NECESSARY CONDITIONS OF OPTIMALITY IN NEUTRAL TYPE OPTIMAL PROBLEMS WITH NON-FIXED INITIAL MOMENT

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Let \( J = [a, b] \) be a finite interval; \( O \subseteq \mathbb{R}^n, G \subseteq \mathbb{R}^r \) be open sets; \( \tau : \mathbb{R}^1 \to \mathbb{R}^1, \eta : \mathbb{R}^1 \to \mathbb{R}^1 \) be absolutely continuous and continuously differentiable functions, respectively, satisfying the conditions: \( \tau(t) \leq t, \tau(t) > 0, \eta(t) < t, \eta(t) > 0; \gamma(t) = \tau^{-1}(t) \); \( \sigma(t) = \eta^{-1}(t) \); \( q^i : J \times O^2 \to \mathbb{R}^1, i = 0, \ldots, l \) be continuously differentiable functions; \( \Delta = \Delta(J, M) \) be the set of continuously differentiable functions \( \varphi : J \to M, J = [0, 1], \rho(t) = \min\{\eta(t), \tau(t)\}, t \in J \); \( \|\varphi\| = \sup\{\rho(a) + \|\varphi(t)\| : t \in J \} \), \( M \subset O \) be a convex bounded set; \( \Omega \) be the set of measurable functions \( u : J \to U \) such that \( \sigma(t) : t \in J \subset G \) is compact, \( U \subset G \) be an arbitrary set; \( \Omega \) be a set of measurable functions \( v : J \to V, V \subset G \) be a convex bounded set; \( A(t, v) \) the an \( n \times n \) dimensional matrix function, continuous on \( J \times V \) and continuously differentiable with respect to \( \nu \in V \).

Next, let the function \( f : J \times O^2 \times G \to \mathbb{R}^n \) satisfy the following conditions:

1) for a fixed \( t \in J \) the function \( f(t, x_1, x_2, u) \) is continuous with respect to \( (x_1, x_2, u) \) \( \in O^2 \times G \) and continuously differentiable with respect to \( (x_1, x_2) \in O^2; \)

2) for a fixed \( (x_1, x_2, u) \) \( \in O^2 \times G \) the functions \( f, f_{x_1}, i = 1, 2, \) are measurable with respect to \( t \); for arbitrary compacts \( K \subset O, W \subset G \) there exists a function \( m_{K,W}(\cdot) \in L_1(J, \mathbb{R}^n_+), \mathbb{R}^n_+ = [0, \infty), \) such that

\[
\|f(t, x_1, x_2, u) + \sum_{i=1}^2 f_{x_i}(\cdot)\| \leq m_{K,W}(t), \quad \forall(t, x_1, x_2, u) \in J \times K^2 \times W.
\]

To every element \( \mu = (t_0, t_1, x_0, \varphi, u, v) \in B = J^2 \times O \times \Delta \times \Omega \times \Omega, t_0 < t_1, \) there corresponds the differential equation

\[
\dot{x}(t) = A(t, v(t)) \dot{x}(\eta(t)) + f(t, x(t), x(\tau(t)), u(t)), \quad t \in [t_0, t_1],
\]

with the initial condition

\[
x(t_0) = \varphi(t_0), \quad t \in [\rho(t_0), t_0], \quad x(t_0) = x_0.
\]

**Definition 1.** The function \( x(t) = x(t, \mu) \in O, t \in [\rho(t_0), t_1], \) said to be a solution corresponding to the element \( \mu \in B, \) if on \( [\rho(t_0), t_1] \) it satisfies the condition (2), while on the interval \([t_0, t_1]\) is absolutely continuous and satisfies the equation (1) almost everywhere.

