AN EXISTENCE THEOREM FOR A CLASS OF OPTIMAL PROBLEMS WITH DELAYED ARGUMENT

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1. Statement of the Problem. An Existence Theorem

Let $J = [a, b]$ be a finite closed interval; $O \subset \mathbb{R}^n$ be an open set; $K_i$, $i = 0, 1$, $U \subset \mathbb{R}^r$, $V \subset \mathbb{R}^p$ be compact sets; for each fixed $(x_1, x_2, u_1, u_2) \in O^2 \times U^2$ let the function $f : J \times O^2 \times U^2 \to \mathbb{R}^n$ be measurable with respect to $t \in J$; for an arbitrary compact $K \subset O$ there exist measurable functions $m_K(t)$, $L_K(t)$, $t \in J$, such that

\[
|f(t, x_1, x_2, u_1, u_2)| \leq m_K(t), \quad \forall (t, x_1, x_2, u_1, u_2) \in J \times K \times U^2,
\]

\[
|f(t, x_1', x_2', u_1', u_2') - f(t, x_1', x_2', u_1, u_2)| \leq L_K(t) \sum_{i=1}^{2} |x_i' - x_i|,
\]

\[
\forall (t, x_1', x_2', u_1', u_2') \in J \times K \times U^4.
\]

Further, let the functions $\tau(t)$, $\theta(t)$, $t \in J$, be absolutely continuous and satisfy the conditions: $\tau(t) \leq t$, $\dot{\tau}(t) > 0$, $\theta(t) \leq t$, $\dot{\theta}(t) > 0$; $\Omega = \Omega(J_0, V, m, L)$ be the set of piecewise continuous functions $v : J_0 = [a_0, b_0] \to V$ satisfying the condition: for each function $v(\cdot) \in \Omega$ there exists a partition $a_0 = \xi_0 < \cdots < \xi_0 = b_0$ such that the restriction of the function $v(t)$ satisfies the Lipschitz condition on the open interval $(\xi, \xi_{i+1})$, $i = 0, \ldots, m$, i.e., $|v(t') - v(t'')| \leq L|t' - t''|$, $\forall t', t'' \in (\xi_i, \xi_{i+1})$, $i = 0, \ldots, m$, where the numbers $n$ and $L$ do not depend on $v \in \Omega$; let $\Omega_0 = \Omega([\tau(a), b], J, K_0, m_0, L_0)$, elements of this set will be denoted by $\varphi(\cdot)$; $\Omega_1 = \Omega([\theta(a), \theta(b)], J, K_1, m_1, L_1)$, its elements being denoted by $u(\cdot)$; let $\varphi^i : J \times O^2 \to \mathbb{R}^r$, $i = 0, \ldots, l$, be continuous functions.

Consider the problem:

\[
\dot{x}(t) = f(t, x(t), x(\tau(t)), u(t), u(\theta(t))), \quad t \in [t_0, t_1] \subset J, \quad u(\cdot) \in \Omega_1
\]

\[
x(t) = x_0, \quad t \in [\tau(t_0), t_0], \quad x(t_0) = x_0, \quad u(\cdot) \in \Omega_0, \quad x_0 \in K_1,
\]

\[
qu'(t_0, t_1, x_0, x(t_0)) = 0, \quad i = 0, \ldots, l,
\]

\[
qu''(t_0, t_1, x_0, x(t_0)) \to \text{min.}
\]

\[
\text{Definition 1.} \quad \text{The function } x(t) = x(t, z) \in O, \ t \in [\tau(t_0), t_1], \text{ is said to be a solution corresponding to the element } z = (t_0, t_1, x_0, \varphi(\cdot), u(\cdot)) \in A = J^2 \times K_1 \times \Omega_0 \times \Omega_1, \text{ if on }
\]

\[
[t_0, t_1] \text{ it satisfies the condition (2), while on the interval } [t_0, t_1] \text{ it is absolutely continuous and the pair } (u(\cdot), x(\cdot)) \text{ is almost everywhere (a.e.) on } [t_0, t_1] \text{ satisfies the equation (1).}
\]

\[
\text{Definition 2.} \quad \text{The element } z \in A \text{ is said to be admissible if the corresponding solution } x(t) \text{ satisfies the condition (3).}
\]

The set of admissible elements will be denoted by $\Delta$.

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**Definition 3.** The element $\mathbf{z} = (t_0, \bar{t}, \bar{x}_0, \bar{y}(\cdot), \bar{u}(\cdot)) \in \Delta$ is said to be optimal if

$$\tilde{t} - I(\mathbf{z}) = \inf_{t \in \Delta} I(t),$$

where

$$I(t) = \varphi^0(t_0, t_1, x_0, x(t_1)), \quad x(t) = x(t, z).$$

**Theorem 1.** Let the following conditions be valid:

1. $\Delta \neq \emptyset$;
2. there exists a compact set $K_2 \subset O$ such that $x(t, z) \in K_2$, $\forall z \in \Delta$.

Then there exists an optimal element.

