ON RELATION BETWEEN STABILITY AND CORRECTNESS OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

(Reported on April 17, 2000)

Consider the problem

\[ dx(t) = dA(t) \cdot p(t) \cdot x(t) + df(t), \quad (1) \]

\[ x(t_0) = c_0, \quad (2) \]

where \( A: \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) and \( f: \mathbb{R}_+ \to \mathbb{R}^n \) are, respectively, the real matrix- and vector-functions with locally bounded variation components. \( p : \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) is a matrix-function locally integrable with respect to \( A; c_0 \in \mathbb{R}^n \) and \( t_0 \in \mathbb{R}_+ \).

Along with the problem (10), (2) let us consider the problem

\[ dx(t) = \tilde{dA}(t) \cdot \tilde{p}(t) \cdot x(t) + \tilde{df}(t), \quad (3) \]

\[ x(t_0) = c_0, \quad (4) \]

where \( \tilde{A} : \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) and \( \tilde{f} : \mathbb{R}_+ \to \mathbb{R}^n \) are, respectively, real matrix- and vector-functions with locally bounded variation components. \( \tilde{p} : \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) is a matrix-function locally integrable with respect to \( \tilde{A}; c_0 \in \mathbb{R}^n \) and \( t_0 \in \mathbb{R}_+ \).

Before passing to the statement of the basic results, we give some notation and definitions.

\( \mathbb{R} \equiv (-\infty, +\infty] \) is the set of real numbers, \([a, b]\) and \([a, b)\) are, respectively, closed and open intervals; \( \mathbb{R}_+ = [0, +\infty] \).

\( \mathbb{R}^{n \times m} \) is the space of all real \( n \times m \)-matrices \( x = (x_{ij})_{i=1}^m \) with the norm \( \|x\| = \max_{i=1, \ldots, m} \sum_{j=1}^n |x_{ij}| \).

\( \mathbb{R}^n - \mathbb{R}^{n \times 1} \) is a space of all real column \( n \)-vectors \( x = (x_i)_{i=1}^n \).

If \( x \in \mathbb{R}^{n \times n} \), then \( x^{-1} \) and \( \det(x) \) are, respectively, the inverse to \( x \) matrix and the determinant of \( x \); \( I_n \) is the identity \( n \times n \) matrix;

\[ V = \sup_{c \in [a, b]} \{ V(x) : c < a < b < d \} \], where \( V \) is the sum of total variations on a closed interval \([a, b]\) of components \( x_{ij} \) \((i = 1, \ldots, n; j = 1, \ldots, m)\) of the matrix-function \( x : [c, d] \to \mathbb{R}^{n \times m}; \)

\[ v(x)(t) = \left( v(x_{ij})(t) \right)_{i=1, j=1}^n \], where \( v(x_{ij})(t) = \left( \int_{c}^{t} v_{ij}(s) \, ds \right)_{i=1, j=1}^n \).

\( x(t-) \) and \( x(t+) \) are the left and the right limits of the matrix-function \( x : [c, d] \to \mathbb{R}^{n \times m} \) at the point \( t \in [c, d] \).

\( x_{ij} \) as a constant outside \([a, b]\) is assumed to be continuous.
$BV_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^{n \times m})$ is the set of all matrix-functions $x : \mathbb{R}^+ \to \mathbb{R}^{n \times m}$ of bounded variations on every closed interval from $\mathbb{R}^+$.

If $g : \mathbb{R}^+ \to \mathbb{R}$ is a nondecreasing function, $x : \mathbb{R}^+ \to \mathbb{R}$ and $0 \leq s < t < +\infty$, then

$$\int_s^t x(\tau) \, dg(\tau) = \int_s^t x(\tau) \, dg_1(\tau) - \int_s^t x(\tau) \, dg_2(\tau) + \sum_{s < \tau \leq t} x(\tau) \, d\gamma(\tau) - \sum_{s < \tau \leq t} x(\gamma) \, d\gamma(\tau),$$

where $g_j : \mathbb{R}^+ \to \mathbb{R}$ ($j = 1, 2$) are continuous nondecreasing functions such that the function $g_1 - g_2$ is identically equal to the continuous part of $g$, and $\int_{s \leq \tau \leq t} \, dg_j(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $[s,t]$ with respect to the measure corresponding to the function $g_j(j = 1, 2)$ (if $s = t$, then $\int_s^t x(\tau) \, dg(\tau) = 0$);

$L_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^n ; g)$ is the set of all functions $x : \mathbb{R}^+ \to \mathbb{R}$ measurable (i.e. measurable with respect to the measures $\mu(g_1)$ and $\mu(g_2)$) and integrable on the closed interval $[0, b]$ for every $b \in \mathbb{R}^+$.

