + \ldots + P_{m} = I$ and $P_{i} P_{j} = 0$ for $i \neq j$. By developing the series for $|\lambda| > \|H\|$, we obtain

$$H = \lambda_{1} P_{1} + \ldots + \lambda_{m} P_{m}.$$
It is left to verify that $\|P_i\| = 1$. The properties of the projections imply that

\[(\lambda - iH)^{-1} = \frac{P_1}{\lambda - iz_1} + \ldots + \frac{P_m}{\lambda - iz_m}.\]

Therefore, by making $\lambda$ converge to $iz_i$, with $\Re(\lambda) > 0$ and $\Im(\lambda) = z_i$, we obtain

\[||P_i|| = \lim_{\lambda \to iz_i} ||(\lambda - iH)^{-1}|| = \Re(\lambda)||(\lambda - iH)^{-1}|| \leq 1,\]

which yields the result because $||P_i||$ is always greater than 1.

**Corollary 2.3.** Let $H \in M_n(\mathbb{C})$ be a Hermitian matrix for some norm. Let $U = (1+iH)(1-iH)^{-1}$ be its Cayley transform. Then $||U^k|| \leq n$, for any $k \in \mathbb{Z}$.

**Proof.** By the previous arguments we have

\[U^k = \left(\frac{1 + iz_1}{1 - iz_1}\right)^k P_1 + \ldots + \left(\frac{1 + iz_n}{1 - iz_n}\right)^k P_n,\]

but

\[\left|\frac{1 + iz_r}{1 - iz_r}\right| = 1, \quad \text{for } r = 1, \ldots, n.\]

Hence $||U^k|| \leq ||P_1|| + \ldots + ||P_n|| = n$. ■

Let us now investigate the more general problem of the conformal transformation of a quasi-dissipative matrix. Let $\Pi$ be the half-plane \{z: $\Re(z) \leq 0$\} and $\Lambda$ be the unit disk \{z: $|z| \leq 1$\}. The transformation

\[\varphi_0(z) = \frac{1 + z}{1 - z}\]

is a conformal transformation of $\Pi$ onto $\Lambda$. Knowing that all the conformal transformations of the unit disk $\Lambda$ are obtained by composition of the transformations

\[\psi_\alpha(z) = \frac{z - \alpha}{1 - \alpha z}, \quad |\alpha| < 1,\]

and the rotations $R_\theta: z \rightarrow e^{i\theta}z$, we can conclude that all conformal transformations of $\Pi$ onto $\Lambda$ are of the form $R_\theta \circ \psi_\alpha \circ \varphi_0$.

**Theorem 2.4.** Let $T$ be a quasi-dissipative operator, with constant $C > 0$, and $\varphi$ be a conformal transformation of $\Pi$ onto $\Lambda$. Then $\varphi(T)$ satisfies the Kreiss condition outside $\Lambda$ with a constant less than or equal to

\[C_\varphi = C \frac{||1 - T|| + r||1 + T||}{1 - r},\]

where $r = |\varphi(-1)|$. 
Proof. By Holomorphic Functional Calculus it is clear that the spectrum of \( \varphi(T) \) is contained in \( A \).

(a) First suppose that \( \varphi = \varphi_0 \). Let \( |\mu| > 1 \), and then \( \mu = \frac{1 + i}{t - \frac{1}{t}} \) with \( \lambda = \frac{\mu + 1}{\mu - 1} \) satisfies \( \Re(\lambda) > 0 \). We have \( \mu - \varphi_0(T) = \frac{\lambda}{\lambda - \frac{1}{\lambda}}(1 - T)^{-1} \), and consequently \( (\mu - \varphi_0(T))^{-1} = \frac{1}{\lambda}(1 - T)(\lambda - T)^{-1} = \frac{\mu + 1}{\mu - 1}(1 - T) \left( \frac{\mu - 1}{\mu + 1} - T \right)^{-1} \); hence, by Lemma 2.1, we have

\[
\|(\mu - \varphi_0(T))^{-1}\| \leq \frac{\|1 - T\|}{|1 + \mu|} \Re\left( \frac{1 - \frac{1}{\mu}}{\frac{1}{\mu} + 1} \right).
\]

Let \( \mu = a + ib \), then

\[
\frac{\mu - 1}{\mu + 1} = \frac{a - 1 + ib}{a + 1 + ib} = \frac{(a - 1 + ib)(a + 1 - ib)}{(a + 1)^2 + b^2};
\]

hence

\[
\Re\frac{\mu - 1}{\mu + 1} = \frac{a^2 + b^2 - 1}{(a + 1)^2 + b^2} = \frac{|\mu|^2 - 1}{|\mu + 1|^2}. \tag{7}
\]

Finally, we obtain

\[
\|(\mu - \varphi_0(T))^{-1}\| \leq C\|1 - T\| |\mu + 1| \frac{|\mu|^2 - 1}{|\mu|^2 - 1} \leq C\|1 - T\| \frac{1 + |\mu|}{|\mu|^2 - 1} = C \frac{1 - T}{|\mu - 1|}.
\]

