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LIDSTONE CONTINUOUS AND
DISCRETE BOUNDARY VALUE PROBLEMS
Abstract. Nonsingular continuous and discrete Lidstone boundary value problems are discussed in this paper. Existence criteria for one or more solutions are presented.

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1. Introduction

In this paper we discuss the existence of one and more solutions to Lidstone continuous and discrete boundary value problems. Problems of this type have become quite popular and many articles have appeared in the literature [4–8, 9, 13–16]. The results presented here extend, complement and improve those in the literature.

Our paper will be divided into two main sections. In section 2 we discuss the Lidstone continuous problem

\[
\begin{cases}
(-1)^n y^{(2n)}(t) = f(t, y(t)), & 0 < t < 1, \\
y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \leq i \leq n-1.
\end{cases}
\]

where \( n \geq 1 \). We begin Section 2 by presenting an existence principle for (1.1). This principle together with Krasnosel’skiǐ’s fixed point theorem in a cone will enable us to establish the existence of one or more solutions to (1.1). Throughout Section 2 we will let \( G_n(t, s) \) denote Green’s function for the boundary value problem

\[
\begin{cases}
y^{(2n)}(t) = 0 & \text{on } (0, 1), \\
y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \leq i \leq n-1.
\end{cases}
\]

Now \( G_n(t, s) \) can be expressed as [5]

\[
G_n(t, s) = \int_0^1 G(t, u) G_{n-1}(u, s) \, du,
\]

where

\[
G_n(t, s) = G(t, s) = \begin{cases} 
  t (s - 1), & 0 \leq t \leq s, \\
  s (t - 1), & s \leq t \leq 1.
\end{cases}
\]

The following inequalities have appeared in the literature [13, 14]

\[
0 \leq (-1)^n G_n(t, s) \leq \frac{1}{6^{n-1}} s (1 - s) \quad \text{for } (t, s) \in [0, 1] \times [0, 1].
\]

and for \( \delta \in (0, \frac{1}{2}) \) fixed,

\[
(-1)^n G_n(t, s) \geq \theta_n s (1 - s) \quad \text{for } (t, s) \in [\delta, 1 - \delta] \times [0, 1],
\]

where \( 0 < \theta_n < \frac{1}{6^{n-1}} \) is given by

\[
\theta_n = \delta^n \left( \frac{4 \delta^3 - 6 \delta^2 + 1}{6} \right)^{n-1}.
\]
In Section 3 we discuss the Lidstone discrete problem
\[
\begin{align*}
(-1)^m \Delta^m y(k) &= f(k, y(k)) \quad \text{for } k \in I_N, \\
\Delta^i y(0) &= \Delta^i y(N + 2m - 2i) = 0, \quad 0 \leq i \leq m - 1.
\end{align*}
\]

Here \( N \in \{1, 2, \ldots\} \), \( m \geq 1 \), \( I_N = \{0, 1, \ldots, N\} \) and \( y : I_{N+2m} = \{0, 1, \ldots, N + 2m\} \rightarrow \mathbb{R} \). Existence of one or more solutions to (1.6) is established in Section 3. Throughout Section 3 we will let \( G_m^1(k, l) \) denote Green's function for
\[
\begin{align*}
\Delta^m y &= 0 \quad \text{on } I_N, \\
\Delta^i y(0) &= \Delta^i y(N + 2m - 2i) = 0, \quad 0 \leq i \leq m - 1.
\end{align*}
\]

Now \( G_m^1 \) can be expressed as [1]
\[
G_m^1(k, l) = \sum_{i=0}^{N+2m-2} G_m(k, i) G_{m-1}(i, l),
\]
where
\[
G_m(k, l) = \begin{cases}
- \frac{(N + 2m - k)(l + 1)}{N + 2m}, & l \in \{0, 1, \ldots, k - 2\} \\
- \frac{k(N + 2m - 1 - l)}{N + 2m}, & l \in \{k - 1, \ldots, N + 2m - 2\}
\end{cases}
\]
and
\[
G_1^1(k, l) = G_1(k, l).
\]

The following inequalities have appeared in the literature [15, 16]:
\[
0 \leq (-1)^m G_m^1(k, l) \leq a_m (l + 1)(N + 1 - l) \quad \text{for } (k, l) \in I_{N+2m} \times I_N
\]
with
\[
a_m = \left[ \prod_{i=1}^{m} (N + 2i) \right]^{m-1} \prod_{i=1}^{m-1} s_{2i}, \quad (1.9)
\]
where for \( j \geq 1 \),
\[
s_j = \sum_{i=0}^{N+j} (i + 1)(N + j + 1 - i) = 4 \left( N + j + 3 \right)^3,
\]
and
\[
(-1)^m G_m^1(k, l) \geq b_m \min\{l + 1, N + 1 - l\} \quad \text{for } (k, l) \in J_N \times I_N, \quad (1.10)
\]
where \( J_n = \{1, \ldots, N + 2m - 1\} \),
\[
b_m = \left[ \prod_{i=1}^{m} (N + 2i) \right]^{m-1} \prod_{i=1}^{m-1} T_{2i-1}, \quad (1.11)
\]
with
\[
T_j = \sum_{i=1}^{N+j} \min\{i+1, N+j+2-i\} =
\begin{cases} 
\frac{(N+j)^2 + 6(N+j) + 1}{4} & \text{if } N+j \text{ odd,} \\
\frac{(N+j)(N+j+6)}{4} & \text{if } N+j \text{ even}
\end{cases}
\]
for \( j \geq 1 \). Finally we state Krasnosel’skiĭ’s Fixed Point Theorem in a cone.