**2. Auxiliary Lemmas**

**Lemma 1.** Let $x_k(t) = x(t, z_k), t \in [\tau(t_0^k), t_1^k]$, be the solution corresponding to the element $z_k \in A$: $t_0^k \to t_0, t_1^k \to t_1$ as $k \to \infty$. $t_0^k \geq t_1, t_1^k \leq t_1$; $K_i \subset O$, $i = 3, 4$. be compact sets with $K_3 \subset \text{int}K_4$ and $x_k(t) \in K_3$, $t \in [t_0^k, t_1^k]$. Then for sufficiently large $k$ the functional differential equation

$$\dot{y}(t) = f(t, y(t), h(t_0^k, y_k(\cdot), y_k(\cdot))(\tau(t)), u_k(t), u_k(\theta(t))),$$

$$y(t_0^k) = x_0^k,$$

where

$$h(t_0^k, y_k(\cdot), y_k(\cdot))(t) = \begin{cases} y(t), & t \in [\tau(a), t_0), \\ y(t), & t \in [t_0, t_1^k], \end{cases}$$

has a solution $y_k(t) = x_k(t) \in K_4$ defined on $[t_0, t_1^k]$ and $y_k(t) = x_k(t), t \in [t_0^k, t_1^k]$.

The proof of this lemma can be carried out in the standard way (for example, see Theorem 2 in [1]), since (5) is an ordinary differential equation for $t < t_0^k$, and is a differential equation with delayed argument for $t > t_0^k$.

**Lemma 2.** Let $v_k(\cdot) \in \Omega, k = 1, 2, \ldots$. Then there exists a subsequence of the sequence $(v_{k}(\cdot))$ such that it converges to some function $v(\cdot) \in \Omega$ for each $t \in J$, except for not more than $(m + 1)$ points.

**Proof.** By assumption the function $v_k(t), t \in (\xi_i^k, \xi_{i+1}^k)$, satisfies Lipschitz condition. From this it immediately follows the existence of one-sided limits

$$\lim_{t \to \xi_i^k-} v_k(t) = v_{k_i^-}, \quad i = 0, \ldots, q - 1, \quad \lim_{t \to \xi_i^k+} v_k(t) = v_{k_i^+}, \quad i = 1, \ldots, q.$$

We set the function

$$\omega_k(t) = \begin{cases} v_{k_i^-}, & t \leq \xi_i^k, \\ v_k(t), & t \in (\xi_i^k, \xi_{i+1}^k), \\ v_{k_i^+}, & t \geq \xi_{i+1}^k. \end{cases}$$

$$\omega_k(t) = \sum_{i=0}^{m} \chi_{k_i}(t) \omega_k(t), \quad t \in [0, t_0], \quad \omega_k(b_0) = \omega_k(b_0 -),$$

where $\chi_{k_i}(t)$ is the characteristic function of the semi-open interval $E_{k_i} = [\xi_i^k, \xi_{i+1}^k)$. Obviously, $\omega_k(\cdot) \in \Omega$ and

$$\omega_k(t) = v_k(t), \quad t \in (\xi_i^k, \xi_{i+1}^k).$$
The sequence \( \{\omega_k(t)\}_{k=1}^{\infty} \) is uniformly bounded and equicontinuous for each \( i = 0, \ldots, m \). Therefore, by virtue of Arzelà-Ascoli’s lemma, from \( \{\omega_k(t)\}_{k=1}^{\infty} \) it can be picked out a uniformly convergent subsequence which again is denoted by \( \{\omega_k(t)\}_{k=1}^{\infty} \).

Thus
\[
\lim_{k \to \infty} \omega_k(t) = \omega(t) \text{ uniformly for } t \in J_0.
\]
Without loss of generality we will assume that
\[
\lim_{k \to \infty} \xi_k^i = \xi_i^0, \quad i = 1, \ldots, q - 1.
\]
Consequently we have
\[
\lim_{k \to \infty} E_i \xi_k^i = E_i, \quad \lim_{k \to \infty} \chi_{\xi_k^i} = \chi_i, \quad t \in \mathbb{R},
\]
where \( E_i \) is an interval and \( \chi_i(t) \) is the characteristic function of the interval \( E_i \).

Therefore for each \( t \in J_0 \)
\[
\lim_{k \to \infty} \omega_k(t) = \omega(t) = \sum_{i=0}^{m} \chi_i(t)\omega_i(t),
\]
besides \( \omega(\cdot) \in \Omega \).

Taking into account (6), we can conclude that
\[
\lim_{k \to \infty} u_k(t) = \omega(t) = u_0(t), \quad t \in (\xi_i, \xi_{i+1}), \quad i = 0, 1, \ldots, m. \quad \square
\]

3. PROOF OF THEOREM

There exists a sequence \( z_k = (t^k_0, t^k_1, x^k_0, \varphi^k(\cdot), u^k(\cdot)) \in \Delta, \quad k = 1, 2, \ldots, \) such that
\[
I(z_k) \to \tilde{I}, \quad t^k_0 \to \tilde{t}_0, \quad t^k_1 \to \tilde{t}_1, \quad x^k_0 \to \tilde{x}_0 \quad \text{as} \quad k \to \infty;
\]
\[
\lim_{k \to \infty} \varphi_k(t) = \overline{\varphi}(t), \quad \text{a.e. on} \quad [\tau(a), b], \quad \overline{\varphi}(\cdot) \in \Omega_0;
\]
\[
\lim_{k \to \infty} u_k(t) = \overline{u}(t), \quad \text{a.e. on} \quad [\theta(a), b], \quad \overline{u}(\cdot) \in \Omega_2;
\]
(see Lemma 2).