A matrix-function $t_0$ is said to be nondecreasing if each of its components is such.

If $G = (g_{ij})_{i,j=1}^{m \times n} : \mathbb{R}^+ \to \mathbb{R}^{m \times n}$ is a nondecreasing matrix-functions, then $L(\mathbb{R}^+, \mathbb{R}^{m \times n} ; G)$ is the set of all matrix-functions $x = (x_{ij})_{i,j=1}^{m \times n} : \mathbb{R}^+ \to \mathbb{R}^{m \times n}$ such that $x_{ij} \in L(\mathbb{R}^+, \mathbb{R}; g_{ij}) (i = 1, \ldots, t ; k = 1, \ldots, n; j = 1, \ldots, m);

$$\int_s^t \, dG(\tau) \cdot x(\tau) = \left( \sum_{k=1}^n \int_s^t x_{kj}(\tau) \, dg_{kk}(\tau) \right)_{i,j=1}^{m \times n} \quad \text{for} \quad 0 \leq s \leq t < +\infty.$$

If $G_j : \mathbb{R}^+ \to \mathbb{R}^n \quad (j = 1, 2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $x : \mathbb{R}^+ \to \mathbb{R}^{n \times m}$, then

$$\int_s^t \, dG(\tau) \cdot x(\tau) = \int_s^t \, dG_1(\tau) \cdot x(\tau) - \int_s^t \, dG_2(\tau) \cdot x(\tau) \quad \text{for} \quad 0 \leq s \leq t < +\infty;$$

$$L(\mathbb{R}^+, \mathbb{R}^{n \times m} ; G) = \bigcap_{j=1}^2 L(\mathbb{R}^+, \mathbb{R}^{n \times m} ; G_j).$$

Under a solution of the system (1) is understood a vector-function $x \in BV_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^n)$ such that

$$x(t) - x(s) = \int_s^t \, dA(\tau) \cdot p(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for} \quad 0 \leq s \leq t < +\infty.$$

We will assume that $f \in BV_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^n)$, $A \in BV_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^{n \times n})$ and $p \in L_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^{n \times n}, A)$ are such that

$$\det(A_0 + (-1)^j \, dA(t) \cdot p(t)) \neq 0 \quad \text{for} \quad t \in \mathbb{R}^+ \quad (j = 1, 2). \quad (5)$$

Then the problem (1), (2) has a unique solution [see [1]].

**Definition 1.** The problem (1), (2) is said to be correct if for every arbitrarily small $\varepsilon > 0$ and arbitrarily large $\rho > 0$ there exists $\delta > 0$ such that for any $t_0 \in \mathbb{R}^+$, $\varepsilon > 0$, $\rho > 0$. 


\[ \bar{A} \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n \times n}), \bar{f} \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \] and \( \bar{p} \in L_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n \times n}, A) \) satisfying the conditions

\[
\begin{align*}
|t_0 - \tilde{t}_0| < \delta, \quad &\|c_0 - \tilde{c}_0\| < \delta, \\
|\bar{M}(t) - \bar{M}(t)| < \delta, \quad &\|\bar{f}(t) - \bar{f}(t)\| < \delta, \quad V(J - \bar{M}) < \rho
\end{align*}
\]

and

\[ \det(t_n + (-1)^j t \cos \bar{\lambda}(t) \cdot \bar{p}(t)) \neq 0 \quad \text{for} \quad t \in \mathbb{R}_+ \quad (j = 1, 2) \]

with

\[
\begin{align*}
M(t) &= \int_0^t dA(\tau) \cdot p(\tau), \quad \bar{\lambda}(t) = \int_0^t d\bar{A}(\tau) \cdot \bar{p}(\tau),
\end{align*}
\]

the inequality

\[ \|x(t) - y(t)\| < \varepsilon \quad \text{for} \quad t \in \mathbb{R}_+ \]

holds, where \( x \) and \( y \) are the solutions of the problems (1), (2) and (3), (4), respectively.