**Theorem 1.1.** Let \( E = (E, \| \cdot \|) \) be a Banach space and let \( K \subset E \) be a cone in \( E \). Assume that \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1 \) and \( \overline{\Omega}_1 \subset \Omega_2 \) and let \( A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K \) be continuous and completely continuous. In addition suppose either
\[
\| Au \| \leq \| u \| \quad \text{for } u \in K \cap \partial \Omega_1 \quad \text{and} \quad \| Au \| \geq \| u \| \quad \text{for } u \in K \cap \partial \Omega_2
\]

or
\[
\| Au \| \geq \| u \| \quad \text{for } u \in K \cap \partial \Omega_1 \quad \text{and} \quad \| Au \| \leq \| u \| \quad \text{for } u \in K \cap \partial \Omega_2
\]

hold. Then \( A \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

2. **Continuous Problem**

In this section we present existence criteria for one or more solutions to (1.1). Our theory will rely on the following existence principle.

**Theorem 2.1.** Assume that
\[
f : [0, 1] \times \mathbb{R} \to \mathbb{R} \text{ is continuous.} \tag{2.1}
\]

\[
\phi \in C(0, 1) \text{ with } \phi > 0 \text{ on } (0, 1) \text{ and } \int_0^1 t (1-t) \phi(t) \, dt < \infty \tag{2.2}
\]

and
\[
\begin{cases} 
\lim_{t \to 0^+} t^2 (1-t) \phi(t) = 0 & \text{if } \int_0^1 (1-t) \phi(t) \, dt = \infty \\
\text{and } \lim_{t \to 1^-} t (1-t)^2 \phi(t) = 0 & \text{if } \int_0^1 t \phi(t) \, dt = \infty
\end{cases}
\tag{2.3}
\]

hold. Suppose there is a constant \( M > 0 \) with
\[
|y|_0 = \sup_{[0,1]} |y(t)| \neq M
\]
for any solution $y \in C^{2n-2}[0, 1] \cap C^2(0, 1)$ to
\[
\begin{aligned}
(-1)^n y^{(2n)}(t) &= \lambda \phi(t) f(t, y(t)), \quad 0 < t < 1, \\
y^{(2i)}(0) &= y^{(2i)}(1) = 0, \quad 0 \leq i \leq n-1
\end{aligned}
\] (2.4)$_\lambda$

for each $\lambda \in (0, 1)$. Then (1.1) has a solution $y \in C^{2n-2}[0, 1] \cap C^2(0, 1)$ with $\|y\|_0 \leq M$.

Proof. Solving (2.4)$_\lambda$ is equivalent to finding a solution $y \in C[0, 1]$ to
\[
y(t) = \lambda \int_0^1 (-1)^n G_n(t, s) \phi(s) f(s, y(s)) ds,
\] (2.5)$_\lambda$

where $G_n(t, s)$ is as in Section 1. \qed

Remark 2.1. From (1.3) we can see that
\[
\int_0^1 (-1)^n G_n(t, s) \phi(s) ds \leq \frac{1}{6^{n-1}} \int_0^1 s (1 - s) \phi(s) ds.
\]

Remark 2.2. Showing the equivalence of (2.4)$_\lambda$ and (2.5)$_\lambda$ is just a matter of modifying slightly the argument in [11, 12] using the ideas in [5 p. 3]. It is enough for us to note that if $y \in C[0, 1]$ and (2.1)-(2.3) are satisfied, then
\[
r_1(t) = \int_0^t (1 - t) s \phi(s) f(s, y(s)) ds + \int_t^1 t (1 - s) \phi(s) f(s, y(s)) ds = \int_0^1 (-1) G_1(t, s) \phi(s) f(s, y(s)) ds \in C[0, 1]
\]

with $r_1(0) = r_1(1) = 0$ and $-r_1'(t) = \phi(t) f(t, y(t))$ for $t \in (0, 1)$. Next note that
\[
r_2(t) = \int_0^1 (-1)^2 G_2(t, s) \phi(s) f(s, y(s)) ds = \int_0^1 G_1(t, x) \left[ \int_0^1 G_1(x, s) \phi(s) f(s, y(s)) ds \right] dx \in C^2[0, 1]
\]
with \( r_2(0) = r_2'(0) = r_2(1) = r_2'(0) = 0 \) and \( r_2'(t) = -r_1(t) \) so \( r_2^{(4)}(t) = \phi(t) f(t, y(t)) \) for \( t \in (0, 1) \). In general,

\[
r_n(t) = \int_0^1 (-1)^n G_n(t, s) \phi(s) f(s, y(s)) \, ds \in C^{n-\gamma}[0, 1]
\]

with \( r_n^{(2i)}(0) = r_n^{(2i)}(1) = 0 \) for \( 0 \leq i \leq n - 1 \) and \( r_n^{(2n)}(t) = (-1)^n \phi(t) \times f(t, y(t)) \) for \( t \in (0, 1) \).

