ON NONNEGATIVE BOUNDED SOLUTIONS OF SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Statement of the Problem and Formulation of the Main Results

Let \( \mathbb{R} \) be the set of real numbers. \( C_{00c}(\mathbb{R}, \mathbb{R}) \) be the space of continuous functions \( u : \mathbb{R} \to \mathbb{R} \) with the topology of uniform convergence on every compact interval \( C_{00c}(\mathbb{R}; \mathbb{R}) = \{ u \in C_{00c}(\mathbb{R}; \mathbb{R}) : u(t) \geq 0 \text{ for } t \in \mathbb{R} \} \). \( L_{00c}(\mathbb{R}, \mathbb{R}) \) be the space of locally summable functions \( u : \mathbb{R} \to \mathbb{R} \) with the topology of convergence in the mean on every compact interval, and \( L_{00c}(\mathbb{R}; \mathbb{R}) = \{ u \in L_{00c}(\mathbb{R}; \mathbb{R}) : u(t) \geq 0 \text{ for almost all } t \in \mathbb{R} \} \). Consider the system of differential equations

\[
x'_i(t) = p_i(t)x_i(t) + \sum_{k=1}^{n} \ell_{ik}(x_k(t)) + q_i(t) \quad (i = 1, \ldots, n),
\]

where \( \ell_{ik} : C_{00c}(\mathbb{R}, \mathbb{R}) \to L_{00c}(\mathbb{R}, \mathbb{R}) (i, k = 1, \ldots, n) \) are linear continuous operators, \( p_i \) and \( q_i \in L_{00c}(\mathbb{R}, \mathbb{R}) (i = 1, \ldots, n) \). Moreover, there exist linear positive operators \( \ell_{ik} : C_{00c}(\mathbb{R}, \mathbb{R}) \to L_{00c}(\mathbb{R}, \mathbb{R}) (i, k = 1, \ldots, n) \) such that for any \( u \in C_{00c}(\mathbb{R}, \mathbb{R}) \) the inequalities

\[
|\ell_{ik}(u)(t)| \leq \ell_{ik}(|u|)(t) \quad (i, k = 1, \ldots, n)
\]

are fulfilled almost everywhere on \( \mathbb{R} \).

The simple but important case of (1) is the system of differential equations with deviating arguments

\[
x'_i(t) = \sum_{k=1}^{n} \sum_{j=1}^{m} p_{ikj}(t)x_k(\tau_{ikj}(t)) + q_i(t) \quad (i = 1, \ldots, n),
\]

where \( q_i \) and \( p_{ikj} \in L_{00c}(\mathbb{R}; \mathbb{R}) \), \( \tau_{ikj} : \mathbb{R} \to \mathbb{R} \) are measurable functions, and \( \tau_{i1}(t) \equiv t \).

A locally absolutely continuous vector function \( \{x_i\}_{i=1}^{n} : \mathbb{R} \to \mathbb{R} \) is called a nonnegative bounded solution of the system (1) if it satisfies this system almost everywhere on \( \mathbb{R} \),

\[
\sup \left\{ \sum_{i=1}^{n} |x_i(t)| : t \in \mathbb{R} \right\} < +\infty,
\]

and

\[
x_i(t) \geq 0 \quad \text{for } t \in \mathbb{R} \quad (i = 1, \ldots, n).
\]

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I. Kiguradze [3, 8] has established optimal in some sense sufficient conditions of the existence and uniqueness of nonnegative bounded solutions of the differential system

$$\frac{dx_i(t)}{dt} = \sum_{h=1}^{n} p_{ih}(t)x_h(t) + q_i(t) \quad (i = 1, \ldots, n).$$

In the present paper these results are generalized for the systems (1) and (1').

Before formulating the main results we want to introduce some notation. 

$\delta_{ik}$ is Kronecker's symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ik} = 0$ for $i \neq k$.

$A = (a_{ik})_{i,k=1}^{n}$ is a $n \times n$ matrix with components $a_{ik}$ ($i, k = 1, \ldots, n$).