**Definition 2.** The problem (1), (2) is said to be **weakly correct** if for arbitrary \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( t_0 \in \mathbb{R}_+ \) and \( \tilde{c}_0 \in \mathbb{R}^n \), \( \bar{A} \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n \times n}), \bar{f} \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \) and \( \bar{p} \in L_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n \times n}, A) \) satisfying the conditions (6), (7) and

\[
\begin{align*}
\frac{t}{V} (M - \bar{M}) < \delta, \quad \frac{t}{V} (\bar{f} - \bar{f}) < \delta,
\end{align*}
\]

where the matrix-functions \( M \) and \( \bar{M} \) are defined by (8), the inequality (9) holds, where \( x \) and \( y \) are the solutions of the problems (1), (2) and (3), (4), respectively.

**Definition 3.** Let \( \xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a nondecreasing function such that \( \lim_{t \to +\infty} \xi(t) = +\infty \). A solution \( x \) of the system (1) is said to be **\( \xi \)-exponentially asymptotically stable** if there exists a positive number \( \eta \) such that for every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that an arbitrary solution \( y \) of the system (1) the satisfying the inequality

\[ \|x(t_0) - y(t_0)\| < \delta \]

for some \( t_0 \in \mathbb{R}_+ \), admits the estimate

\[ \|x(t) - y(t)\| < \varepsilon \exp(-\eta (\xi(t) - \xi(t_0))) \quad \text{for} \quad t \geq t_0 \]

The uniform stability of the solution \( x \) is defined just in the same way as for systems of ordinary differential equations (see, e.g., [2] or [3]).

**Definition 4.** The system (1) is said to be **uniformly stable (\( \xi \)-exponentially asymptotically stable)** if every solution of that system is uniformly stable (\( \xi \)-exponentially asymptotically stable).

**Definition 5.** A pair \((A, p)\) of matrix-functions \( A \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n \times n}) \) and \( p \in L_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n \times n}, A) \) satisfying the condition (5) is said to be **uniformly stable (\( \xi \)-exponentially asymptotically stable)** if the system

\[ dx(t) = dA(t) \cdot p(t) \cdot x(t) \]

is uniformly stable (\( \xi \)-exponentially asymptotically stable).
Theorem 1. Let $A \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n})$, $f \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^n)$, $p \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n}, A)$, and let the condition (5) hold. Moreover, let the pair $(A, p)$ be $\xi$-exponentially asymptotically stable and the conditions

$$\lim_{t \to +\infty} \sup \frac{v'(t)}{V(t)} (B) < +\infty,$$

and

$$\lim_{t \to +\infty} \frac{v'(t)}{V(t)} (B) = 0$$

hold, where

$$v(t) - \sup\{\tau \geq t : \xi(\tau) \leq \xi(t) + 1\},$$

$$B(A, p)(t) = \int_0^t dA(\tau) \cdot p(\tau) + \sum_{0 \leq \tau < t} d_1 A(\tau) \cdot p(\tau) (I_n - d_1 A(\tau) \cdot p(\tau))^{-1} \cdot d_1 A(\tau) \cdot (\tau) - \sum_{0 \leq \tau < t} d_2 A(\tau) \cdot p(\tau) (I_n + d_2 A(\tau) \cdot p(\tau))^{-1} \cdot d_2 A(\tau) \cdot (\tau),$$

$$\tilde{B}(A, p, f)(t) = f(t) + \sum_{0 \leq \tau \leq t} d_1 A(\tau) \cdot p(\tau) (I_n - d_1 A(\tau) \cdot p(\tau))^{-1} \cdot d_1 f(\tau) - \sum_{0 \leq \tau < t} d_2 A(\tau) \cdot p(\tau) (I_n + d_2 A(\tau) \cdot p(\tau))^{-1} \cdot d_2 f(\tau).$$

Then the problem (1), (2) is correct.

Theorem 2. Let $A \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n})$, $f \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^n)$, $p \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n}, A)$, and let the condition (5) hold. Moreover, the pair $(A, p)$ be uniformly stable. Then the problem (1), (2) is weakly correct.

References


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