Let \( N : C[0, 1] \rightarrow C[0, 1] \) be given by

\[
N y(t) = \int_0^1 (-1)^n G_n(t, s) \phi(s) f(s, y(s)) \, ds.
\]

We now show that \( N : C[0, 1] \rightarrow C[0, 1] \) is continuous and completely continuous. The continuity follows immediately from the Lebesgue dominated convergence theorem since

\[
|N y_n(t) - N y(t)| \leq \frac{1}{6^{n-1}} \int_0^1 s (1-s) \phi(s) |f(s, y_n(s)) - f(s, y(s))| \, ds
\]

for \( y_n, y \in C[0, 1] \). To show the complete continuity, we will use the Arzela–Ascoli theorem. To see this, let \( \Omega \subseteq C[0, 1] \) be bounded, i.e., suppose that there exists \( r_\Omega > 0 \) with \( |u| \leq r_\Omega \) for each \( u \in \Omega \). Also there exists a constant \( K_0 \) with \( |f(s, u(s))| \leq K_0 \) for \( s \in [0, 1] \) and for all \( u \in \Omega \). Now if \( u \in \Omega \) and \( t \in [0, 1] \), we have

\[
|N u(t)| \leq \frac{K_0}{6^{n-1}} \int_0^1 s (1-s) \phi(s) \, ds \quad (2.6)
\]

with

\[
|N u^{(n)}(t)| \leq L_n \int_0^t x \, dx + L_n \int_t^1 (1-x) \, dx \equiv \eta_n(t) \quad \text{if} \ n > 1 \quad (2.7)
\]

and

\[
|N u^{(n)}(t)| \leq K_0 \int_0^t s \phi(s) \, ds + K_0 \int_t^1 (1-s) \phi(s) \, ds \equiv \eta_n(t) \quad \text{if} \ n = 1; \quad (2.8)
\]

here

\[
L_n = K_\Omega \sup_{x \in [0, 1]} \int_0^1 (-1)^n G_{n-1}(x, s) \phi(s) \, ds \quad \text{if} \ n > 1.
\]
Remark 2.3. Note that (2.7) is immediate since if \( n > 1 \),

\[
N u(t) = \int_0^1 G(t, x) \left( \int_0^1 (-1)^n G_{n-1}(x, s) \phi(s) f(s, u(s)) \, ds \right) \, dx =
\]

\[= (1 - t) \int_0^1 x \left( \int_0^1 (-1)^n G_{n-1}(x, s) \phi(s) f(s, u(s)) \, ds \right) \, dx +
\]

\[+ t \int_1^t (1 - x) \left( \int_0^1 (-1)^n G_{n-1}(x, s) \phi(s) f(s, u(s)) \, ds \right) \, dx
\]

and (2.8) is immediate since if \( n = 1 \),

\[
N u(t) = (1 - t) \int_0^1 \phi(s) f(s, u(s)) \, ds + t \int_1^t (1 - s) \phi(s) f(s, u(s)) \, ds.
\]

Note that \( \eta_n \in L^1[0, 1] \) for \( n \geq 1 \). Now (2.6) together with (2.7) and (2.8) imply that \( N \Omega \) is a bounded, equicontinuous family on \([0, 1]\), so the Arzela–Ascoli theorem guarantees that \( N : C[0, 1] \to C[0, 1] \) is completely continuous. Let

\[
U = \{ u \in C[0, 1] : \| u \| < M \}.
\]

The nonlinear alternative of Leray–Schauder [3, 12] guarantees that \( N \) has a fixed point in \( U \), i.e., (1.1) has a solution \( y \in C^{2n-2}[0, 1] \cap C^{2n}(0, 1) \) with \( \| y \| \leq M \). \( \Box \)

We are now in a position to establish the existence of one or more non-negative solutions to (1.1). First we present two results which guarantee the existence of at least one solution.

**Theorem 2.2.** Suppose the following conditions are satisfied:

\[
\begin{align*}
\{ f : [0, 1] \times [0, \infty) &\to [0, \infty) \, \text{ is continuous with} \\
f(t, u) > 0 \, \text{ for } (t, u) \in [0, 1] \times (0, \infty),
\end{align*}
\]

(2.9)

\[
\phi \in C(0, 1) \, \text{ with } \phi > 0 \, \text{ on } (0, 1) \, \text{ and } \int_0^1 t (1 - t) \phi(t) \, dt < \infty, \quad (2.10)
\]

\[
\begin{align*}
\lim_{t \to 0^+} t^2 (1 - t) \phi(t) &= 0 \, \text{ if } \int_0^1 (1 - t) \phi(t) \, dt = \infty \\
\text{and } \lim_{t \to 1^-} t (1 - t) \phi(t) &= 0 \, \text{ if } \int_0^1 t \phi(t) \, dt = \infty.
\end{align*}
\]

(2.11)
\[
\begin{cases}
 f(t, u) \leq w(u) \text{ on } [0, 1] \times [0, \infty) \text{ with } w \geq 0 \\
 \text{continuous and nondecreasing on } [0, \infty)
\end{cases}
\] (2.12)

and

\[
\exists r > 0 \text{ with } \frac{r}{w(r) \sup_{t \in [0,1]} \int_0^1 (-1)^n G_n(t, s) \phi(s) \, ds} \geq 1. \tag{2.13}
\]

Then (1.1) has a solution \(y_1 \in C^{2n-2}[0,1] \cap C^{2n}(0,1)\) with \(y_1 \geq 0\) on \([0,1]\) and \(\|y_1\| < r\).