**Definition 2.** The element \( \mu \in B \) is said to be admissible, if the corresponding solution \( x(t) \) satisfies the conditions

\[
q^i(t_0, t_1, x(t_1)) = 0, \quad i = 1, \ldots, l.
\]

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Definition 3. The element \( \tilde{\mu} = (\tilde{t}_0, \tilde{t}_1, \tilde{z}_0, \tilde{\phi}, \tilde{\psi}) \) in \( B_0 \) is said to be locally optimal, if there exist a number \( \delta > 0 \) and a compact set \( X \subset O \) such that for an arbitrary element \( \mu \in B_0 \) satisfying
\[
|\tilde{f}_0 - t_0| + |\tilde{f}_1 - t_1| + |\tilde{z}_0 - x_0| + |\tilde{\phi} - \phi| + \|\tilde{f} - f\|_X + \sup_{t \in J} |\tilde{\psi}(t) - \psi(t)| \leq \delta
\]
the inequality
\[
q^0(\tilde{f}_0, \tilde{f}_1, \tilde{z}_0, \tilde{\phi}(\tilde{t}) \leq q^0(t_0, t_1, x_0, \phi(t_1))
\]
is fulfilled.

Here
\[
\|\tilde{f} - f\|_X = \int_H H(t; f, X).
\]

\[
H(t; f, X) = \sup \left\{ |\tilde{f}(t, x_1, x_2) - f(t, x_1, x_2)| + \sum_{i=1}^2 |\tilde{f}_i(\cdot) - f_{x_i}(\cdot)| : x_1, x_2 \in X^2 \right\};
\]

\[
\tilde{f}(t, x_1, x_2) = f(t, x_1, x_2, \tilde{u}(t)), \quad f(t, x_1, x_2) = f(t, x_1, x_2, u(t)), \quad \tilde{u}(t) = x(t, \tilde{\mu}).
\]

The problem of optimal control consists in finding a locally optimal element.

Theorem 1. Let \( \tilde{\mu} \in B_0, \tilde{t}_i \in (a, b), i = 0, 1, \) be a locally optimal element, \( \tilde{\psi}(t) \) be a piecewise continuous function; \( \gamma_0 = \gamma(\tilde{t}_0) \in (\tilde{t}_0, \tilde{t}_1), \sigma_0 = \sigma(\tilde{t}_0) \in (\tilde{t}_0, \tilde{t}_1) \); there exist integer numbers \( m_i \geq 0, i = 1, 2, \) such that \( \gamma_0 \in \{ \eta^{m_1+1}(\tilde{t}_1), \eta^{m_2}(\tilde{t}_1) \}, \sigma_0 \in \{ \eta^{m_2+1}(\tilde{t}_1), \eta^{m_2}(\tilde{t}_1) \} \) \( (\eta^q(t) = \eta^q(\tilde{t}) \) and there exist the finite limits:
\[
\lim_{\omega \to \gamma_0} \tilde{f}(\omega) = f_{\gamma_0}, \quad \omega = (t, x_1, x_2) \in \mathbb{R}_{\gamma_0}^2 \times O^2, \mathbb{R}_{\gamma_0}^2 = (\infty, \tilde{t}_0), \nu_0 = (\tilde{t}_0, \tilde{x}_0, \tilde{\psi}(\tilde{t}_0));
\]
\[
\lim_{(\omega_1, \omega_2) \to (\nu_1, \nu_2)} [\tilde{f}(\omega_1) - \tilde{f}(\omega_2)] = f', \quad \omega_i \in \mathbb{R}_{\nu_i}^2 \times O^2, i = 1, 2, \quad \nu_i = (\gamma_0, \tilde{x}(\gamma_0), \tilde{x} \tilde{e});
\]
\[
\nu_2 = (\gamma_0, \tilde{x}(\gamma_0), \tilde{\psi}(\tilde{t}_0)), \quad \lim_{t \to \gamma_0} \tilde{\psi}(t) = \tilde{\psi} \tilde{t}_1;
\]
\[
\lim_{\omega \to \nu_0} \tilde{f}(\omega) = f_{\nu_0}, \quad \omega \in \mathbb{R}_{\nu_0}^2 \times O^2, \nu_0 = (\tilde{t}_0, \tilde{x}(\tilde{t}_0), \tilde{\psi}(\tilde{t}_0));
\]
\[
\lim_{t \to \gamma_0} \tilde{\psi}(t) = \tilde{\psi} \tilde{t}_1;
\]
\[
\lim_{t \to \gamma_0} \tilde{\psi}(t) = \tilde{\psi} \tilde{t}_1;
\]
\[
\lim_{t \to \gamma_0} \tilde{\psi}(t) = \tilde{\psi} \tilde{t}_1;\]