Consider the case where \( t^k_0 \geq \tilde{t}_0, \quad t^k_1 \leq \tilde{t}_1 \). The remaining cases can be considered analogously.

Let \( K_5 \in O \) be a compact set, \( K_2 \in \text{int} K_5 \). For sufficiently large \( k \geq k_0 \) there exists the solution \( y_k(t) \in K_5 \) of the equation (5) defined on \([\tilde{t}_0, \tilde{t}_1]\) and \( y_k(t) = x_k(t), \quad t \in [t^k_0, t^k_1] \); (see Lemma 1).

Obviously
\[
h(t^k_0, \varphi_k(\cdot), u_k(\cdot)) \in K_6, \quad k \geq k_0, \quad t \in [\tau(\tilde{t}_0), \tilde{t}_1], \quad K_6 = K_5 \cup K_0,
\]
therefore
\[
|\tilde{y}(t)| \leq m_{K_6}(t), \quad t \in [\tilde{t}_0, \tilde{t}_1];
\]

Thus the sequence \( \{y_k(\cdot)\}_{k=0}^{\infty} \) is uniformly bounded and equicontinuous. Without loss of generality we can assume that
\[
\lim_{k \to \infty} y_k(t) = \tilde{y}(t) \text{ uniformly with } t \in [\tilde{t}_0, \tilde{t}_1].
\]

Consequently,
\[
\lim_{k \to \infty} f_k[t] = \tilde{f}[t], \quad \text{a.e.} \quad t \in [\tilde{t}_0, \tilde{t}_1],
\]
where
\[
f_k[t] = f(t, y_k(t), h(t^k_0, \varphi_k(\cdot), y_k(\cdot))(\tau(t)), u_k(t), u_k(\theta(t))),
\]
\[
\tilde{f}[t] = f(t, y(t), h(\tilde{t}_0, \tilde{\varphi}(\cdot), \tilde{y}(\cdot))(\tau(t)), \tilde{u}(t), \tilde{u}(\theta(t))).
\]
Further,
\[ y_k(t) = x_0^k + \int_{i_0}^{t} f(s) ds + \alpha_k + \beta_k(t), \quad (7) \]
where
\[ \alpha_k = \int_{i_0}^{i_1} f_k(t) dt, \quad \beta_k(t) = \int_{t_0}^{t} [f_k(s) - f(s)] ds. \]

Evidently
\[ \lim_{k \to \infty} \alpha_k = 0, \quad |\beta_k(t)| \leq \int_{t_0}^{i_1} |f_k(s) - f(s)| ds. \]

By virtue of Lebesgue’s theorem on passage to limit under the integral sign we have
\[ \lim_{k \to \infty} \beta_k(t) = 0 \text{ uniformly with } t \in [i_0, i_1]. \]

From (7) as \( k \to \infty \) we get
\[ \hat{y}(t) = \hat{x}_0 + \int_{i_0}^{t} f(s) ds. \]

It is easy to see that
\[ \lim_{k \to \infty} y_k(t) = \hat{y}(i_1), \]

therefore
\[ q'(\tilde{t}_0, i_1, \tilde{x}_0, \hat{y}(i_1)) = 0, \quad i = 1, \ldots, l, \quad \hat{I} = q^0(\tilde{t}_0, i_1, \tilde{x}_0, \hat{y}(i_1)). \]

Introduce the function
\[ \hat{x}(t) = \begin{cases} \hat{y}(t), & t \in [\tau(i_0), \tilde{t}_0), \\ \hat{y}(t), & t \in [\tilde{t}_0, i_1]. \end{cases} \]

Obviously \( \hat{z} = (\tilde{z}_0, i_1, \tilde{x}_0, \hat{x}(\cdot)) \in \Delta \) and \( \hat{I}(\tilde{z}) = \hat{I}. \)

Finally, note that the proved theorem is also valid in the case where the right-hand side of the equation (1) has the form
\[ f(t, x(\tau_1(t)), \ldots, x(\tau_s(t)); u(\theta_1(t)), \ldots, u(\theta_n(t))), \]

where the functions \( \tau_i(t), i = 1, \ldots, s, \theta_i(t), i = 1, \ldots, n, \) are absolutely continuous and satisfy the conditions \( \tau_i(t) \leq t, \quad \theta_i(t) > 0 \text{ for } t \in [i_0, i_1]. \)

If \( \Omega, \ U \) are convex sets and the points of discontinuity of the functions from the set \( \Omega, \ i = 0, 1, \) are fixed beforehand, then for the problem (1)–(4) necessary conditions of optimality are valid in the form given in [2]. In the class of measurable functions the problem of existence is studied in [3–4].

References


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