**Proof.** We will use Theorem 2.1. The idea is to look at the boundary value problem

\[
\begin{cases}
 (-1)^n y'^{2n}(t) = \lambda \phi(t) f^*(t, y(t)), & 0 < t < 1. \\
y^{2i}(0) = y^{2i}(1) = 0, & 0 \leq i \leq n - 1
\end{cases}
\] (2.14)_\lambda

for \(0 < \lambda < 1\); here

\[
f^*(t, u) = \begin{cases}
 f(t, u), & u \geq 0, \\
 f(t, 0), & u < 0.
\end{cases}
\]

Let \(y\) be any solution of (2.14)_\lambda. Then \(y(t) \geq 0\) for \(t \in [0,1]\) and

\[
y(t) = \lambda \int_0^1 (-1)^n G_n(t, s) \phi(s) f^*(s, y(s)) \, ds \leq \leq w(\|y\|) \sup_{t \in [0,1]} \int_0^1 (-1)^n G_n(t, s) \phi(s) \, ds
\]

for \(t \in [0,1]\). Consequently

\[
\frac{\|y\|}{w(\|y\|) \sup_{t \in [0,1]} \int_0^1 (-1)^n G_n(t, s) \phi(s) \, ds} \leq 1. \tag{2.15}
\]

Now (2.13) and (2.15) imply \(\|y\| \neq r\). Thus Theorem 2.1 guarantees that (2.14)_1 has a solution \(y_1\) with \(\|y_1\| < r\) (note that \(\|y_1\| \leq r\) by Theorem 2.1 but \(\|y\| \neq r\) by an argument similar to the one above). In fact, \(0 \leq y_1(t) \leq r\) for \(t \in [0,1]\) and so \(y_1\) is a solution of (1.1).

In Theorem 2.2 note that it is possible to have \(y_1\) with \(\|y_1\| = 0\) in some application. We remove this situation in the next theorem.
Theorem 2.3. Suppose (2.9)-(2.13) are satisfied. In addition assume that the following conditions hold:

\[
\begin{align*}
\text{there exists } \delta \in \left(0, \frac{1}{2}\right) \text{ (choose and fix it) and } \tau & \in C[\delta, 1 - \delta] \\
\text{with } \tau > 0 \text{ on } [\delta, 1 - \delta] \text{ and with } \phi(t)f(t,u) \geq \tau(t)u(u) & \quad \text{(2.16)}
\end{align*}
\]

and

\[
\exists R > r \text{ with } \frac{R}{w(6^n - 1, \theta_n, R)} \leq \int_{\delta}^{1-\delta} (-1)^n G_n(\sigma, s) \tau(s) \, ds; \quad \text{(2.17)}
\]

here \(0 \leq \sigma \leq 1\) is such that

\[
\int_{\delta}^{1-\delta} (-1)^n G_n(\sigma, s) \tau(s) \, ds = \sup_{\tau \in [0, 1]} \int_{\delta}^{1-\delta} (-1)^n G_n(t, s) \tau(s) \, ds \quad \text{(2.18)}
\]

and \(0 < \theta_n < \frac{1}{\sqrt{1-\delta}}\) is as in (1.5). Then (1.1) has a solution \(y_2 \in C^{2n-2}[0, 1] \cap C^2(0, 1)\) with \(y_2 \geq 0\) on \([0, 1]\), \(y_2(t) > 0\) for \(t \in [\delta, 1 - \delta]\) and \(r < |y_2|_0 \leq R\).

Proof. To show the existence of \(y_2\), we use Theorem 1.1. Let \(E = (C[0, 1], \|\cdot\|_0)\) and

\[
K = \{u \in C[0, 1] : u(t) \geq 0 \text{ for } t \in [0, 1] \text{ and } \min_{t \in [\delta, 1-\delta]} u(t) \geq 6^n - 1, \theta_n|u|_0\}.
\]

Clearly \(K\) is a cone of \(E\). Let \(A : K \rightarrow C[0, 1]\) be defined by

\[
Au(t) = \int_{0}^{1} (-1)^n G_n(t, s) \phi(s) f(s, u(s)) \, ds.
\]

The argument in Theorem 2.1 implies that \(A : K \rightarrow C[0, 1]\) is continuous and completely continuous. We now show that \(A : K \rightarrow K\). If \(u \in K\), then clearly \(Au(t) \geq 0\) for \(t \in [0, 1]\). Also for \(t \in [0, 1]\) we have from (1.3) that

\[
Au(t) \leq \frac{1}{6^n - 1} \int_{0}^{1} s(1-s) \phi(s) f(s, u(s)) \, ds,
\]

