SPLITTABILTY FOR FINITE PARTIALLY-ORDERED SETS

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Abstract

Arhangel’skii and his co-workers have defined and developed notions of split-
tability or cleavability for topological spaces. We apply the analogous idea in
partially-ordered sets (posets) and identify those posets that are splittable over a
finite chain and those posets over which a given chain (finite or infinite) is splittable.

1. Introduction

A partial order on a non-empty set E is a binary relation on E (usually denoted
by ≤) that is reflexive, transitive and antisymmetric. A mapping f from one partially-
ordered set (poset) E to another E’ is called order-preserving or increasing if x ≤ y
in E implies f(x) ≤ f(y) in E’. The basic form of Arhangel’skii’s definition [1] in
topology is that a (topological) space X is splittable over a space X’ if, to every
subset A of X, there corresponds a continuous mapping from X into X’ under
which the images of A and of X\A are disjoint; by direct analogy, let us say that
the poset E is splittable over the poset E’ if, to every subset A of E, there corresponds
an order-preserving mapping from E into E’ under which the images of A and of
E\A are disjoint. The fundamental problem is to determine convenient, necessary
and sufficient conditions for this to happen and, as in topology, it appears to be a
difficult problem. The purpose of this article is to solve it in the special cases where
(i) E is a chain or
(ii) E’ is a finite chain.

A poset is called a chain when, for every two of its elements x and y, either x ≤ y or
y ≤ x is true.) The discussion falls naturally into two sections, according to whether
it is the domain or the codomain that is taken to be a chain. We shall discuss the
codomain case first.

2. Splittability over a chain

For each n ≥ 1 let us denote by C_n the archetypal n-point chain {1, 2, 3, ..., n}
under its natural order. It is easy to see that C_{n+1} is not splittable over C_n since,
choosing A to be the subset {1, 3, 5, ...} of C_{n+1}, any order-preserving map f from
C_{n+1} to C_n for which f(A) was disjoint from f(C_{n+1} \ A) would need to be one-to-one,
which is clearly impossible. Also, if C_n ⊕ C_n denotes the direct sum of two copies
of C_n (that is, the set {1, 2, ..., n, 1’, 2’, ..., n’} ordered by taking the natural orders
within {1, 2, ..., n} and {1’, 2’, ..., n’}) then C_n ⊕ C_n is not splittable over C_n: for,
letting B denote the subset {1, 3, 5, ..., 2’, 4’, 6’, ...} of C_n ⊕ C_n, an order-preserving
map from C_n ⊕ C_n into C_n that ‘kept B and its complement apart’ would have to be