ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let \( \mathbb{R} \) be the set of real numbers, \( \mathbb{R}^+ = [0, +\infty) \), \( \mathbb{R}^+ = [0, +\infty) \), \( a, b \in \mathbb{R}^+ \), \( p \geq 1 \).

\( L_p([a, b]) \) is the space of functions \( f : [a, b] \to \mathbb{R} \) such that \( |f(x)|^p \) is integrable on \([a, b]\).

\[ \|f\|_{L_p} = \left( \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p} \]

\( \hat{C}_p([a, b]) \) is the space of functions \( u : [a, b] \to \mathbb{R} \) such that \( u \in L_p([a, b]) \), \( \|u\|_{\hat{C}_p} = \|u(a)\| + \|u'\|_{L_p} \).

\( C([a, b]) \) is the space of continuous functions \( u : I \to \mathbb{R} \), \( \|u\|_C = \sup \{ |u(t)| : t \in I \} \).

\( \hat{C}_p^2([a, b]) \) is the set of functions \( u \in \hat{C}_1([a, b]) \) such that \( u' \in \hat{C}_p([a, b]) \).

Consider the boundary value problem

\[ u''(t) - H(u, u', u''(t)) = 0, \quad t \in [a, b] \]
\[ u(a) = 0, \quad u(b) = 0, \quad \tag{1} \]

Under a solution of equation (1) we mean a function \( u \in \hat{C}_p([a, b]) \) satisfying a.e. equation (1).

Below two theorems on the solvability of the problem (1), (2) are given.

**Theorem 1.** Let the inequality

\[ -g(t) \leq H(x, x', z)\cdot \text{sign} x(t), \quad t \in [a, b], \quad (x, z) \in \hat{C}_p^2([a, b]) \times L_p([a, b]) \]

be fulfilled, where \( g \in L_p([a, b]) \). Moreover, let for any \( r > 0 \) there exist \( \gamma_r, \alpha_r \in \mathbb{R}^+ \), \( f_r \in C([a, b]) \) such that

\[ \|H(x, x', z)\|_{L_p} \leq \alpha_r \cdot f_r \left( \|z\|_{L_p} \right) \quad \text{for} \quad \|x\|_C \leq r, \quad \|z\|_{L_p} \geq \gamma_r \]

and

\[ \liminf_{p \to +\infty} \frac{\rho}{f_r(p)} > \alpha_r. \]

Then the problem (1), (2) is solvable.

**Theorem 2.** Let the condition (3) be fulfilled. Moreover, let for any \( r \in \mathbb{R}^+ \), \( \alpha \in [0, (b - a)r] \), \( \beta \in [0, \alpha] \) there exist \( \gamma_r, \alpha_r \in \mathbb{R}^+ \), \( f_r, g_r \in C([0, b]) \) and \( h_r(t) \in L_p([a, b]) \) such that

\[ h_r(t) > 0 \quad \text{for} \quad t \in [a, b], \quad h_r(0) = 0, \]

\[ \|H(x, x', z)\|_{L_p} \leq \gamma_r \left( \|x\|_C \right) \cdot f_r \left( \|z\|_{L_p} \right) + \alpha_r \quad \text{for} \quad \|x\|_C < \alpha, \]

\[ \|z\|_{L_p} \leq r, \quad \|x\|_{L_p} \geq \gamma_r, \]

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where have solutions satisfying the boundary conditions (2).

Then the problem (1), (2) is solvable.

Let us give some examples.

Let

\[ G_1 \in L_p([a, b] \times [a, b]; \mathbb{R}^+), \quad K(x, y) (t) \cdot \text{sign} x(t) \geq -g(t), \quad t \in [a, b], \]

where

\[ K : C([a, b]) \times C([a, b]) \to L_p([a, b]), \quad q, g \in L_p([a, b]), \quad k \in \mathbb{N}, \]

\[ 0 < G_2(t, s) \leq g_1(t), \quad (t, s) \in [a, b] \times [a, b], \quad g_1 \in L_p([a, b]). \quad (4) \]

Consider the equation

\[
\begin{align*}
    u''(t) &= u^{2k+1}(t) \left[ \int_{a}^{b} G_2(t, s) \left( 1 + |u''(s)|^p \right)^{ \frac{1}{(k+1)p}} \left[ \int_{a}^{b} G_2(t, \tau) \cdot |u''(\tau)|^p \frac{d \tau}{d \tau} \right] ds \right. \\
    & \quad + K(u, u')(t) + q(t),
\end{align*}
\]

where \( \alpha \in \mathbb{R}^+ \), \( p, \lambda \alpha \leq 1 \). Then according to Theorem 2, the problem (6), (2) is solvable.

Analogously, the equations

\[
\begin{align*}
    u''(t) &= u^{2k+1}(t) \left( 1 + |u'(t)|^p \right)^{ \frac{1}{(k+1)p}} \left[ \int_{a}^{b} G_2(t, s) \cdot |u''(s)|^p \frac{d s}{d s} \right] + \\
    & \quad + K(u, u')(t) + q(t), \quad \text{for } \alpha \in \mathbb{R}^+, \quad \varepsilon < \frac{1}{p}
\end{align*}
\]

and

\[
\begin{align*}
    u''(t) &= u^{2k+1}(t)\|u''(t)\|_{C} \left[ \int_{a}^{b} G_2(t, s) \cdot |u''(s)|^{\|u''(s)\|_{C} + p} ds \right] + K(u, u')(t) + q(t),
\end{align*}
\]

where

\[ p \geq (b-a) \int_{a}^{b} |g(s)| + |q(s)| ds + \varepsilon, \quad \varepsilon > 0 \]

have solutions satisfying the boundary conditions (2).

Suppose now that the conditions (4) are fulfilled, and

\[ 0 \leq G_2(t, s) \leq g_1(t), \quad (t, s) \in [a, b] \times [a, b], \quad g_1 \in L_p([a, b]), \]

\[ \lambda \alpha < 1, \quad \lambda \leq p, \quad \beta > 0, \quad 0 < \alpha < p, \quad g_0 \in L_p([a, b]), \]

and

\[
\begin{align*}
    \liminf_{\rho \to +\infty} \frac{f_\rho(p)}{f_\rho(p)} > 0, \quad \limsup_{\rho \to +\infty} g_\rho(p) = +\infty.
\end{align*}
\]
Then by Theorem 1, the equations

\[ u''(t) = u^{2k+1}(t) \int_{a}^{b} G_1(t, s) \cdot [u'(s)] \left[ \int_{a}^{b} G_2(s, \tau) \cdot [u(\tau)]^p \cdot [u''(\tau)]^p \, d\tau \right]^p \, ds + \]

\[ + K(u, u')(t) + q(t), \]

\[ u''(t) = u^{2k+1}(t) \cdot [u'(t)] \ln \left( 1 + \int_{a}^{b} G_2(t, \tau) [u(\tau)]^p \cdot [u''(\tau)]^p \, d\tau \right) + K(u, u')(t) + q(t) \]

have solutions satisfying the boundary conditions (2).

**References**


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