and so

\[
|Au|_0 \leq \frac{1}{6^n - 1} \int_{0}^{1} s(1-s) \phi(s) f(s, u(s)) \, ds. \quad \text{(2.19)}
\]
In addition, (1.4) and (2.19) yield
\[
\min_{t \in [\delta, 1]} A u(t) = \min_{t \in [\delta, 1-\delta]} \int_{0}^{1} (1)^n G_n(t, s) \phi(s) f(s, u(s)) \, ds \geq \\
\geq \theta_n \int_{0}^{1} s (1-s) \phi(s) f(s, u(s)) \, ds \geq 6^{n-1} \theta_n |A u|_0.
\]
Consequently \( A u \in K \), so \( A : K \to K \). Let
\[
\Omega_1 = \{ u \in C[0, 1] : |u|_0 < r \} \quad \text{and} \quad \Omega_2 = \{ u \in C[0, 1] : |u|_0 < R \}.
\]
We first show
\[
|A u|_0 \leq |u|_0 \quad \text{for} \quad u \in K \cap \partial \Omega_1. \quad (2.20)
\]
To see this, let \( u \in K \cap \partial \Omega_1 \), so \( |u|_0 = r \). Then (2.12) and (2.13) imply for all \( t \in [0, 1] \) that
\[
A u(t) \leq w(|u|_0) \int_{0}^{1} (1)^n G_n(t, s) \phi(s) \, ds \leq \\
\leq w(r) \sup_{t \in [0, 1]} \int_{0}^{1} (1)^n G_n(t, s) \phi(s) \, ds < r = |u|_0.
\]
Thus \( |A u|_0 < |u|_0 \), and so (2.20) is true. Next we show
\[
|A u|_0 \geq |u|_0 \quad \text{for} \quad u \in K \cap \partial \Omega_2. \quad (2.21)
\]
To see this, let \( u \in K \cap \partial \Omega_2 \), so \( |u|_0 = R \) and \( \min_{t \in [\delta, 1-\delta]} u(t) \geq 6^{n-1} \theta_n |u|_0 = 6^{n-1} \theta_n R \) so in particular \( u(t) \in [6^{n-1} \theta_n R, R] \) \( t \in [\delta, 1-\delta] \). Now with \( \sigma \) as defined in (2.18) we have from (2.16) and (2.17) that
\[
A u(\sigma) = \int_{0}^{1} (1)^n G_n(\sigma, s) \phi(s) f(s, u(s)) \, ds \geq \\
\geq \int_{\delta}^{1-\delta} (1)^n G_n(\sigma, s) \phi(s) f(s, u(s)) \, ds \geq \\
\geq \int_{\delta}^{1-\delta} (1)^n G_n(\sigma, s) \phi(s) \tau(s) w(u(s)) \, ds \geq \\
\geq w(6^{n-1} \theta_n R) \int_{\delta}^{1-\delta} (1)^n G_n(\sigma, s) \phi(s) \tau(s) \, ds \geq R = |u|_0.
\]
Thus $|Au| \geq |u|$ and so (2.21) holds. Now Theorem 1.1 implies that $A$ has a fixed point $y_2 \in K \cap (\Omega_2 \setminus \Omega_1)$, i.e., $r \leq |y_2|_0 \leq R$. In fact, $r < |y_1|_0$ (argue as in the first part of the theorem). Also $y_2 \geq 0$ on $[0, 1]$ and since $y_2 \in K$, we have $y_2(t) > 0$ for $t \in [\delta, 1 - \delta]$ since $|y_2|_0 > r$. □

Remark 2.4. If in (2.17) we have $R < r$, then (1.1) has a solution $y \in C[0, 1]$ with $R \leq |y|_0 < r$. The argument is essentially the same as that in Theorem 2.3 except here we use the other half of Theorem 1.1.

Theorem 2.4. Suppose (2.9)-(2.13), (2.16) and (2.17) hold. Then (1.1) has two solutions $y_1, y_2 \in C^{2, \mu-2}[0, 1] \cap C^{2, \mu}(0, 1)$ with $y_1, y_2 \geq 0$ on $[0, 1]$, $y_2(t) > 0$ for $t \in [\delta, 1 - \delta]$ and $0 \leq |y_1|_0 < r < |y_2|_0 \leq R$.

Proof. The existence of $y_1$ follows from Theorem 2.2 and of $y_2$ from Theorem 2.3. □

In Theorem 2.4 it is possible to have $|y_1|_0$ to be zero in some applications. Our next theorem guarantees the existence of two solutions $y_1, y_2 \in C^{2, \mu-2}[0, 1] \cap C^{2, \mu}(0, 1)$ with $0 < |y_1|_0 < r < |y_2|_0 \leq R$.

Theorem 2.5. Suppose (2.9)-(2.13), (2.16) and (2.17) hold. In addition assume that

$$\exists L, 0 < L < r \text{ with } \frac{L}{u(6^2-1)|y_1|_0} \leq \int_0^1 (-1)^{2\nu}G_\nu(s)\tau(s)ds \quad (2.22)$$

is satisfied. Then (1.1) has two solutions $y_1, y_2 \in C^{2, \mu-2}[0, 1] \cap C^{2, \mu}(0, 1)$ with $y_1, y_2 \geq 0$ on $[0, 1]$, $y_2(t) > 0$ for $t \in [\delta, 1 - \delta]$ and $0 < L \leq |y_1|_0 < r < |y_2|_0 \leq R$.

Proof. The existence of $y_2$ follows from Theorem 2.3 and of $y_1$ from Remark 2.4. □

Remark 2.5. It is easy to use Theorem 2.3 and Remark 2.4 to write a theorem which guarantees the existence of more than two solutions to (1.1). We leave the details to the reader.

Example. Consider the boundary value problem

$$\begin{cases}
y^{(6)} + (y^\alpha + y^\beta + 1) = 0 \quad \text{on} \quad (0, 1), \\
y(0) = y'(0) = y^{(4)}(0) = y(1) = y'(1) = y^{(4)}(1) = 0
\end{cases} \quad (2.23)$$

with $0 < \alpha < 1 < \beta$. Then (2.23) has two solutions $y_1, y_2 \in C^4[0, 1] \cap C^0(0, 1)$ (in fact in $C^6[0, 1]$) with $y_1 > 0$ on $[0, 1]$, $y_2 > 0$ on $[0, 1]$ and $0 < |y_1|_0 < 1 < |y_2|_0$. 