(b) Since the Kreiss condition depends only on \( |\mu| \), it suffices to assume, without loss of generality, that \( \varphi = \psi_\varphi \circ \varphi_0 \), with \( 0 < |\varphi| < 1 \). We set \( A = \varphi_0(T) \), so \( \varphi(T) = \psi_\varphi(A) \). Let \( |\mu| > 1 \) with \( \mu \neq -\frac{1}{t} \), then \( \mu = \psi_\varphi(t) \), where \( t = \frac{\mu + 1}{\mu - 1} \), with \( |t| > 1 \). Thus we obtain

\[
\mu - \varphi(T) = \frac{(1 - |\varphi|^2)}{1 - |\varphi|^2}(t - A)(1 - \overline{A}A)^{-1},
\]

hence

\[
(\mu - \varphi(T))^{-1} = \frac{1 - \overline{\varphi}t}{1 - |\varphi|^2}(1 - |\varphi|^2)(t - A)(1 - \overline{A}A)^{-1}
= \frac{1 - \overline{\varphi}t}{1 - |\varphi|^2}(1 - \overline{\varphi}(1 + T)(1 - T)^{-1} - (t - (1 + T)(1 - T)^{-1})
= \frac{1 - \overline{\varphi}t}{1 - |\varphi|^2}(1 - \overline{\varphi} - (1 + \overline{\varphi})T)(t - (1 + t)T)^{-1}
= \frac{1 - \overline{\varphi}t}{1 - |\varphi|^2}(1 - \overline{\varphi} - (1 + \overline{\varphi})T) \frac{1}{1 + t} \left( \frac{t - 1 - T}{t + 1} \right)^{-1}.
\]

Thus, for \( |\mu| > 1 \), \( \mu \neq -\frac{1}{t} \), we have

\[
\|(\mu - \varphi(T))^{-1}\| \leq \frac{\|1 - \overline{\varphi} - (1 + \overline{\varphi})T\| |\mu - \overline{\varphi}| \frac{C}{1 + t} \Re\left( \frac{1 + \frac{1}{\mu}}{\frac{1}{\mu} + 1} \right)}{1 - |\varphi|^2}.
\]
Therefore, by formula (7), we have
\[
\| (\mu - \varphi(T)^{-1} \| \leq \frac{C\| 1 - \overline{\varphi} - (1 + \overline{\varphi})T \|}{1 - |\varphi|} \frac{|\mu|}{|\mu| - 1} = C\| 1 - \overline{\varphi} - (1 + \overline{\varphi})T \| \frac{|\mu|}{|\mu| + 1 + 2|\varphi|\|T\|} \frac{1 + |\varphi|}{|\mu| - 1} \leq C\| 1 - \overline{\varphi} - (1 + \overline{\varphi})T \| \frac{1 + |\varphi|}{|\mu| - 1} \frac{1}{|\mu| - 1}.
\]

Since |\mu + z| - |1 + \varphi| = |\mu + z| - |\varphi| |\mu + \frac{1}{2}| and \varphi and \frac{1}{2} are colinear, we can assume, without loss of generality, that \varphi is real positive in order to find a majorant for \frac{1}{|\mu| - 1}, which depends only on |\mu|. Suppose \varphi > 0, and let \mu = a + ib, then
\[
\frac{1}{|\mu + z|^2 - |x\mu + 1|^2} = \frac{1}{(1 - \varphi^2)(|\mu|^2 - 1)},
\]
which implies that
\[
\frac{1}{|\mu + z| - |x\mu + 1|} = \frac{|\mu + z| + |x\mu + 1|}{|\mu + z|^2 - |x\mu + 1|^2} \leq \frac{(1 + z)(1 + |\mu|)}{(1 - \varphi^2)(|\mu|^2 - 1)} = \frac{1}{(1 - \varphi^2)(|\mu|^2 - 1)}.
\]
This proves that for |\mu| > 1, and \mu \neq -\frac{1}{2}, we have
\[
||(\mu - \varphi(T))^{-1}|| \leq \frac{C\| 1 - \overline{\varphi} - (1 + \overline{\varphi})T \|}{1 - |\varphi|} \frac{1}{|\mu| - 1} = C\| 1 - \overline{\varphi} - (1 + \overline{\varphi})T \| \frac{1}{|\mu| - 1} \frac{1}{|\mu| - 1}.
\]
By continuity we obtain the same inequality at \frac{1}{2}. Hence the result. ■

Remark. In Theorem 2.4 the constant \( C_\varphi \) tends to +\( \infty \) when |\varphi| \( \rightarrow \) 1. Unfortunately, it is not possible to improve this. We can see this by taking \( \varphi_n = 1 - \frac{1}{n} \) and \( \varphi_n = \psi_n \). If |\mu| > 1 is fixed, then we have
\[
\lim_{n \to \infty} ||(\mu - \varphi_n(T))^{-1}||(|\mu| - 1) = +\infty.
\]

Corollary 2.5. Let \( T \in M_n(\mathbb{C}) \) be a quasi-dissipative matrix for some norm, and let \( \varphi \) be a conformal transformation of \( \Pi \) onto \( \Lambda \). Then ||\varphi(T)^k|| is bounded.

Proof. This follows from the previous theorem and from results in the first section. ■
Many results of this paper can be extended to some operators in infinite dimension, for example Riesz operators (see [14, theorem 7]) or meromorphic operators (see [9]).

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