$r(A)$ is the spectral radius of the matrix $A$.

$\mathcal{P}_{\mathbb{R}}$ is the set of linear operators mapping $C_{a.e.}(\mathbb{R}; \mathbb{R}^+) \rightarrow L_{a.e.}(\mathbb{R}; \mathbb{R}^+)$.

If $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \ldots, n$), then

$$\mathcal{N}_0(t_1, \ldots, t_n) = \{i : t_i \in \mathbb{R}\}.$$

If $u \in L_{a.e.}(\mathbb{R}, \mathbb{R})$, then

$$\eta(u)(t, s) = \int_{t}^{s} u(\xi) \, d\xi$$

for $t$ and $s \in \mathbb{R}$.

For $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \ldots, n$) put

$$\sigma_i(t) = \text{sgn}(t - t_i) \quad \text{if } t_i \in \mathbb{R},$$

$$\sigma_i(t) \equiv 1 \quad \text{if } t_i = -\infty, \quad \sigma_i(t) \equiv -1 \quad \text{if } t_i = +\infty.$$

**Theorem 1.** Let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \ldots, n$), a matrix $A = (a_{ik})_{i,k=1}^{n} \in \mathbb{R}^{n \times n}$, and a nonnegative number $\alpha$ such that

$$r(A) < 1,$$

$$\left| \int_{t_i}^{t} \exp \left( \int_{s}^{t} p_i(\xi) \, d\xi \right) \|a_k(1)\| \, ds \right| \leq a_{ik} \quad \text{for } t \in \mathbb{R} \quad (i, k = 1, \ldots, n),$$

$$\sum_{i=1}^{n} \left| \int_{t_i}^{t} \exp \left( \int_{s}^{t} p_i(\xi) \, d\xi \right) \|a_i(1)\| \, ds \right| \leq \alpha \quad \text{for } t \in \mathbb{R}$$

and

$$\sup_{t_i \in \mathcal{N}_0(t_1, \ldots, t_n)} \left\{ \int_{t_i}^{t} p_i(\xi) \, d\xi : t \in \mathbb{R} \right\} < +\infty \quad \text{for } i \in \mathcal{N}_0(t_1, \ldots, t_n).$$

Let, moreover, $\sigma_i \in \mathcal{P}_{\mathbb{R}}$; $\sigma_i \in L_{a.e.}(\mathbb{R}; \mathbb{R}^+)$. Then for any $c_i \in \mathbb{R}_+$ ($i \in \mathcal{N}_0(t_1, \ldots, t_n)$) the system (1) has at least one nonnegative bounded solution satisfying

$$z_i(t_i) = c_i \quad \text{for } i \in \mathcal{N}_0(t_1, \ldots, t_n).$$

**Theorem 2.** Let all the assumptions of Theorem 1 be fulfilled and

$$\liminf_{t \to t_i} \int_{t_i}^{t} p_i(\xi) \, d\xi = -\infty \quad \text{for } i \in \mathbb{N} \setminus \mathcal{N}_0(t_1, \ldots, t_n).$$
Then for any \( c_i \in \mathbb{R}_+ \ i \in N_0[t_1, \ldots, t_n] \) the system (1) has a unique bounded solution satisfying (6), and this solution is nonnegative.

If \( t_i \in \{-\infty, +\infty\} \ i = 1, \ldots, n \), then \( N_0(t_1, \ldots, t_n) = \emptyset \). In that case in Theorems 1 and 2 the conditions (5) and (6) become unnecessary. Consequently, these theorems are formulated as follows:

**Corollary 1.** Let there exist \( t_i \in \{-\infty, +\infty\} \ i = 1, \ldots, n \), a matrix \( A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n} \) and a nonnegative number a such that the conditions (2) – (4) are fulfilled. Let moreover, \( \sigma_i \in P_1 \sigma_i q_i \in L_{ loc}(\mathbb{R}; \mathbb{R}_+) \). Then the system (1) has at least one nonnegative bounded solution.