To show the above, we will apply Theorem 2.5 with \( \phi = \tau = 1 \), \( n = 3 \), 
\( w(x) = x^\alpha + x^\beta + 1 \). \( r = 1 \) and \( \delta = \frac{1}{2} \). Note that (2.9), (2.10), (2.11),
(2.12) and (2.16) hold. Also since
\[( -1 )^3 G_3(t, s) \leq \frac{1}{36} s (1 - s) . \]
we have
\[ \sup_{t \in [0, 1]} \int_0^1 ( -1 )^3 G_3(t, s) \phi(s) ds \leq \frac{1}{36} \int_0^1 s (1 - s) ds = \frac{1}{216} . \]
Next note that (2.13) holds with \( r = 1 \) since
\[ \frac{r}{w(r) \sup_{t \in [0, 1]} \int_0^1 ( -1 )^3 G_3(t, s) \phi(s) ds} = \frac{216}{3} > 1 . \]
Now since \( \beta > 1 \), we have
\[ \lim_{x \to \infty} \frac{x}{w(36 \theta_3 x)} = \lim_{x \to \infty} \frac{x}{(36 \theta_3 x)^\alpha + (36 \theta_3 x)^\beta + 1} = 0 , \]
so there exists \( R > r = 1 \) with (2.17) holding. Finally note that
\[ \lim_{x \to 0} \frac{x}{w(36 \theta_3 x)} = \lim_{x \to 0} \frac{x}{(36 \theta_3 x)^\alpha + (36 \theta_3 x)^\beta + 1} = 0 , \]
so there exists \( L, 0 < L < 1 \), with (2.22) holding. Theorem 2.5 now guarantees that (2.23) has two solutions \( y_1, y_2 \in C^4([0, 1]) \cap C^5(0, 1) \) with \( y_1 \geq 0, y_2 \geq 0 \) on \( [0, 1] \), \( y_1(t) > 0 \) and \( y_2(t) > 0 \) for \( t \in [\frac{1}{3}, \frac{2}{3}] \) and \( 0 < |y_1|_0 < 1 < |y_2|_0 \). The extra regularity and the fact that \( y_1(t) > 0 \) and \( y_2(t) > 0 \) for \( t \in (0, 1) \) follows immediately from the integral representation of \( y_1 \) and \( y_2 \).

3. Discrete Problem

In this section we discuss the discrete problem (1.6). We first obtain an existence principle for (1.6). For convenience we note here that by a solution to (1.6) we mean a \( w \in C(I_{N+2m}) \) such that \( w \) satisfies the difference equation and the boundary data in (1.6). Recall that \( C(I_{N+2m}) \) denotes the class of maps \( w \) continuous on \( I_{N+2m} \) (discrete topology) with the norm \( |w|_0 = \max_{x \in I_{N+2m}} |w(k)| \).

**Theorem 3.1.** Assume that \( f : I_N \times \mathbb{R} \to \mathbb{R} \) is continuous (i.e., continuous as a map from the topological space \( I_N \times \mathbb{R} \) into the topological space \( \mathbb{R} \)) (of course the topology on \( I_N \) is the discrete topology). Suppose there is a constant \( M > 0 \) with
\[ |y|_0 = \max_{k \in I_{N+2m}} |y(k)| \neq M . \]
for any solution $y \in C(I_{N+2m})$ to

$$\begin{cases} (-1)^m \Delta^2 y(k) = \lambda f(k, y(k)) & \text{for } k \in I_N, \\
\Delta^2 y(0) = \Delta^2 y(N + 2m - 2i) = 0, & 0 \leq i \leq m - 1 \end{cases}$$

(3.1)\lambda

for each $\lambda \in (0, 1)$. Then (1.6) has a solution $y \in C(I_{N+2m})$ with $|y|_0 \leq M$.

Proof. Solving (3.1)\lambda is equivalent to finding a $y \in C(I_{N+2m})$ to

$$y(k) = \lambda \sum_{l=0}^{N} (-1)^m G^1_{m}(k, l) f(l, y(l)) \quad \text{for } k \in I_{N+2m}, \quad (3.2)\lambda$$

where $G^1_{m}$ is as in Section 1. Define the operator $N : C(I_{N+2m}) \to C(I_{N+2m})$ by setting

$$N y(t) = \sum_{l=0}^{N} (-1)^m G^1_{m}(k, l) f(l, y(l)).$$

It is easy to see [2, 3] that $N : C(I_{N+2m}) \to C(I_{N+2m})$ is continuous and completely continuous. Let

$$U = \{u \in C(I_{N+2m}) : |u|_0 < M\} \quad \text{and} \quad E = C(I_{N+2m}).$$

The nonlinear alternative of Leray-Schauder [3, 12] guarantees that $N$ has a fixed point in $\overline{U}$, i.e., (1.6) has a solution $y \in C(I_{N+2m})$ with $|y|_0 \leq M$. \qed

Remark 3.1. It is clear that an existence principle could also be established for

$$\begin{cases} (-1)^m \Delta^2 y(k) = \lambda f(k, y(k), y(k + 1), \ldots, y(k + 2 m - 1)) & \text{for } k \in I_N, \\
\Delta^2 y(0) = \Delta^2 y(N + 2m - 2i) = 0, & 0 \leq i \leq m - 1. \end{cases}$$

We leave the details to the reader.