Then there exists a non-zero vector \( \pi = (\pi_0, \ldots, \pi_1), \pi_0 \leq O, \) and solutions \( \psi(t), \chi(t) \) of the system
\[
\begin{cases}
\chi(t) = -\psi(t) \tilde{f}_1, |t| - \psi(\gamma(t)) \tilde{f}_2, |\gamma(t)| \tilde{\psi}(t), \\
\psi(t) - \chi(t) + \psi(\sigma(t)) \tilde{A}(\sigma(t)) \tilde{\psi}(t), \quad t \in [\tilde{t}_0, \tilde{t}_1], \quad \psi(t) = 0, \quad t > \tilde{t}_1,
\end{cases}
\]

and the following conditions are fulfilled:
\[
\int_{\tilde{t}_0}^{\tilde{t}_1} \psi(\gamma(t)) \tilde{f}_1 \tilde{\psi}(t) dt + \int_{\tilde{t}_0}^{\tilde{t}_1} \psi(\sigma(t)) \tilde{A}(\sigma(t)) \tilde{\psi}(t) dt \geq \]
\[
\begin{align*}
\int_{\tau(t_0)}^{t_0} \psi(\eta(t)) f_x \psi(\eta(t)) \varphi(t) dt + \int_{\tau(t_0)}^{t_0} \psi(\eta(t)) \hat{A}(\sigma(t)) \dot{\varphi}(t^+) dt, \quad \forall \varphi \in \Delta, \\
\int_{t_0}^{t_1} \psi(t) f(t) dt \geq \int_{t_0}^{t_1} \psi(t) f(t, \tilde{x}(t), \tilde{\tau}(t), \tilde{u}(t)) dt, \quad \forall u \in \Omega_1; \\
\int_{t_0}^{t_1} \psi(t) \hat{A}_x(t) \dot{\tilde{\eta}}(t) \tilde{\varphi}(t) dt \geq \int_{t_0}^{t_1} \psi(t) \hat{A}_x(t) \dot{\tilde{\eta}}(t) \psi(t^+) dt, \quad \forall \psi \in \varphi_2; \\
\pi \hat{Q}_{x_0} = -\chi(t_0), \quad \pi \hat{Q}_{x_1} = \chi(t_1), \\
\pi \hat{Q}_{t_0} \geq \chi(\tau(t_0)) A_{\tau(t_0)}^+ \dot{\psi}(\eta(t_0)) + \\
\int_{\tau(t_0)}^{t_0} + \psi(\eta_0) A_{\eta_0}^+ [A_{\eta_0}^+ \dot{\psi}(\eta(t_0))] + f^+_{\eta_0} \psi(t_0) |\hat{\sigma}(t_0) + \psi(\eta_0) f^+_{\eta_0} \gamma^+, \\
\pi \hat{Q}_{t_1} \geq -\psi(t_1) A_{t_1}^+ \dot{\tilde{\eta}}(t_1^-) + f^+_{\eta_1}.
\end{align*}
\]

Here \( Q = (q^0, \ldots, q^f)^T \), the tilde over \( Q \) means that the corresponding gradient is calculated at the point \((\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{x}(\tilde{t}_1))\); \( f_x, [t] = f_x(t, \tilde{x}(t), \tilde{\tau}(t)) \), \( f^+_t = f(t, \tilde{x}(t), \tilde{\tau}(t)) \).

**Theorem 2.** Let \( \hat{\mu} \in B_0 \), \( \tilde{t}_i \in (0, b) \), \( i = 0, 1 \), be a locally optimal element, \( \tilde{v}(t) \) be a piecewise continuous function; \( \gamma_0 \in (\tau(t_0), \tau(t_1)) \), \( \sigma_0 \in (\tau(t_0), \tau(t_1)) \); there exist integer numbers \( m_j \geq 0 \), \( i = 1, 2 \), such that \( \gamma_0 \in (\eta^{m_1+1}(t_i), \eta^{m_1}(t_i)) \), \( \sigma_0 \in (\eta^{m_2+1}(t_i), \eta^{m_2}(t_i)) \) and there exist the finite limits:

\[
\begin{align*}
\lim_{\omega \to \tau(\gamma_0)} f(\omega) = f^+_0, \quad \omega \in R^{n} \times O_2, \quad \lim_{\omega \to \tau(\gamma_0)} \hat{A}(\omega) = A^+_{\gamma_0}, \quad i = 0, 1; \\
\lim_{\omega \to \tau(\gamma_0)} f(\omega) = f^+_0, \quad \omega \in R^{n} \times O_2, \quad \lim_{\omega \to \tau(\gamma_0)} \hat{A}(\omega) = A^+_{\sigma_0}, \quad i = 1, 2; \\
\lim_{t \to \tau(\gamma_0)} \hat{A}(t) = A^+_{\gamma_0}, \quad t \in R^{n} \times O_2, \quad i = 1, \ldots, m_1; \\
\lim_{t \to \tau(\sigma_0)} \hat{A}(t) = A^+_{\sigma_0}, \quad t \in R^{n} \times O_2, \quad i = 0, \ldots, m_2.
\end{align*}
\]

Then there exists a non-zero vector \( \pi = (\pi_0, \ldots, \pi_1) \), \( \tau_0 \leq 0 \), and solutions \( \psi(t) \), \( \chi(t) \) of the system (3) such that the conditions (4) - (7) are fulfilled. Moreover,

\[
\pi \hat{Q}_{t_0} \leq \chi(\tau(t_0)) [A_{\eta_0}^+ \dot{\psi}(\eta(t_0)) + f^+_0 + \psi(\eta_0) A_{\eta_0}^+ \dot{\psi}(\eta(t_0)) + f^+_0 + \dot{\tilde{\psi}}(t_1)] |\hat{\sigma}(t_0) + \psi(\eta_0) f^+_0 \gamma^+, \\
\pi \hat{Q}_{t_1} \leq -\psi(t_1) A_{t_1}^+ \dot{\tilde{\eta}}(t_1^-) + f^+_1.
\]

**Theorem 3.** Let \( \hat{\mu} \in B_0 \), \( \tilde{t}_i \in (0, b) \), \( i = 0, 1 \), be a locally optimal element, \( \tilde{v}(t) \) be a piecewise continuous function; \( \gamma_0 \in (\tau(t_0), \tau(t_1)) \), \( \sigma_0 \in (\tau(t_0), \tau(t_1)) \); there exist integer numbers \( m_j \geq 0 \), \( i = 1, 2 \), such that \( \gamma_0 \in (\eta^{m_1+1}(t_i), \eta^{m_1}(t_i)) \), \( \sigma_0 \in (\eta^{m_2+1}(t_i), \eta^{m_2}(t_i)) \); the function \( \tilde{\eta}(t) \) is continuous at the points \( \tau(t_0), \tau(t_1) \); \( \sigma^1(\eta_0), i = 1, \ldots, m_1; \sigma^1(\eta_0), i = 0, \ldots, m_2 \); the function \( \dot{\tilde{\eta}}(t) \) is continuous at the point \( t_1 \).
Then there exists a non-zero vector $\pi = (\pi_0, \ldots, \pi_t)$, $\pi_0 \leq 0$, and solutions $\psi(t)$, $\chi(t)$ of the system (3) such that the conditions (4)-(7) are fulfilled. Moreover,

$$
\pi Q_{t_0} - \chi(t_0)\psi(\sigma_0)\tilde{\psi}(\eta(t_0)) + f(\nu_0) + \psi(\sigma_0)\tilde{\psi}(\eta(t_0)) + f(\nu_0) - \\
- \psi(t_0)\tilde{\psi}(\eta(t_0)) - \psi(t_0)\tilde{\psi}(\eta(t_0)) + f(\nu_0) - \\
\pi Q_{t_1} - \psi(t_1)\tilde{\psi}(\eta(t_1)) + f(\nu_1).
$$

Finally we note that the theorems formulated above are analogues of the theorems given in [1]. These theorems are proved using formulas for the differential of the solution with respect to the initial data and the right-hand side given in [2], by the scheme described in [3].

References


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