**Corollary 2.** Let all the assumptions of Corollary 1 be fulfilled and

\[
\liminf_{t \to t_i} \int_0^t p_i(\xi) d\xi = -\infty \quad (i = 1, \ldots, n).
\]

Then the system (1) has a unique bounded solution, and this solution is nonnegative.

The above theorems yield the following statements for the system \((5.1')\).

**Corollary 1'.** Let \( t_i \in \mathbb{R} \cup \{-\infty, +\infty\} \ i = 1, \ldots, n \),

\[
(1 - \delta_i \delta_j) \sigma_{ij} p_{ij} q_{ij} \in L_{ loc}(\mathbb{R}; \mathbb{R}_+) \quad \sigma_i q_i \in L_{ loc}(\mathbb{R}; \mathbb{R}_+) \quad (i, k = 1, \ldots, n; \ m = 1, 2, \ldots,)
\]

there exist a matrix \( A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n} \) and a nonnegative number a such that \( r(A) < 1 \)

\[
\sum_{j=1}^m \int_{t_i}^t \exp \left( \int_s^t p_{ij}(\xi) d\xi \right)(1 - \delta_i \delta_j) p_{ij}(s) ds \leq a_i \quad \text{for } t \in \mathbb{R}
\]

\[
(i, k = 1, \ldots, n),
\]

\[
\sum_{j=1}^n \int_{t_i}^t \exp \left( \int_s^t p_{ij}(\xi) d\xi \right) q_i(s) ds \leq a \quad \text{for } t \in \mathbb{R}
\]

and

\[
\sup_{t \in [t_i, t]} \int_{t_i}^t p_{ii}(\xi) d\xi < +\infty \quad \text{for } i \in N_0(t_1, \ldots, t_n).
\]

Then for any \( c_i \in \mathbb{R}_+ \ i \in N_0(t_1, \ldots, t_n) \) the system \((5.1')\) has at least one nonnegative bounded solution satisfying the conditions (6).

**Corollary 2'.** Let all the assumptions of Corollary 1' be fulfilled and

\[
\liminf_{t \to t_i} \int_0^t p_{ii}(\xi) d\xi = -\infty \quad \text{for } i \in \{1, \ldots, n\} \setminus N_0(t_1, \ldots, t_n).
\]

Then for any \( c_i \in \mathbb{R}_+ \ i \in N_0(t_1, \ldots, t_n) \) the system \((5.1')\) has a unique bounded solution satisfying the conditions (6), and this solution is nonnegative.
Corollary 3'. Let there exist \( t_i \in \mathbb{R} \cup \{-\infty, +\infty\}, \), \( b_i \in [0, +\infty], \), \( b_{ik} \in [0, +\infty] \) \((i, k = 1, \ldots, n)\) such that the condition (7) is fulfilled, the real part of every eigenvalue of the matrix \((-\delta_i b_i + b_{ik})_{i,k=1}^{n} \) is negative and the inequalities
\[
\sigma_i(t)p_i(t) \leq -b_i, \quad \sum_{j=1}^{m} (1 - \delta_i \delta_j) \sigma_i(t)p_{ij}(t) \leq b_{ik} \quad (i, k = 1, \ldots, n)
\]
hold almost everywhere on \( \mathbb{R} \). Moreover, let
\[
\lim\sup_{t \to t_i} \int_{t}^{t_i} p_{ij}(s) \, ds : t \in R < +\infty \quad (i=1, \ldots, n).
\]
Then for any \( c_i \in \mathbb{R}^n \) \((i \in \mathbb{N}_0(t_1, \ldots, t_n))\) the system (5.1') a unique bounded solution satisfying conditions (6), and this solution is nonnegative.