Theorem 3.2. Suppose the following conditions are satisfied:

$$f : I_N \times [0, \infty) \to [0, \infty) \text{ is continuous with } f(i, u) > 0$$

for $(i, u) \in I_N \times (0, \infty)$, \quad (3.3)

$$\begin{cases} f(k, u) \leq q(k) u(k) & \text{on } I_N \times [0, \infty) \text{ with } q : I_N \to (0, \infty) \\
\text{and } w \geq 0 \text{ continuous and nondecreasing on } [0, \infty) \end{cases}$$

(3.4)

and

$$\exists \ r > 0 \text{ with } \frac{w(r) \max_{k \in I_{N+2m}} \sum_{l=0}^{r} (-1)^m G^1_{m}(k, l) q(l)}{r} > 1. \quad (3.5)$$

Then (1.6) has a solution $y \in C(I_{N+2m})$ with $y_i \geq 0$ on $I_{N+2m}$ and $|y|_0 < r$. 
Proof. The idea is to use Theorem 3.1, so look at
\[
\begin{cases}
(-1)^\mu \Delta^2 y(k) = \lambda f^*(k, y(k)) & \text{for } k \in I_N, \\
\Delta^2 y(0) = \Delta^2 y(N + 2m - 2i) = 0, & 0 \leq i \leq m - 1
\end{cases}
\] (3.6)
for $0 < \lambda < 1$; here
\[
f^*(k, u) = \begin{cases}
f(k, u), & u \geq 0, \\
f(k, 0), & u < 0.
\end{cases}
\]

Let $y$ be any solution of (3.6)\(_\lambda\). Then
\[
y(k) = \lambda \sum_{i=0}^{N} (-1)^\mu G_{m}(k, l) f^*(l, y(l)),
\]
so $y(k) \geq 0$ for $k \in I_{N+2m}$ and
\[
|y(k)| \leq w(|y|) \max_{k \in I_{N+2m}} \sum_{i=0}^{N} (-1)^\mu G_{m}(k, l) q(l) \text{ for } k \in I_{N+2m}.
\]
Consequently
\[
\frac{|y|}{w(|y|) \max_{k \in I_{N+2m}} \sum_{i=0}^{N} (-1)^\mu G_{m}(k, l) q(l)} \leq 1. \tag{3.7}
\]
Now (3.5) and (3.7) imply $|y|_0 \neq r$. Thus Theorem 3.1 guarantees that (3.6)\(_1\) has a solution $y_1 \in C(I_{N+2m})$ with $|y_1|_0 < r$ (note that $|y|_0 \neq r$ by an argument similar to the one above).

Note that in some application $|y_1|_0$ may be zero in Theorem 3.2. We remove this situation in the next result.

**Theorem 3.3.** Suppose (3.3)–(3.5) are satisfied. In addition assume that the following conditions hold:
\[
\begin{cases}
\text{there exists } \tau : K_N = \{1, 2, \ldots, N\} \rightarrow (0, \infty) \\
\text{with } f(i, u) \geq \tau(i) u(u) \text{ on } K_N \times (0, \infty)
\end{cases}
\] (3.8)
and
\[
\exists R > r \text{ with } \frac{R}{w(t_{\frac{3m}{\infty}, c_0} R)} \leq \sum_{i=1}^{N} (-1)^\mu G_{m}(\sigma, l) \tau(l); \tag{3.9}
\]
here $\sigma \in J_N = \{1, \ldots, N + 2m - 1\}$ is such that
\[
\sum_{i=1}^{N} (-1)^\mu G_{m}(\sigma, l) \tau(l) = \max_{k \in J_N} \sum_{i=1}^{N} (-1)^\mu G_{m}(k, l) \tau(l). \tag{3.10}
\]
and

\[ c_0 = \min_{l \in I_N} \left[ \frac{\min\{l + 1, N + 1 - l\}}{(l + 1)(N + 1 - l)} \right] > 0 \]  \hspace{1cm} (3.11)

with \( a_m \) as in (1.9) and \( b_m \) as in (1.11). Then (1.6) has a solution \( y_2 \in C(I_{N+2m}) \) with \( y_2(k) > 0 \) for \( k \in J_N \) and \( r < |y_2|_0 \leq R \).

**Proof.** To show the existence of \( y_2 \), we use Theorem 1.1. Let \( E = (C(I_{N+2m}), \cdot, | \cdot |_0) \) and

\[ K = \left\{ u \in C(I_{N+2m}) \mid u(i) \geq 0 \text{ for } i \in I_{N+2m} \right\} \]

Let \( K = C(I_{N+2m}) \) be defined by

\[ K = \left\{ u \in C(I_{N+2m}) \mid u(i) \geq 0 \text{ for } i \in I_{N+2m} \right\} \]

To show \( A : K \rightarrow C(I_{N+2m}) \) be defined by

\[ A u(k) = \sum_{l=0}^{N} (-1)^m G_m^l(k, l) f(l, u(l)) \]

To show \( A : K \rightarrow K \), let \( u \in K \). Then \( A u(k) \geq 0 \) for \( k \in I_{N+2m} \). Also (1.8) implies for \( k \in I_{N+2m} \) that

\[ A u(k) \leq a_m \sum_{l=0}^{N} (l + 1)(N + 1 - l) f(l, u(l)) \]

and so

\[ |A u|_0 \leq a_m \sum_{l=0}^{N} (l + 1)(N + 1 - l) f(l, u(l)) \]  \hspace{1cm} (3.12)

In addition (1.10) and (3.12) imply

\[ \min_{k \in J_N} A u(k) = \min_{k \in J_N} \sum_{l=0}^{N} (-1)^m G_m^l(k, l) f(l, u(l)) \geq \]

\[ \geq b_m \sum_{l=0}^{N} \min\{l + 1, N + 1 - l\} f(l, u(l)) \geq \]

\[ \geq b_m c_0 \sum_{l=0}^{N} (l + 1)(N + 1 - l) f(l, u(l)) \geq \frac{b_m}{a_m} c_0 |A u|_0 \]

Consequently \( A u \in K \) so \( A : K \rightarrow K \). Let

\[ \Omega_1 = \{ u \in C(I_{N+2m}) \mid |u|_0 < r \} \quad \text{and} \quad \Omega_2 = \{ u \in C(I_{N+2m}) \mid |u|_0 < R \} \]

We first show

\[ |A u|_0 \leq |u|_0 \quad \text{for } u \in K \cap \partial \Omega_1. \]  \hspace{1cm} (3.13)
Let \( u \in K \cap \partial \Omega_1 \), so \(|u|_0 = r\). Now (3.4) and (3.5) imply for \( k \in I_{N+2m} \) that

\[
A u(l) \leq \sum_{l=0}^{N} (-1)^m G_{m}^1(k, l) q(l) u(l) \leq w(r) \sup_{k \in I_{N+2m}} \sum_{l=0}^{N} (-1)^m G_{m}^1(k, l) q(l) < r = |u|_0.
\]

Thus \(|Au|_0 < r = |u|_0\) and so (3.13) is true. Next we show

\[
|Au|_0 \geq |u|_0 \quad \text{for} \quad u \in K \cap \partial \Omega_2.
\]

(3.14)

Let \( u \in K \cap \partial \Omega_2 \), so \(|u|_0 = R\), and \( \min_{k \in J_N} u(k) \geq \frac{b_{2m}}{a_{m}} c_0 R \), in particular,

\[
u(k) \in \left[ \frac{b_{2m}}{a_{m}} c_0 R, R \right] \quad \text{for} \quad k \in J_N.
\]

It is easy to see that \( 0 < \frac{b_{2m}}{a_{m}} c_0 < 1\). Now (3.8) and (3.9) (here \( \sigma \) is as in (3.10)) imply

\[
A u(\sigma) = \sum_{l=0}^{N} (-1)^m G_{m}^1(\sigma, l) f(l, u(l)) \geq \sum_{l=1}^{N} (-1)^m G_{m}^1(\sigma, l) f(l, u(l)) \geq \sum_{l=1}^{N} (-1)^m G_{m}^1(\sigma, l) \tau(l) u(l) \geq w\left(\frac{b_{m}}{a_{m}} c_0 R\right) \sum_{l=1}^{N} (-1)^m G_{m}^1(\sigma, l) \tau(l) \geq R = |u|_0.
\]

Thus \(|Au|_0 \geq |u|_0\) and so (3.14) is true. Now Theorem 1.1 guarantees that \( A \) has a fixed point \( y_2 \in K \cap (\Omega_2 \setminus \Omega_1) \), i.e., \( r \leq |y_2|_0 \leq R \). In fact \(|y_2|_0 > r\) (argue as in the first part of the theorem). Also \( y_2 \geq 0 \) on \( I_{N+2m} \) and \( y_2(k) > 0 \) for \( k \in J_N \) since \( y_2 \in K \) and \(|y_2|_0 > r\). \( \square \)

Remark 3.2. If in (3.9) we have \( R < r \), then (1.6) has a solution \( y_2 \in C(I_{N+2m}) \) with \( R \leq |y_2|_0 < r\).

**Theorem 3.4.** Suppose (3.3)-(3.5), (3.8) and (3.9) hold. Then (1.6) has two solutions \( y_1, y_2 \in C(I_{N+2m}) \) with \( y_1 \geq 0 \) on \( I_{N+2m} \), \( y_2(k) > 0 \) for \( k \in J_N \) and \( 0 \leq |y_1|_0 < r < |y_2|_0 \leq R \).

**Proof.** The existence of \( y_1 \) follows from Theorem 3.2 and of \( y_2 \) from Theorem 3.3. \( \square \)

In Theorem 3.4 it is possible for \(|y_1|_0 \) to be zero.
Theorem 3.5. Suppose (3.3)–(3.5), (3.8) and (3.9) hold. In addition assume that

\[ \exists L, \ 0 < L < r \quad \text{with} \quad \frac{L}{\nu \left( \frac{a_m}{c_0} L \right)} \leq \sum_{i=1}^{N} (-1)^m C_m^1 \sigma_i \tau_j \]  

(3.15)

is satisfied; here \( \sigma \) is as in (3.10), \( c_0 \) is as in (3.11), \( a_m \) is as in (1.9) and \( b_m \) is as in (1.11). Then (1.6) has two solutions \( y_1, y_2 \in C(I_{N+2m}) \) with \( y_1(k) > 0, \ y_2(k) > 0 \) for \( k \in J_N \) and \( 0 < L \leq \|y_1\|_0 < r < \|y_2\|_0 \leq R \).

Proof. The existence of \( y_2 \) follows from Theorem 3.3 and of \( y_1 \) from Remark 3.2. \( \square \)

References


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