Corollary 4'. Let there exist \( t_i \in \{-\infty, +\infty\}, \) \((i=1, \ldots, n)\), a matrix \( A = (a_{ij})_{i,k=1}^{n} \in \mathbb{R}_{+}^{n \times n} \) and a nonnegative number \( r \) such that \( r(A) < 1 \) and the conditions (7) - (9) be fulfilled. Then the system (5.1') has at least one nonnegative bounded solution.

Corollary 5'. Let all the assumptions of Corollary 4' be fulfilled and
\[
\lim\inf_{t \to t_i} \int_{0}^{t} p_{ij}(s) \, ds = -\infty \quad (i=1, \ldots, n).
\]
Then the system (5.1') has a unique bounded solution, and this solution is nonnegative.

Corollary 6'. Let there exist \( \sigma_i \in [-1, 1], \) \( b_i \in [0, +\infty], \) \( b_{ik} \in [0, +\infty] \) \((i, k = 1, \ldots, n)\) such that the condition (7) is fulfilled, the real part of every eigenvalue of the matrix \((-\delta_i b_i + b_{ik})_{i,k=1}^{n} \) is negative and the inequalities
\[
\sigma_i p_{ij}(t) \leq -b_i, \quad \sum_{j=1}^{m} (1 - \delta_i \delta_j) \sigma_i p_{ij}(t) \leq b_{ik} \quad (i, k = 1, \ldots, n)
\]
hold almost everywhere on \( \mathbb{R} \). Moreover, if the conditions [10] are fulfilled, then the system (5.1') has a unique bounded solution, and this solution is nonnegative.

2. Proof of the Main Results

Proof of Theorem 1. By Theorem 1.1 in [1] we obtain that under the assumptions of Theorem 1 there exists at least one bounded solution \( (x_i)_{i=1}^{n} \) of the equation (1), which is a uniform limit of the sequence of functions
\[
x_{im}(t) = c_{im}(y_{im})(t) \quad (i = 1, \ldots, n; \quad m = 1, 2, \ldots),
\]
where \( (y_{im})_{i=1}^{n} \) is the solution of the problem
\[
y_i'(t) - p_i(t)y_i(t) + \sum_{k=1}^{n} e_{ik}(y_k)(t) + q_{im}(t),
\]
on the segment \([a_m, b_m], \) \( e_{im} = \lim_{m \to +\infty} e_{im}, \) \( b_m = \lim_{m \to +\infty} b_m \) are sequences of real numbers such that \( a_m < b_m, \) \( t_i \in [a_m, b_m] \) for \( i \in \mathbb{N}_0(t_1, \ldots, t_n) \) \((m = 1, 2, \ldots), \)
\[
\lim_{m \to +\infty} a_m = -\infty, \quad \lim_{m \to +\infty} b_m = +\infty.
\]
$p_{im}$ and $q_{im}$ are the restrictions of the functions $p_i$ and $q_i$ on the segment $[a_m, b_m]$,

$$\xi_{im}(u)(t) \equiv \xi_i(e_m(u))(t).$$

where

$$e_m(u)(t) \begin{cases} u(t) & \text{for } a_m \leq t \leq b_m \\ u(a_m) & \text{for } t < a_m \\ u(b_m) & \text{for } t > b_m \end{cases}$$

$c_{im} - c_i$ if $i \in \mathcal{N}_0(t_1, \ldots, t_n)$, $c_{im} = 0$ if $i \in \{1, \ldots, n\} \setminus \mathcal{N}_0(t_1, \ldots, t_n)$, $t_{im} = t_k$ if $t_k \in \mathbb{R}$, $t_{im} = a_m$ if $t_k = -\infty$, $t_{im} = b_m$ if $t_k = +\infty$ ($i, k = 1, \ldots, n; m = 1, 2, \ldots$).

On the other hand, we have

$$y_{im}(t) \geq 0 \quad \text{for } t \in [a_m, b_m] \quad (i = 1, \ldots, n; m = 1, 2, \ldots).$$

Consequently,

$$x_i(t) \geq 0 \quad \text{for } t \in \mathbb{R} \quad (i = 1, \ldots, n). \quad \square$$

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References


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